PRACTICE PROBLEMS

1. Vector geometry

1.1. Find the equation of the plane that passes through $A(1,2,0)$, $B(0,1,-2)$, $C(4,0,1)$.

1.2. Find the distance from the point $(2,1,-2)$ to the plane

$$x - 2y + 2z + 5 = 0$$

1.3. Given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, do the equations

$$\overrightarrow{v} \times \overrightarrow{a} = \overrightarrow{b} \quad \text{and} \quad \overrightarrow{v} \cdot \overrightarrow{a} = \|a\|$$

determine the vector $\overrightarrow{v}$ uniquely? If so, find an explicit formula of $\overrightarrow{v}$ in terms of $\overrightarrow{a}$ and $\overrightarrow{b}$.

2. Tangent planes & lines

2.1. Find the points on the surface $z = x^2y^2 + y + 1$ where the tangent plane (to the surface) is parallel to the plane $-2x - 3y + z = 1$.

2.2. Find the tangent plane to the graph of the function

$$f(x,y) = \frac{x}{x+y}$$

at the point $(x_0, y_0) = (1,0)$.

2.3. Find a unit vector normal to the graph of $f(x,y) = e^x y$ at the point $(-1,1)$.

2.4. Show that the graphs of $f(x,y) = x^2 + y^2$ and $g(x,y) = -x^2 - y^2 + xy^3$ are tangent at $(0,0)$.

2.5. Consider the surface $z^2 = x^2 + y^2$. Is the tangent plane at the point $(0,0,0)$ well defined?

3. Curves in $\mathbb{R}^3$

3.1. The curve $\mathbf{c}(t) = (t, t^2, t^3)$ crosses the plane $4x+2y+z = 24$ at a single point. Find that point and calculate the cosine of the angle between the tangent vector at $\mathbf{c}$ at that point and the normal vector to the plane.

3.2. A particle travels on the surface of a fixed sphere of radius R centered at the origin, i.e.

$$\|\gamma(t)\| = R, \quad \forall t$$

where $\gamma(t) \in \mathbb{R}^3$ is the position of the particle at time t. Prove that the velocity is always perpendicular on the position vector, i.e.

$$\gamma'(t) \cdot \overrightarrow{\gamma(t)} = 0, \quad \forall t$$

3.3. **a)** Let $V : \mathbb{R}^3 \to \mathbb{R}$ given by $V(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, or in other words $V(\overrightarrow{r}) = \frac{1}{r}$. Compute ∇V.

b) The equation of motion of a planet that orbits the Sun satisfies the equation of motion

$$c''(t) = -GM\frac{\overrightarrow{c(t)}}{\|c(t)\|^3}$$

where $c(t) = (x(t), y(t), z(t))$ is the position of the planet at time t. Prove that the vector (angular momentum)

$$\overrightarrow{L} = \overrightarrow{c(t)} \times c'(t)$$

is independent of time.

c) Prove that the quantity (energy)

$$E = \frac{1}{2}m\|c'(t)\|^2 - \frac{GMm}{\|c(t)\|}$$

is independent of time

3.4. The position of a particle in time is given by

$$c(t) = (\cos(\pi t), \sin(\pi t), \pi t)$$

At time $t_0 = \frac{5\pi}{2}$ the particle is freed of any constraints and starts travelling along the tangent at the constant speed $v = v(t_0)$. Determine how long does it take (starting from t_0) the particle to hit the wall given by the equation $x = 0$.

4. Limits

4.1. Determine whether the following limit exists:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x) - 2x + y}{x^3 + y}$$

5. Differentiability

5.1. Compute f_x, f_y, f_z and evaluate them at the indicated points

a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$; $(3,0,4)$

b) $f(x,y,z) = x\sqrt{x^2+y^2+z^2}$; $(2,1,3)$

c) $f(x,y,z) = xy + e^x \cos y$; $(1, \frac{\pi}{2}, 0)$

5.2. Compute the matrix of partial derivatives of the function $f(x,y) = (x+y, x-y, xy)$.

5.3. **a)** True or false: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$ exist, then f is differentiable at $(1,0)$.

b) True or false: if $\partial_{\overrightarrow{v}} f$ is the directional derivative of f along the vector $\overrightarrow{v}$, then $\partial_{\overrightarrow{j}} f(1,0,1) = f_y(1,0,1)$.

c) True or false: if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(0,1)$ exist, then f is continuous at $(1,0)$.

d) Give the definition of the directional derivative $\partial_{\overrightarrow{v}} f(x_0,y_0)$ where $\overrightarrow{v}$ is some unit vector in $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ a function.

e) Given a function $f:\mathbb{R}^2\to\mathbb{R}$, give the definition of the tangent plane at $(1,0)$ to the graph of f (provided it exists).

f) Assume $f:\mathbb{R}^2\to\mathbb{R}$ is *differentiable* and $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$ exist.
True or false: if $\overrightarrow{v}=p\,\overrightarrow{i}+q\,\overrightarrow{j}$ is a unit vector, then $\partial_{\overrightarrow{v}}f(1,0)=p$.

g) Assume $f:\mathbb{R}^2\to\mathbb{R}$ is *continuous* and $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$ exist.
True or false: if $\overrightarrow{v}=p\,\overrightarrow{i}+q\,\overrightarrow{j}$ is a unit vector, then $\partial_{\overrightarrow{v}}f(1,0)=\nabla f(1,0)\cdot\overrightarrow{v}$.

5.4. **a)** Argue that the function $f:\mathbb{R}^2\to\mathbb{R}^2$ given by $f(x,y,z)=(x+e^z+y,yx^2)$ is differentiable.

b) Compute an approximate value for $f(1.1,0,-0.9)$.

5.5. $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq 0\\ 0, & (x,y)=(0,0)\end{cases}$.

a) Compute $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$, and $\partial_{\overrightarrow{v}}f(0,0)$ for any unit vector $\overline{v}$.

b) Is f differentiable at $(0,0)$?

6. Chain rule

6.1. Use the chain rule to find u_x,u_y,u_z for $u=e^x\cos(yz^2)$.

6.2. Let $g:\mathbb{R}^2\to\mathbb{R}$ a differentiable map, such that $g_z(0,1)=1$, $g_y(0,1)=2$. Determine the rate of change of g (at (1,0)) along the circle centered at the origin and radius 1. In other words, compute $\frac{\partial g}{\partial\theta}$ at $(1,0)$ where $x=r\cos\theta$ and $y=r\sin\theta$.

6.3. Let $(x(t),y(t))$ a path in the plane $0\le t\le 1$, and let $f(x,y)$ a C^1 function of two variables. Assume that

$$x'(t)f_x(x(t),y(t))+y'(t)f_y(x(t),y(t))\le 0$$

Prove that $f(x(1),y(1))\le f(x(0),y(0))$.

6.4. Let $f:\mathbb{R}\to\mathbb{R}$ a differentiable function and $h=f(\frac{x+y}{x-y})$. Prove that $h(z,y)$ satisfies the equation

$$x\frac{\partial h}{\partial x}+y\frac{\partial h}{\partial y}=0$$

6.5. Let $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2+y$ and $h:\mathbb{R}^2\to\mathbb{R}$, $h(u)=(\sin(3u),\cos(8u))$. Let $g(u)=f(h(u))$, for $u\in\mathbb{R}$. Compute $g'(0)$ both directly *and* by using the chain rule.

7. Gradients

7.1. Let $f(x,y,z)=(\sin(xy))e^{-z^2}$. In what direction from $(1,\pi,0)$ should one proceed to increase f most rapidly? Express your answer as a unit vector.