TWO VECTOR FIELDS

1. Setup

1.1. In $\mathbb{R}^3$:

$$\begin{array}{ccccccc} \text{Functs. } \phi & \xrightarrow{\nabla} & \text{V. fields } \overrightarrow{F} & \xrightarrow{\text{curl}} & \text{V. fields } \overrightarrow{G} & \xrightarrow{\text{div}} & \text{Functs. } f \\ \text{Pts.} & \xleftarrow{\partial} & \text{Curves } \gamma & \xleftarrow{\partial} & \text{Surfaces } \Sigma & \xleftarrow{\partial} & \text{Solid domains } W \end{array}$$

Note that

$$\partial(\partial W) = 0, \quad \partial(\partial \Sigma) = 0$$

and the analogue of this is

$$\operatorname{curl}(\nabla \phi) = 0, \quad \operatorname{div}(\operatorname{curl} \overrightarrow{F}) = 0$$

The three theorems of "integrals on the boundary" are:

$$\int_\gamma \nabla \phi = \phi(Q) - \phi(P) \quad P, Q = \partial \gamma = \text{endpoints of } \gamma$$

$$\int_{\partial \Sigma} \overrightarrow{F} = \int\int_\Sigma \operatorname{curl} \overrightarrow{F} \cdot dS \quad \text{[Stokes]}$$

$$\int\int_{\partial W} \overrightarrow{G} \cdot dS = \int\int\int_W (\operatorname{div} \overrightarrow{G})\, dV \quad \text{[Gauss]}$$

2. Two vector fields

2.1. Consider the vector field $\overrightarrow{G} = \dfrac{\mathbf{r}}{r^3}$ (the negative of the gravitational vector field).

a) Prove that $\operatorname{div} \overrightarrow{G} = 0$ ($\overrightarrow{G}$ is incompressible).

b) Prove that

$$\int\int_{\Sigma_R} \overrightarrow{G} \cdot dS = 4\pi$$

where Σ_R is the sphere of radius R centered at the origin.

c) Prove (using Stokes theorem) that $\overrightarrow{G}$ is not the curl of another vector field $\overrightarrow{F}$ (well-defined on $\mathbb{R}^3 - (0,0,0)$) such that $\operatorname{curl} \overrightarrow{F} = \overrightarrow{G}$.

d) What is so special about the geometry of $\mathbb{R}^3 - (0,0,0)$ that a vector field with the properties of $\overrightarrow{G}$ exists?

e) (Gauss's law) Assume Σ is a closed (simple, oriented) surface in $\mathbb{R}^3$, not passing through the origin $(0,0,0)$. Use Gauss' divergence theorem to prove that

$$\int\int_\Sigma \overrightarrow{G} \cdot dS = \begin{cases} 4\pi, & \text{if } (0,0,0) \text{ is in the interior of } \Sigma \\ 0, & \text{otherwise} \end{cases}$$

2.2. (Side computation) Let $\overrightarrow{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ a vector field in $\mathbb{R}^2$. Prove that $\operatorname{curl} F = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x})\mathbf{k}$.

2.2.1. Consider the following vector field in $\mathbb{R}^2$:

$$\overrightarrow{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

a) Prove that curl $\overrightarrow{F} = 0$ (i.e. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} = 0$)

b) Compute the work of $\overrightarrow{F}$ along C_R, the circle of radius R oriented counterclockwise.

c) Prove that $\overrightarrow{F}$ is not a conservative vector field. In other words, prove that there does not exist a (potential) function ϕ (well defined on $\mathbb{R}^2 - (0,0)$) such that $\overrightarrow{F} = \nabla\phi$.

d) What is so special about the geometry of $\mathbb{R}^2 - (0,0)$ that a vector field with the properties of $\overrightarrow{F}$ exists?

e) Let γ an arbitrary simple (without self-intersections) closed curve in $\mathbb{R}^2$, not passing through the origin (oriented counter-clockwise). Use Green's formula to prove that

$$\frac{1}{2\pi}\int_\gamma \overrightarrow{F} = \begin{cases} 1, & \gamma \text{ circles around the origin } (0,0) \\ 0, & \text{otherwise} \end{cases}$$