CONTINUITY OF MULTIVARIABLE FUNCTIONS. EXAMPLES

1. Definitions

1.1. **Limit.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ some function, $\mathbf{x_0} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y_0} = (y_1, \ldots, y_m) \in \mathbb{R}^m$. Then

$$\lim_{\mathbf{x} \to \mathbf{x_0}} f(\mathbf{x}) = \mathbf{y_0}$$

if and only for $\mathbf{x}$ "close to" $\mathbf{x_0}$, $f(\mathbf{x})$ is "close to" $\mathbf{y_0}$. In other words, given $\epsilon > 0$, there exists δ depending on ϵ and $\mathbf{x_0}$ (so we write $\delta = \delta(\epsilon, \mathbf{x_0})$) such that

$$d_{\mathbb{R}^n}(\mathbf{x}, \mathbf{x_0}) < \delta \Rightarrow d_{\mathbb{R}^m}(f(\mathbf{x}), \mathbf{y_0}) < \epsilon$$

where the distance on the left is taken in $\mathbb{R}^n$, whereas the one on the right-hand side is taken in the target space $\mathbb{R}^m$. We will drop the subscript $\mathbb{R}^n$ from d, and we will often write $\|\mathbf{x} - \mathbf{x_0}\|$ instead of $d(\mathbf{x}, \mathbf{x_0})$. We will also drop the bold-face notation.

Note: a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is clearly given by a row vector $f = (f_1, \ldots, f_m)$ where f_i's are the components of f. In fact, $f_i = p_i \circ f$ (see below). Then $\lim_{x \to x_0} f(x) = y_0$ if and only if $\lim_{x \to x_0} f_i(x) = y_i$, $i = 1, \ldots, m$.

1.2. **Continuity.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in \mathbb{R}^n$ iff

$$\lim_{x \to x_0} f(x) = f(x_0)$$

We say that f is continuous (everywhere) if it is continuous at every point of the domain $\mathbb{R}^n$.

Note: if $f = (f_1, \ldots, f_m)$, where $f_1, \ldots, f_m$ are the components of f, then f is continuous iff $f_1, \ldots, f_m$ are continuous, as functions : $\mathbb{R}^n \to \mathbb{R}$.

2. Tools

Operations with limits: addition, subtraction, multiplication, division (when possible), etc.

Components. If $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$, then $\lim_{x \to x_0} f(x) = y_0 = (y_1, \ldots, y_m)$ iff $\lim_{x \to x_0} f_i(x) = y_i$, for every $i = 1, \ldots, m$.

Projections. The functions $p_i : \mathbb{R}^n \to \mathbb{R}$, $p_i(x_1, \ldots, x_n) = x_i$ are continuous, and this can be easily checked with the (ϵ, δ)- definition of continuity.

Composition of functions. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $g : \mathbb{R}^m \to \mathbb{R}^N$ is continuous, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^N$ is continuous, where $g \circ f(x) := g(f(x))$.

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^N$$

3. Examples

1. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x \sin(z)$.
Then $f = p_1 \cdot (\sin \circ p_3) =$ continuous.

2. $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (xy, x + y + z^2)$.

- $f_1(x, y) = xy = p_1 \cdot p_2 =$ cont.
- $f_2(x, y) = x + y + z^2 = p_1 + p_2 + (p_3)^2 =$ cont.

Therefore $f = (f_1, f_2)$ is continuous.

3. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ Proof that f is continuous.

On $\mathbb{R}^2 - (0, 0)$, f is continuous since it is a ratio of continuous functions (with non-vanishing denominator), namely $f = \frac{p_1^2}{\sqrt{\cdot} \circ (p_1^2 + p_2^2)} =$ continuous.
NB: To make sure we understand why is the denominator continuous, we can write the denominator as the composition of functions

$$\mathbb{R}^2 \xrightarrow{p_1^2+p_2^2} [0, \infty) \xrightarrow{\sqrt{\cdot}} \mathbb{R}$$

We are thus left to show that f is continuous at $(0, 0)$, in other words to prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2 + y^2}} = 0$$

Squeeze:

$$0 \leq \frac{x^2}{\sqrt{x^2 + y^2}} \leq \frac{x^2}{\sqrt{x^2}} = |x|$$

However, as $(x, y) \to (0, 0)$ necessarily $x \to 0$ and hence $|x| \to 0$, therefore by the Pinching Lemma $\frac{x^2}{\sqrt{x^2+y^2}} \to 0$ as $(x, y) \to 0$.

Different way of saying it ((ϵ, δ)-proof). Note that

$$d((x, y), (0, 0)) \leq \epsilon \Leftrightarrow \sqrt{x^2 + y^2} \leq \epsilon \Rightarrow |x| \leq \epsilon \Rightarrow \frac{x^2}{\sqrt{x^2 + y^2}} \leq |x| \leq \epsilon$$

4. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$.

Proof that f is not continuous at $(0, 0)$. Consider the followin sequences of points in $\mathbb{R}^2$:

$$x_k = (\frac{1}{k}, 0), \quad k \geq 1$$
$$y_k = (\frac{1}{k}, \frac{1}{k}), \quad k \geq 1$$

Then $x_k \to (0, 0)$ and $y_k \to (0, 0)$ as $k \to \infty$, and yet

$$f(x_k) = 0 \to 0, \quad k \to \infty$$
$$f(y_k) = \frac{1}{2} \to \frac{1}{2}, \quad k \to \infty$$

which shows that the limit

$$\lim_{x\to(0,0)} f(x)$$

does **not** exist. In particular, the function f is not continuous at $(0, 0)$.

5. Let

$$g(x,y) = \begin{cases} x\sin(1/y), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

Determine the points $(x,y) \in \mathbb{R}^2$ where g is continuous.

6. Tricky example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that $\lim_{k\to\infty} f(x_k) = 0$, for any sequence $x_k \to (0,0)$ such that $x_k, k \geq 1$ is on a fixed line passing through $(0,0)$. Is f continuous at $(0,0)$?