Solutions for 110.202 Practice Test I

1 Calculate the length of the curve $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$ defined by the function

$$\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$$

as t varies between $t = 0$ and $t = \frac{1}{2}\pi$.

solution The velocity vector is

$$\mathbf{c}'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t) \ ,$$

which has magnitude (also known as speed)

$$|c'(t)| = [9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t]^{1/2} = 3|\sin t \cos t|[\cos^2 t + \sin^2 t]^{1/2} \ ;$$

using the double-angle formula for the sine, we can rewrite this as

$$|c'(t)| = \frac{3}{2}|\sin 2t| \ .$$

The sine is positive in the interval between 0 and π, so we can drop the absolute-value bars and write the arc-length formula as

$$\int_0^{\pi/2} |c'(t)| \ dt = \frac{3}{2} \int_0^{\pi/2} \sin 2t \ dt = -\frac{3}{4} \cos 2t|_0^{\pi/2} \ ;$$

this can be evaluated as

$$-\frac{3}{4}[\cos \pi - \cos 0] = -\frac{3}{4}[(-1) - 1] = \mathbf{\frac{3}{2}} \ .$$

2 The curve $\mathbf{a}(t) = (t, t^2, t^3)$ crosses the plane $4x + 2y + z = 24$ at a single point. Find that point, and then calculate the cosine of the angle between the the tangent vector to $\mathbf{a}$ at that point, and the normal vector to the plane.

solution: To find the point where the curve crosses the plane, we must find t such that $x = t, y = t^2, z = t^3$ lies on the plane: in other words, we must solve the cubic equation

$$4t + 2t^2 + t^3 = 24 \ ,$$

Normally cubic equations are hard to solve, but in this case trial and error ($t = 0$? absurd. $t = 1$? too small. $t = 2$? Just right: $4 \cdot 2 + 2 \cdot 2^2 + 2^3 = 3 \cdot 8 = 24!$) does the trick.

So the tangent direction to $\mathbf{a}(t)$ at $t = 2$ is $\mathbf{a}'(t) = (1, 2t, 3t^2) \mapsto (1, 4, 12)$. On the other hand, the normal direction to the plane is $(4, 2, 1)$, so the cosine of the angle between these vectors is given by the ratio

$$\frac{(4,2,1) \cdot (1,4,12)}{[(4^2 + 2^2 + 1^2) \cdot (1^2 + 4^2 + 12^2)]^{1/2}} ,$$

which equals

$$\frac{(4 + 8 + 12)}{(21 \cdot 161)^{1/2}} = \frac{24}{7\sqrt{69}} = \frac{\mathbf{8}}{\mathbf{161}}\sqrt{\mathbf{69}} .$$

3 Calculate the derivative of the function

$$F(t) = f(x(t), y(t), z(t))$$

defined by the path

$$t \mapsto (x, y, z) = (t, t^2, t^3)$$

and the function

$$f(x, y, z) = x^2 y + z \cos x ,$$

first **directly**, and then by using the chain rule.

solution: Substituting in the equations for the path, we get

$$F(t) = t^2 \cdot t^2 + t^3 \cos t = t^4 + t^3 \cos t ;$$

differentiating this yields

$$F'(t) = 4t^3 + 3t^2 \cos t - t^3 \sin t .$$

On the other hand, the gradient of f is

$$\nabla f(x, y, z) = (2xy - z \sin x, x^2, \cos x) ,$$

while $(x'(t), y'(t), z'(t)) = (1, 2t, 3t^2)$. But the chain rule says

$$F'(t) = \nabla f(x(t), y(t), z(t)) \cdot (1, 2t, 3t^2) = (2t^3 - t^3 \sin t, t^2, \cos t) \cdot (1, 2t, 3t^2) ,$$

which equals

$$2t^3 - t^3 \sin t + 2t \cdot t^2 + 3t^2 \cos t = 4t^3 - t^3 \sin t + 3t^2 \cos t ,$$

in agreement with the previous calculation.

4 Find the points on the surface

$$S = \{(x, y, z) \mid z = g(x, y) = x^2 y^2 + y + 1\}$$

where the tangent plane is parallel to the plane

$$-2x - 3y + z = 1 \ .$$

solution S is the level surface

$$0 = G(x, y, z) = z - g(x, y) = z - x^2y^2 - y - 1$$

so the tangent plane at (x, y, z) is perpendicular to the gradient

$$\nabla G(x, y, z) = (-2xy^2, -2x^2y - 1, 1)$$

of the defining function F. To find points where the tangent plane is parallel to the plane $-2x-3y+z = 1$ is thus the same as finding points where the gradient vector is parallel to the normal vector $(-2, -3, 1)$ of that surface; thus we need to solve the equations

$$-2xy^2 = -2, \ -2x^2y - 1 = -3 \ .$$

We can rewrite these equations as $xy^2 = 1, \ x^2y = 1$; from which it follows that neither x nor y can vanish, so we can divide one into the other to conclude that $x = y$. But then we must have $x^3 = y^3 = 1$, and we can factor the equation $x^3 - 1$ as

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = 0 \ ,$$

so it follows that either $x = 1$ (and hence $y = 1$) and $z = 3$ or that

$$x = \frac{1}{2}(1 \pm \sqrt{(-3)}) \ ,$$

which is a complex number, not relevant to our problem.