Solutions to Selected Exercises in §2.5

Exercise 8. *Suppose that a function is given in terms of rectangular coordinates by $u = f(x, y, z)$. If $x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$, express $\partial u/\partial\rho, \partial u/\partial\theta$, and $\partial u/\partial\phi$ in terms of $\partial u/\partial x, \partial u/\partial y$, and $\partial u/\partial z$.*

Solution. By the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial\rho} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\rho} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial\rho}\\
&= \frac{\partial u}{\partial x}\cdot\cos\theta\sin\phi + \frac{\partial u}{\partial y}\cdot\sin\theta\sin\phi + \frac{\partial u}{\partial z}\cdot\cos\phi\\
\frac{\partial u}{\partial\theta} &= \frac{\partial u}{\partial x}\cdot(-\rho\sin\theta\sin\phi) + \frac{\partial u}{\partial y}\cdot\rho\cos\theta\sin\phi + \frac{\partial u}{\partial z}\cdot 0\\
&= -\sin\theta\sin\phi\rho\frac{\partial u}{\partial x} + \cos\theta\sin\phi\rho\frac{\partial u}{\partial y}\\
\frac{\partial u}{\partial\phi} &= \frac{\partial u}{\partial x}\cdot\rho\cos\theta\cos\phi + \frac{\partial u}{\partial y}\cdot\rho\sin\theta\cos\phi + \frac{\partial u}{\partial z}\cdot(-\rho\sin\phi)\\
&= \rho\cos\theta\cos\phi\frac{\partial u}{\partial x} + \rho\sin\theta\cos\phi\frac{\partial u}{\partial y} - \rho\sin\phi\frac{\partial u}{\partial z}. \quad \diamond
\end{aligned}$$

Exercise 12. *Suppose that the temperature at the point (x, y, z) in space is $T(x, y, z) = x^2 + y^2 + z^2$. Let a particle follow the right circular helix $\sigma(t) = (\cos t, \sin t, t)$ and let $T(t)$ be its temperature at time t.*

(a) *What is $T'(t)$?*

(b) *Find an approximate value for the temperature at $t = (\pi/2) + 0.01$.*

Solution.

(a) $T(t) = T(\sigma(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$. $T'(t) = 2t$.

(b) By the linear approximation, an approximate value is

$$T\left(\frac{\pi}{2}\right) + T'\left(\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{2} + 0.01 - \frac{\pi}{2}\right) = 1 + \left(\frac{\pi}{2}\right)^2 + 2\cdot\frac{\pi}{2}\cdot 0.01 \approx 3.4988 \quad \diamond$$

Exercise 13. *Suppose that a duck is swimming in the circle $x = \cos t, y = \sin t$, while the water temperature is given by the formula $T = x^2e^y - xy^3$. Find dT/dt : (a) by the chain rule; (b) by expressing T in terms of t and differentiating.*

Solution. (Also sc

(a) We have

By the chain rule, d

$\frac{dT}{dt} = (2xe^y - y^3)$

$= (2\cos t\, e^{\sin t}$

$= -2\cos t \sin$

(b) Substituting for .

and differentiating th

$\frac{dT}{dt} = 2\cos t(-$

$+ \sin t$ s

$= -2\cos$

which is the same as

Exercise 27. *Use the to show that*

Solution. Followin

so that the left hand of $F(x, x)$. We next c

Indeed, this just follo

Now it just remains last expression. First

Suggested Classroom Examples.

2.6.1 Example. Find ∇f if $f(x,y,z) = xy - z^2$.

Solution. Substituting the partial derivatives of f into the formula for the gradient of f, we find $\nabla f(x,y,z) = y\mathbf{i} + x\mathbf{j} - 2z\mathbf{k}$. ◊

2.6.2 Example. Find ∇f for the function

$$f(x,y,z) = e^{xy} - x\cos(yz^2).$$

Solution. Here $f_x(x,y,z) = ye^{xy} - \cos(yz^2)$, $f_y(x,y,z) = xe^{xy} + xz^2\sin(yz^2)$, and $f_z(x,y,z) = 2xyz\sin(yz^2)$, so

$$\nabla f(x,y,z) = \left[ye^{xy} - \cos(yz^2)\right]\mathbf{i} + \left[xe^{xy} + xz^2\sin(yz^2)\right]\mathbf{j} + \left[2xyz\sin(yz^2)\right]\mathbf{k}. \quad \Diamond$$

2.6.3 Example. Sketch the gradient vector field of the function

$$f(x,y) = \frac{x^2}{10} + \frac{y^2}{6}.$$

Solution. The partial derivatives are $f_x(x,y) = x/5$ and $f_y(x,y) = y/3$. Evaluating these for various values of x and y and plotting, we obtain the sketch in Figure 2.6.1. For instance, $f_x(2,2) = \frac{2}{5}$ and $f_y(2,2) = \frac{2}{3}$; thus the vector $\frac{2}{5}\mathbf{i} + \frac{2}{3}\mathbf{j}$ is plotted at the point $(2,2)$, as indicated in the figure. ◊

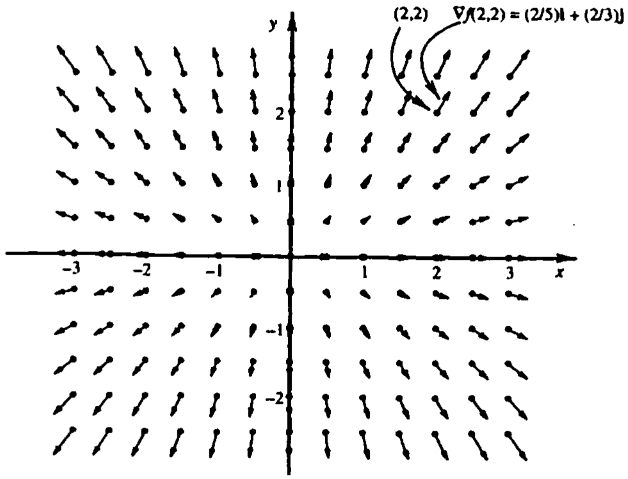

FIGURE 2.6.1. The gradient vector field ∇f, where $f(x,y) = x^2/10 + y^2/6$.

2.6.4 Example. Sketch the gradient vector field of

$$f(x,y) = \frac{x^2}{8} - \frac{y^2}{12}.$$

Solution. $f_x = x/4$ and $f_y = -y/6$, so the gradient vector field is $(x/4)\mathbf{i} - (y/6)\mathbf{j}$. At each point (x_0, y_0), we sketch $(x_0/4, -y_0/6)$. A sketch of this field is shown in Figure 2.6.2. ◊

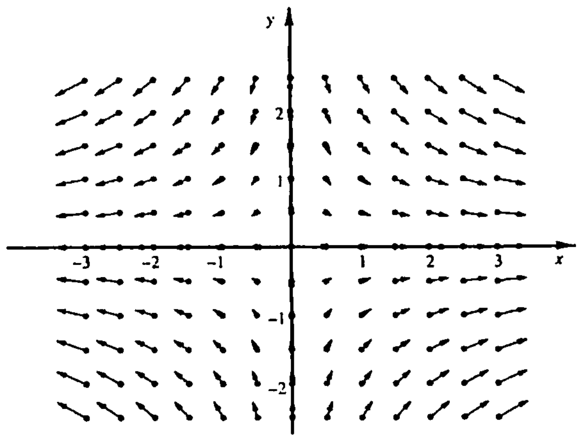

FIGURE 2.6.2. The gradient field of $f(x,y) = x^2/8 - y^2/12$.

2.6.5 Example. Compute the directional derivative of the function

$$f(x,y,z) = e^{-(x^2+y^2+z^2)}$$

at the point $(x_0, y_0, z_0) = (1, 10, 100)$ in the direction $\mathbf{d} = (1, -1, -1)/\sqrt{3}$.

Solution. Note that $\mathbf{d}$ is indeed a unit vector, so we can compute the directional derivative as follows:

$$\nabla f(x,y,z) = \left(-2xe^{-(x^2+y^2+z^2)}, -2ye^{-(x^2+y^2+z^2)}, -2ze^{-(x^2+y^2+z^2)}\right),$$
$$\nabla f(1,10,100) = \left(-2e^{-10101}, -20e^{-10101}, -200e^{-10101}\right),$$
$$\nabla f(1,10,100)\cdot\mathbf{d} = \frac{-2e^{-10101}}{\sqrt{3}} + \frac{20e^{-10101}}{\sqrt{3}} + \frac{200e^{-10101}}{\sqrt{3}}. \quad \Diamond$$

The following example is done at various points in the text in conjunction with computing the gravitational potential and comes up in quite a few of the problems, so is worth considering doing explicitly in lectures.

2.6.6 Example. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$. (Note that r is a scalar and $\mathbf{r}$ is a vector.) Show that

(a) $\nabla r = \dfrac{\mathbf{r}}{r}$ and (b) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{r}}{r^3}$, $r \neq 0$.

What is $\|\nabla(1/r)\|$?

Solution. (a) By the definition of the gradient,

$$\begin{aligned}\nabla r &= \frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}\\ &= \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\mathbf{r}}{r},\end{aligned}$$

where $\mathbf{r}$ is the point (x, y, z). Note that ∇r is the unit vector in the direction of (x, y, z).

(b)

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\mathbf{k}.$$

The partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{x}{r^3},$$

and, similarly,

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}.$$

Thus,

$$\nabla\left(\frac{1}{r}\right) = -\frac{x}{r^3}\mathbf{i} - \frac{y}{r^3}\mathbf{j} - \frac{z}{r^3}\mathbf{k} = -\frac{1}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -\frac{\mathbf{r}}{r^3},$$

as required. Finally,

$$\left\|\nabla\left(\frac{1}{r}\right)\right\| = \left|-\frac{1}{r^3}\right|\|\mathbf{r}\| = \frac{r}{r^3} = \frac{1}{r^2} = \frac{1}{x^2+y^2+z^2}. \quad \diamond$$

2.6.7 Example. Let

$$f(x, y, z) = (\sin xy)e^{-z^2}.$$

In what direction from $(1, \pi, 0)$ should one proceed to increase f most rapidly?

Solution. We compute the gradient:

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\\ &= y\cos(xy)e^{-z^2}\mathbf{i} + x\cos(xy)e^{-z^2}\mathbf{j} + (-2z\sin xy)e^{-z^2}\mathbf{k}.\end{aligned}$$

At $(1, \pi, 0)$ this becomes

$$\pi\cos(\pi)\mathbf{i} + \cos(\pi)\mathbf{j} = -\pi\mathbf{i} - \mathbf{j}.$$

Thus, one should proceed in the direction of the vector $-\pi\mathbf{i} - \mathbf{j}$. $\diamond$

2.6.8 Example. Find the equation for the tangent plane to the surface

$$x^2 + 2y^2 + 3z^2 = 10$$

at the point $(1, \sqrt{3}, 1)$.

Solution. First one checks that the point is indeed on the surface by substituting $x = 1$, $y = \sqrt{3}$, and $z = 1$. Indeed we get $x^2 + 2y^2 + 3z^2 = 10$. Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$ so that the surface is the level set $f = 10$. Thus,

$$f_x = 2x, \quad f_y = 4y, \quad \text{and} \quad f_z = 6z,$$

which, at $(1, \sqrt{3}, 1)$, gives $f_x = 2$, $f_y = 4\sqrt{3}$ and $f_z = 6$, and so the tangent plane is

$$2(x-1) + 4\sqrt{3}(y-\sqrt{3}) + 6(z-1) = 0$$

i.e., $2x + 4\sqrt{3}y + 6z = 20$. $\diamond$

Solutions to Selected Exercises in §2.6

Exercise 2b. *Compute the gradient of the function*

$$f(x, y) = \log(\sqrt{x^2+y^2}).$$

Solution. (Also solved in the Student Guide) Using the chain rule, we get

$$f_x(x, y) = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2},$$

and similarly

$$f_y(x, y) = \frac{y}{x^2+y^2}.$$

Thus,

$$\nabla f(x, y) = \frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j}. \quad \diamond$$

Exercise 3(c). *Compute the directional derivative of the function*

$$f(x, y, z) = xyz$$

at the point $(x_0, y_0, z_0) = (1, 0, 1)$ *in the direction of the unit vector parallel to the vector* $\mathbf{d} = (1, 0, -1)$.

Solution. First, we compute the gradient:

$$\nabla f(x,y,z) = (yz, xz, xy).$$

The directional derivative is therefore

$$\nabla f(1,0,1) \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|} = (0,1,0) \cdot \frac{1}{\sqrt{2}}(1,0,-1)$$
$$= \frac{1}{\sqrt{2}}(0+0+0) = 0. \quad \diamond$$

Exercise 16(a). *Captain Ralph is in trouble near the sunny side of Mercury and notices that the hull of his ship is beginning to melt. The temperature in his vicinity is given by*

$$T = e^{-x} + e^{-2y} + e^{3z}.$$

If he is at the point $(1,1,1)$, *in what direction should he proceed in order to cool fastest?*

Solution. In order to cool the fastest, the captain should proceed in the direction in which T is decreasing the fastest; that is, in the direction of the *negative* gradient at the point $(1,1,1)$, namely $-\nabla T(1,1,1)$. Since

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} = -e^{-x}\mathbf{i} - 2e^{-2y}\mathbf{j} + 3e^{3z}\mathbf{k},$$

we have

$$-\nabla T(1,1,1) = e^{-1}\mathbf{i} + 2e^{-2}\mathbf{j} - 3e^{3}\mathbf{k},$$

which is the direction required. $\diamond$

Selected Revie

Exercise 20. *Let* $(x$
$f(x,y)$ *be a* C^1 *functio*

Show that $f(x(1),y(1))$

Solution. Let $g(t) =$

$g'(t$

By the fundamental the

$g($

Therefore,

$f(x(1),y(1,$

Exercise 22. *Find th*

increases most rapidly at
at this point? Interpret th

Solution. We compute

so $\nabla w(-1,1) = (-1,-1)$.
fastest increase, the requi
at this point is

$\|\nabla w($

which is equal to the rat
$\mathbf{n} = (-1,-1)/\sqrt{2}$ is the un
$(-1,1)$,

$\nabla w \cdot \mathbf{n} =$

verifying the general fact
equals the rate of change o
gradient. $\diamond$

Exercise 42. *Use the ch*