202 Final Solutions

All the questions on this test concern a certain geometric object C (a cone centered around the z-axis) and a certain vector field $\mathbf{V}$ (a whirlpool or vortex line centered around the y-axis). Here are the details:

The function

$$\Phi : (r, \theta, t) \in [0, \infty) \times [0, 2\pi] \times [1, \infty) \to \mathbb{R}^3$$

defined by

$$\Phi(r, \theta, t) = (r \cos\theta,\ r \sin\theta,\ tr)$$

parametrizes a solid cone centered around the z-axis, with vertex at the origin; it is the solid of revolution obtained by rotating the line $y = z$ around the vertical axis.

C is the part of this cone below the plane $z = 1$: it is parametrized by the region where $1 \leq t \leq 1/r$ and $0 \leq r \leq 1$. Its surface ∂C has two parts: the top disk D (where $z = tr = 1$) is parametrized by

$$(r, \theta) \mapsto \Phi_D(r, \theta) = \Phi(r, \theta, 1/r) = (r \cos\theta,\ r \sin\theta,\ 1)\ ,$$

and the remaining part S of the surface of the cone, where $t = 1$, is parametrized by

$$(r, \theta) \mapsto \Phi_S(r, \theta) = \Phi(r, \theta, 1) = (r \cos\theta,\ r \sin\theta,\ r)\ .$$

The vector field

$$\mathbf{V}(x, y, z) = (x^2 + z^2)^{-1}(z, 0, -x)$$

represents a whirlpool or vortex centered along the y-axis.

The formula

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \log \left|\frac{a + u}{a - u}\right|$$

may be useful for calculating some of the integrals which come up below.

This problem was suggested by the lower panel of Marcel Duchamp's 'Large Glass',

`http : //www.valdostamuseum.org/hamsmith/Duch.html` .

1 **[15]** Calculate the Jacobian determinant for Φ, and use that to calculate the volume of the cone C.

2 **[20]** Calculate the surface integral

$$\int\int_S \mathbf{V} \cdot d\mathbf{S} \ .$$

3 **[15]** Reduce the corresponding surface integral

$$\int\int_D \mathbf{V} \cdot d\mathbf{S}$$

to a similar, explicit double integral. Do not waste your time evaluating it directly, though.

4 **[10]** Use Gauss's theorem to evaluate the integral in Problem 3.

5 **[10 + 5]** Calculate the curl $\nabla \times \mathbf{A}$ and the divergence $\nabla \cdot \mathbf{A}$ of the vector field

$$\mathbf{A}(x, y, z) = (yx^{-1}, -\frac{1}{2}\log(1 + x^{-2}z^2),\ 0) \ .$$

6 **[10]** Use Stokes' theorem (and the results of the previous problems) to evaluate the line integral $\int \mathbf{A} \cdot d\mathbf{s}$ around the circular loop

$$t \mapsto (\cos t,\ \sin t,\ 1) \ ,$$

where $0 \leq t \leq 2\pi$.

7 **[15]** Use Gauss's theorem (and the results of Problem 1) to evaluate the integral

$$\int\int_{\partial C} \mathbf{A} \cdot d\mathbf{S}$$

over the entire surface of the cone.

[In this problem the vector field $\mathbf{A}$ has no direct physical interpretation, so if the answer you get looks strange, that's not necessarily anything to worry about.]

Solutions

1. Calculate the Jacobian determinant for Φ, and use that to calculate the volume of the cone C.

Solution: The Jacobian is r^2, and the volume is given by the integral

$$\int_0^{2\pi} \int_{r=0}^{r=1} \int_{t=1}^{t=1/r} r^2 \, dt \, dr \, d\theta \, .$$

The integrand is independent of θ, so this boils down to

$$2\pi \int_{r=0}^{r=1} [1/r - 1] r^2 \, dr = 2\pi[\frac{1}{2}r^2 - \frac{1}{3}r^3]|_0^1 = 2\pi[\frac{1}{2} - \frac{1}{3}] = \frac{\pi}{3} \, .$$

2. Calculate the surface integral

$$\int \int_S \mathbf{V} \cdot d\mathbf{S} \, .$$

Solution:

The normal vector on the surface S of the cone is given by

$$\mathbf{N}_S(r, \theta) = r(-\cos\theta, -\sin\theta, \ 1)$$

while the vector field is given in this parametrization by

$$(\mathbf{V} \circ \Phi_S)(r, \theta) = \frac{1}{r^2 \cos^2\theta + r^2)}(r, \ 0, -r\cos\theta) = \frac{1}{r(1 + \cos^2\theta)}(1, \ 0, -\cos\theta) \, .$$

The surface integral is by definition the double integral

$$\int_0^{2\pi} \int_{r=0}^{r=1} (\mathbf{V} \circ \Phi) \cdot \mathbf{N} \, dr \, d\theta$$

with integrand

$$(\mathbf{V} \circ \Phi_S) \cdot \mathbf{N}_S = \frac{1}{r(1 + \cos^2\theta}(1, \ 0, -\cos\theta) \cdot r(-\cos\theta, -\sin\theta, \ 1) = \frac{-2\cos\theta}{1 + \cos^2\theta} \, ,$$

so the problem reduces to the evaluation of

$$-2 \int_0^{2\pi} \frac{\cos\theta \, d\theta}{1 + \cos^2\theta} = -2 \int \frac{d\sin\theta}{1 + 1 - \sin^2\theta} = -2 \int \frac{du}{2 - u^2}$$

after a change $u = \sin\theta$ of variable. According to the hint, this then equals the difference of

$$-2\frac{1}{2\sqrt{2}} \log|\frac{\sqrt{2} + u}{\sqrt{2} - u}|$$

evaluated at $u = \sin 0 = 0$ and $u = \sin 2\pi = 0$, which is **zero**.

3. Reduce the corresponding surface integral

$$\int\int_D \mathbf{V} \cdot d\mathbf{S}$$

to a similar, explicit double integral.

Solution: The normal vector associated to the parametrization Φ_D of the top disk is

$$\mathbf{N}_D(r, \theta) = r\ (0,\ 0,\ 1)\ ,$$

while

$$(\mathbf{V} \circ \Phi_D)(r, \theta) = \frac{1}{r^2 \cos^2 \theta + 1}(1,\ 0, -r \cos \theta)\ .$$

Their dot product is thus

$$(\mathbf{V} \circ \Phi_D) \cdot \mathbf{N}_D = \frac{-r^2 \cos \theta}{1 + r^2 \cos^2 \theta}\ ,$$

so the integral over the disk is

$$-\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} \frac{r^2 \cos \theta\ d\theta\ dr}{1 + r^2 \cos^2 \theta}\ .$$

We can rewrite the interior integral using the substitution $u = \sin \theta$ as

$$\int \frac{r^2\ du}{1 + r^2(1 - u^2)} = \int \frac{du}{a^2 - u^2}$$

where now $a^2 = 1 - r^{-2}$ to simplify notation. According to the hint, this equals the difference of

$$\frac{1}{2a} \log |\frac{a + u}{a - u}|$$

evaluated at $u = \sin 0 = 0$ and $u = \sin 2\pi = 0$, which is again zero. [This time we still have one further integration, over r, to perform; but the integral of zero remains zero.]

4. Use Gauss's theorem to evaluate the integral in problem three.

Solution: That theorem states that

$$\int\int_{\partial C} \mathbf{V} \cdot d\mathbf{S} = \int\int\int_C \nabla \cdot \mathbf{V}\ d\mathrm{Vol}\ .$$

But now $\mathbf{V} = f\mathbf{W}$ with $f = (x^2 + z^2)^{-1}$ and $\mathbf{W} = (z,\ 0, -x)$, so the product formula for the divergence tells us that

$$\nabla \cdot \mathbf{V} = \nabla f \cdot \mathbf{W} + f \nabla \cdot \mathbf{W}\ .$$

We have

$$\nabla \cdot \mathbf{W} = \partial_x(z) + \partial_y(0) + \partial_z(-x) = 0 \ ,$$

while

$$\nabla(x^2 + z^2)^{-1} = -(x^2 + z^2)^{-2}(2x,\ 0,\ 2z) \ ;$$

but

$$(2x,\ 0,\ -2z) \cdot (z,\ 0, -x) = 0 \ ,$$

so the divergence $\nabla \cdot \mathbf{V} = 0$, which implies that the volume integral is zero. But that volume integral is the sum of two surface integrals, one for the lid of the cone and the other for its side. In problem two we showed that the integral over the side is zero, so the integral over the lid must be zero as well.

5. Calculate the curl $\nabla \times \mathbf{A}$ and the divergence $\nabla \cdot \mathbf{A}$ of the vector field

$$\mathbf{A}(x, y, z) = (yx^{-1}, -\frac{1}{2}\log(1 + x^{-2}z^2),\ 0) \ .$$

Solution: $\nabla \times \mathbf{A} = \mathbf{V}$, while $\nabla \cdot \mathbf{A} = -yx^{-2}$.

6. Use Stokes' theorem (and the results of the previous problems) to evaluate the line integral $\int \mathbf{A} \cdot d\mathbf{s}$ around the circular loop

$$t \mapsto (\cos t,\ \sin t,\ 1) \ ,$$

where $0 \leq t \leq 2\pi$.

Solution: The loop in question is the boundary ∂D of the disk D, so we can state Stokes' theorem in this case as the assertion

$$\int\int_D (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\partial D} \mathbf{A} \cdot d\mathbf{s} \ .$$

Now $\nabla \times \mathbf{A} = \mathbf{V}$ by Problem 5, and we have shown in Problem 4 that

$$\int\int_D \mathbf{V} \cdot d\mathbf{S} = 0 \ .$$

According to the definition of the path integral, this implies that

$$\int_{\partial D} \mathbf{A} \cdot d\mathbf{s} = \int_{t=0}^{t=2\pi} [\tan t \ d\cos t - \frac{1}{2}\log(1 + \cos^{-2} t) \ d\sin t] = 0 \ .$$

I haven't tried to verify this directly; it looks pretty strenuous . . .

7. Use Gauss's theorem (and the results of Problem 1) to evaluate the integral

$$\int\int_{\partial C} \mathbf{A} \cdot d\mathbf{S}$$

over the entire surface of the cone C.

Solution: By Gauss's theorem, this surface integral equals

$$\int\int\int_C (\nabla \cdot \mathbf{A}) \, d\mathrm{Vol} \, .$$

Using the change of variables theorem and the results of Problems 1 and 5, we see that this integral equals

$$-\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{t=1}^{t=1/r} \frac{r \sin\theta}{r^2 \cos^2\theta} \cdot r^2 \, dt \, dr \, d\theta \, .$$

The integral over t is trivial but its limits of integration are not, so this reduces to the double integral

$$-\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \frac{r \sin\theta}{\cos^2\theta} [r^{-1} - 1] dr \, d\theta = (r - \frac{1}{2} r^2)|_0^1 \cdot \int_0^{2\pi} \frac{d \cos\theta}{\cos^2\theta} \, .$$

The final integral then becomes $-\sec\theta|_0^{2\pi} = -1 + 1 = 0$, but maybe one shouldn't look too closely at this calculation ...