THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
1 MIDTERM EXAM - FALL SESSION 2005
110.211 - HONORS CALCULUS III.

Examiner: Professor C. Consani
Duration: 1 HOUR (12pm-1pm), October 19, 2005.

No calculators allowed. **Total Marks = 100**

SOLUTIONS

1. [25 marks] Consider the curve C in $\mathbb{R}^2$ having polar equation

$$\rho = a(1+\cos\theta), \qquad a \in \mathbb{R},\ a>0, \quad \theta \in [0,2\pi).$$

1a. [10 marks] Graph C and determine the description of the corresponding path

$$\underline{x} = \underline{x}(\theta)$$

using the basic conversion relations "polar to Cartesian coordinates".

$$x = x(\theta), \quad y = y(\theta).$$

Sol.

$$\underline{x} : [0,2\pi) \longrightarrow \mathbb{R}^2, \qquad \underline{x}(\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta) =$$
$$= (a(1+\cos\theta)\cos\theta, a(1+\cos\theta)\sin\theta).$$

1b. [5 marks] Is $\underline{x}(\theta)$ a C^1-path on $[0,2\pi)$? Does C present any interesting symmetry? Explain in details your answers.

Sol. $\underline{x}(\theta)$ is a piecewise C^1-path on $[0,2\pi)$: $||\underline{x}'(\theta)|| = \sqrt{\rho^2(\theta) + (\frac{d\rho}{d\theta})^2}$ and $\rho^2(\theta) + (\frac{d\rho}{d\theta})^2 (= a^2(1+\cos\theta)^2 + a^2\sin^2\theta) = 0$ iff $\theta = \pi$.
Note that the tangent line to C at $(0,0)$ does not exist but $\lim_{\theta\to 0^-} \frac{x(\pi+\theta)-x(\pi)}{\theta}$ exists as well as $\lim_{\theta\to 0^+} \frac{x(\pi+\theta)-x(\pi)}{\theta}$ and similar conclusion holds for $y(\theta)$.
One sees that $x(2\pi-\theta) = x(\theta)$ and $y(2\pi-\theta) = y(\theta)$, it follows that C is symmetric with respect to the X-axis.

1c. [10 marks] Compute the length $L(\underline{x})$ of the path.

Sol. It follows from part 1b. (i.e. symmetry) that

$$L(\underline{x}) = 2a\int_0^{\pi} \sqrt{2(1+\cos\theta)}d\theta = 4a\int_0^{\pi} \cos(\frac{\theta}{2})d\theta = 8a.$$

Use the trig. formula $\sqrt{\frac{1+\cos\theta}{2}} = +\cos\frac{\theta}{2}$ (the sign $+$ is dictated by the assumption $\theta \in [0,\pi)$)

2. [25 marks] Consider the following scalar-valued real function

$$f(x, y) = \frac{|x| + |y|}{x^2 + y^2}.$$

2a. [5 marks] Determine the domain and the range of $f(x, y)$.

Sol. $f : dom(f) \to \mathbb{R}$, with $dom(f) = \mathbb{R}^2 - \{(0, 0)\}$ and $range(f) = \mathbb{R}_{>0}$.

2b. [15 marks] Compute, if exists

$$\lim_{(x,y)\to(0,0)} f(x, y).$$

Sol. Note that $f(0, y) = \frac{|y|}{y^2}$ and $\lim_{y\to 0} \frac{|y|}{y^2} = +\infty$. Use polar coordinates (centered at the origin $(0, 0)$) to show that $\lim_{(x,y)\to(0,0)} f(x, y) = +\infty$. In fact, we have

$$g(\rho) \le f(\rho\cos\theta, \rho\sin\theta) > 0, \qquad \text{and} \qquad \lim_{\rho\to 0^+} g(\rho) = +\infty$$

where

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{|\cos\theta| + |\sin\theta|}{\rho} \ge \frac{1}{\rho} = g(\rho)$$

Note that $(|\cos\theta| + |\sin\theta|)^2 = 1 + 2|\cos\theta\sin\theta| \ge 1$, hence $(|\cos\theta| + |\sin\theta|) \ge 1$.

2c. [5 marks] Determine, if exists, a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$g(x, y) = f(x, y), \qquad \forall (x, y) \ne (0, 0).$$

Explain in details your answer.

Sol. It follows from 2b. that such $g(x, y)$ (continuous) does not exist.

3. [25 marks] Consider the following scalar-valued real function

$$f(x,y) = \begin{cases} \dfrac{(|x|+y)e^x}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

3a. [20 marks] Study the continuity and the differentiability of $f(x,y)$ at the points: $0 = (0,0)$, $P = (0,1)$ and $Q = (1,1)$.

Sol. $Dom(f) = \mathbb{R}^2$.
-Study of the continuity and differentiability of f at $0 = (0,0)$: if $y \neq 0$, $f(0,y) = \frac{1}{y}$ and $\lim_{y\to 0} f(0,y)$ does not exist, hence f is not continuous at $0 = (0,0)$ then f is not differentiable there.
-Study of the continuity and differentiability of f at $P = (0,1)$: $f(0,1) = 1 = \lim_{(x,y)\to(0,1)} f(x,y)$ as outside the origin, f is a composite of continuous functions.

$$\frac{\partial f}{\partial x}(0,1) = \lim_{x\to 0} \frac{f(x,1)-f(0,1)}{x} = \lim_{x\to 0}\left(\frac{(|x|+1)e^x}{x^2+1} - 1\right)\frac{1}{x} =$$

$$= \lim_{x\to 0} \frac{1}{x^2+1}\left(\frac{|x|}{x}e^x + \frac{e^x-1}{x} - x\right)$$

this limit does not exist as the right-limit is different from the left-limit. It follows that f is not differentiable at P.
-Study of the continuity and differentiability of f at $Q = (1,1)$: $f(1,1) = e = \lim_{(x,y)\to(1,1)} f(x,y)$. Observe that $f_x(x,y)$ and $f_y(x,y)$ are both continuous in an open neighborhood of Q in $\mathbb{R}^2$, hence f is differentiable there.

3b. [5 marks] Determine the tangent plane to the graph of $z = f(x,y)$ at the point $(1,1,f(1,1))$.

Sol. $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (\frac{1}{2}e, -\frac{1}{2}e)$ and $f(1,1) = e$. The equation of the tangent plane is

$$ex - ey - 2z + 2e = 0.$$

4. [25 marks] Consider the following scalar-valued real function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{y - \arctan x} & \text{if } y \neq \arctan x, \\ 0 & \text{if } y = \arctan x. \end{cases}$$

Determine all unit vectors $\underline{v}$ such that the derivative $D_{\underline{v}} f(0,0)$ exists.

Sol. $dom(f) = \mathbb{R}^2$. A unit vector $\underline{v}$ (with beginning point at the origin) is described by (the end point) $Q = (\cos\theta, \sin\theta)$, for $\theta \in [0, 2\pi)$. To find the vectors $\underline{v}$ such that $D_{\underline{v}} f(0,0)$ exists is equivalent to finding the values of θ such that

$$\lim_{h \to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h}$$

exists and is finite.

$$\lim_{h \to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^2 \cos\theta \sin^2\theta}{h\sin\theta - \arctan(h\cos\theta)}.$$

Use the Taylor expansion of $\arctan\theta$ at $\theta = 0$

$$\arctan(h\cos\theta) \sim h\cos\theta + \frac{(h\cos\theta)^3}{c}, \quad \text{for some} \quad c \in \mathbb{R}.$$

Then

$$-\arctan(h\cos\theta) + h\sin\theta \sim h(-\cos\theta + \sin\theta) - \frac{h^3\cos^3\theta}{c}.$$

When $\sin\theta = \cos\theta$

$$\lim_{h \to 0} (h\sin\theta - \arctan(h\cos\theta)) = 0$$

of order 3, and $\lim_{h\to 0}(h^2 \cos\theta \sin^2\theta) = 0$ of order 2.

Therefore $D_{\underline{v}} f(0,0)$ exists for all $Q = (\cos\theta, \sin\theta)$, with $\theta \in [0, 2\pi) - \{\frac{\pi}{4}, \frac{5}{4}\pi\}$