Page 427-430

4.

$$\mathbf{X}(s,t) = (s^2\cos t, s^2\sin t, s) \quad -3 \le s \le 3, 0 \le t \le 2\pi$$

(a)

$$\mathbf{X}_s = (2s\cos t, 2s\sin t, 1)$$

$$\mathbf{X}_t = (-s^2\sin t, s^2\cos t, 0)$$

so the normal vector is :

$$\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = (-s^2\cos t, -s^2\sin t, 2s^3)$$

at the point $(-1,0)$ $\mathbf{N}$ is $(-1,0,-2)$

(b) We have : $\mathbf{X}(-1,0) = (1,0,-1)$ and so from a the equation of the tangent plane at the point $(1,0,-1)$ is :

$$-(x-1) - 2(z+1) = -x - 2z - 1 = 0$$

(c)

$$z^4 = x^2 + y^2$$

10.

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

(a) When $\theta = \frac{\pi}{3}$ then we have :

$$\mathbf{X}(r,\frac{\pi}{3}) = (\frac{r}{2}, \frac{\sqrt{3}r}{2}, \frac{\pi}{3})$$

which is the equation of the line $y = \sqrt{3}x$ in the plane $z = \frac{\pi}{3}$. For any other fixed θ the r - coordinates curve is the line $y = \tan\theta x$ in the plane $z = \theta$.

(b) When $r = 1$ then

$$\mathbf{X}(1,\theta) = (\cos\theta, \sin\theta, \theta)$$

which is the equation of the helix. For any other fixed r the θ - coordinates curve is the helix $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta, \theta)$.

Page 448-449

2.

$$\mathbf{X}(s,t) = (s+t, s-t, st)$$

$$\mathbf{X}_s = (1,1,t)$$

$$\mathbf{X}_t = (1, -1, s)$$
$$\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = (s+t, t-s, -2)$$
$$\|\mathbf{N}\| = \sqrt{2s^2 + 2t^2 + 4}$$

(a)

$$\int\int_{\mathbf{X}} f dS = \int\int_{\mathbf{D}} 4\|\mathbf{N}\| dsdt = \int\int_{\mathbf{D}} 4\sqrt{2s^2 + 2t^2 + 4} dsdt$$

If we use the polar coordinates for the region D we have :

$$\int_0^{\frac{\pi}{2}} \int_0^1 4r\sqrt{2r^2+4} drd\theta = \frac{\pi}{3}(6^{\frac{3}{2}} - 8)$$

(b)

$$\int\int_{\mathbf{X}} \mathbf{F}.d\mathbf{S} = \int\int_{\mathbf{D}} \mathbf{F}(\mathbf{X}(s,t)).\mathbf{N}(s,t) dsdt = \int\int_{\mathbf{D}} (s+t, s-t, st).(s+t, t-s, -2) dsdt$$

$$\int\int_{\mathbf{D}} 2stdsdt = \int_0^{\frac{\pi}{2}} \int_0^1 2(r\cos\theta)(r\sin\theta) rdrd\theta = \frac{1}{4}$$

4.

(a) $\mathbf{X}, \mathbf{Y}$ are both parameterizations for the surface $z = 3(x^2 + y^2)$ when $x^2 + y^2 \leq 4$.

(b)

$$\mathbf{X}_s = (\cos t, \sin t, 6s)$$
$$\mathbf{X}_t = (-s\sin t, s\cos t, 0)$$
$$\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = (-6s^2\cos t, -6s^2\sin t, s)$$
$$\int\int_{\mathbf{X}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}).d\mathbf{S} = \int\int_{\mathbf{D}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}).\mathbf{N} dsdt$$
$$= \int\int_{\mathbf{D}} 9s^5 dsdt = \int_0^{2\pi} \int_0^2 9s^5 dsdt = 192\pi$$

$$\mathbf{Y}_s = (2\cos t, 2\sin t, 24s)$$
$$\mathbf{Y}_t = (-2s\sin t, 2s\cos t, 0)$$
$$\mathbf{N} = \mathbf{Y}_s \times \mathbf{Y}_t = (-48s^2\cos t, -48s^2\sin t, 4s)$$
$$\int\int_{\mathbf{Y}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}).d\mathbf{S} = \int\int_{\mathbf{D}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}).\mathbf{N} dsdt$$

$$= \int\int_{\mathbf{D}} 576s^5 ds dt = \int_0^{4\pi}\int_0^1 576s^5 ds dt = 384\pi$$

21.

(b) For $0 \le z \le 1$ we can parameterize S by

$$\mathbf{X}(s,t) = (\cos s, \sin s, t), \quad 0 \le s \le 2\pi, 0 \le t \le 1$$

$$\mathbf{X}_s = (-\sin s, \cos s, 0)$$

$$\mathbf{X}_t = (0,0,1)$$

$$\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t = (\cos s, \sin s, 0)$$

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = (\cos s, \sin s, 0)$$

For $1 \le z \le 9$ we can parameterize S by

$$\mathbf{Y}(s,t) = (s\cos t, s\sin t, s), \quad 0 \le s \le 9, 0 \le t \le 2\pi$$

$$\mathbf{Y}_s = (\cos t, \sin t, 1)$$

$$\mathbf{Y}_t = (-s\sin t, s\cos t, 0)$$

$$\mathbf{N} = \mathbf{Y}_t \times \mathbf{Y}_s = (s\cos t, s\sin t, -s)$$

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{2}}(\cos t, \sin t, -1)$$

(c)

$$\int\int_{\mathbf{X}} \mathbf{F}.d\mathbf{S} = \int_0^{2\pi}\int_0^1 \mathbf{F}.\mathbf{N} dt ds = 0$$

$$\int\int_{\mathbf{Y}} \mathbf{F}.d\mathbf{S} = \int_0^{2\pi}\int_1^9 \mathbf{F}.\mathbf{N} ds dt = \int_0^{2\pi}\int_1^9 -s^2 ds dt = -\frac{1456}{3}\pi$$

$$\int\int_S \mathbf{F}.d\mathbf{S} = \int\int_{\mathbf{X}} \mathbf{F}.d\mathbf{S} + \int\int_{\mathbf{Y}} \mathbf{F}.d\mathbf{S} = -\frac{1456}{3}\pi$$

Page 464-469

4. We can parameterize S by :

$$\mathbf{X}(\theta,\phi) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi) \quad 0 \le \theta \le 2\pi, \frac{\pi}{2} \le \phi \le \pi$$

then

$$\mathbf{X}_\phi = (2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi)$$

$$\mathbf{X}_\theta = (-2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0)$$

$$\mathbf{N} = \mathbf{X}_\phi \times \mathbf{X}_\theta = (4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi)$$

and so

$$\int_S \nabla\times\mathbf{F}.d\mathbf{S} = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}(5,2,-1).\mathbf{N}d\phi d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}(20\sin^2\phi\cos\theta + 8\sin^2\phi\sin\theta - 4\sin\phi\cos\phi)d\theta d\phi = -4\pi$$

on the other hand we can parameterize the boundary of S by :

$$\mathbf{r}(\theta) = (\cos(-\theta), \sin(-\theta), 0) = (\cos\theta, -\sin\theta, 0) \quad 0\leq\theta\leq 2\pi$$

therefore

$$\oint_C \mathbf{F}.d\mathbf{r} = \int_0^{2\pi}(2\sin\theta)(-2\sin\theta) + (2\cos\theta + 4\sin^2\theta)(2\cos\theta)d\theta = -4\pi$$

9. The boundary of D is $S_a\cup S_b$. We can parameterize S_b by :

$$\mathbf{X}(\theta,\phi) = (b\sin\phi\cos\theta, b\sin\theta\sin\phi, b\cos\phi)$$

$$\mathbf{X}_\phi = (b\cos\phi\cos\theta, b\cos\phi\sin\theta, -b\sin\phi)$$

$$\mathbf{X}_\theta = (-b\sin\phi\sin\theta, b\sin\phi\cos\theta, 0)$$

$$\mathbf{N} = \mathbf{X}_\phi\times\mathbf{X}_\theta = (b^2\sin^2\phi\cos\theta, b^2\sin^2\phi\sin\theta, b^2\sin\phi\cos\phi)$$

so

$$\int_{S_b}\mathbf{F}.d\mathbf{S} = \int_0^{\pi}\int_0^{2\pi}\mathbf{F}(\mathbf{X}(\theta,\phi)).\mathbf{N}d\theta d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}(\sin\phi\cos\theta, \sin\theta\sin\phi, \cos\phi).(b^2\sin^2\phi\cos\theta, b^2\sin^2\phi\sin\theta, b^2\sin\phi\cos\phi)d\theta d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}b^2(\sin^3\phi\cos^2\theta+\sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi)d\theta d\phi = \int_0^{\pi}\int_0^{2\pi}b^2\sin\phi d\theta d\phi = 4\pi b^2$$

similarly we have :

$$\int_{S_a}\mathbf{F}.d\mathbf{S} = -4\pi a^2$$

and so :

$$\int_{\partial D}\mathbf{F}.d\mathbf{S} = 4\pi(b^2-a^2)$$

on the other hand :

$$\nabla.\mathbf{F} = 2\sqrt{x^2+y^2+z^2} = \frac{2}{\rho}$$

$$\int_D \nabla.\mathbf{F}dV = \int_0^{\pi}\int_0^{2\pi}\int_a^b(\frac{2}{\rho})\rho^2 d\rho d\theta d\phi = 4\pi(b^2-a^2)$$

12. If we define D to be the region :

$$\{(x,y,z) : x^2+y^2 \leq 1, 1 \leq z \leq e^{1-x^2-y^2}\}$$

then the boundary of D is $S \cup S'$ where

$$S' = \{(x,y,1) : x^2+y^2 \leq 1\}$$

We can parameterize S' by :

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 1)$$

$$\mathbf{X}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\mathbf{X}_r = (\cos\theta, \sin\theta, 0)$$

$$\mathbf{N} = \mathbf{X}_\theta \times \mathbf{X}_r = -r\mathbf{k}$$

$$\int_{S'} \mathbf{F}.d\mathbf{S} = \int_0^{2\pi}\int_0^1 \mathbf{F}(\mathbf{X}(r,\theta)).\mathbf{N} dr d\theta = \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, 0).(0,0,-r) dr d\theta = 0$$

on the other hand $\nabla.F = 0$ and so from the Gauss theorem we have :

$$\int_S \mathbf{F}.d\mathbf{S} + \int_{S'} \mathbf{F}.d\mathbf{S} = \int_D \nabla.\mathbf{F} dV = 0$$

and so $\int_S \mathbf{F}.d\mathbf{S} = 0$