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15.

$$f(x,y,z) = x^3 + x^2y - yz^2 + 2z^3$$

$$\mathbf{H}f = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 6x+2y & 2x & 0 \\ 2x & 0 & -2z \\ 0 & -2z & 12z \end{pmatrix}$$

so at the point $\mathbf{a} = (1,0,1)$ we have :

$$\mathbf{H}f(\mathbf{a}) = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{pmatrix}$$

19. (a)

$$f(x_1, x_2, ..., x_n) = e^{x_1+2x_2+...+nx_n}$$
$$f_{x_i}(\mathbf{0}) = i \qquad i = 1, 2, ..., n$$
$$f_{x_ix_j}(\mathbf{0}) = ij \qquad i,j = 1, 2, ..., n$$

so

$$\mathbf{H}f(\mathbf{0}) = (f_{x_ix_j}(\mathbf{0}))_{n\times\ n} = (ij)_{n\times n}$$

(b)

$$p_1(x_1, ..., x_n) = f(\mathbf{0}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{0})x_i = 1 + \sum_{i=1}^{n} ix_i$$

$$p_2(x_1, ..., x_n) = f(\mathbf{0}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{0})x_i + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_ix_j}(\mathbf{0})x_ix_j$$
$$= 1 + \sum_{i=1}^{n} ix_i + \frac{1}{2}\sum_{i,j=1}^{n} ijx_ix_j$$

(c)

$$p_1(x_1, ..., x_n) = 1 + Df(\mathbf{0})\mathbf{x}$$

where

$$Df(\mathbf{0}) = (f_{x_1}(\mathbf{0}), f_{x_2}(\mathbf{0}), ..., f_{x_n}(\mathbf{0})) = (1, 2, ..., n)$$

and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

$$p_2(x_1,...,x_n) = 1 + Df(\mathbf{0})\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{H}f(\mathbf{0})\mathbf{x}$$

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7.

$$f(x,y) = xy + \frac{8}{x} + \frac{1}{y}$$

$$f_x(x,y) = y - \frac{8}{x^2} \quad , \quad f_y(x,y) = x - \frac{1}{y^2}$$

$$f_x(x,y) = f_y(x,y) = 0 \Rightarrow (x,y) = (4,\frac{1}{2})$$

$$f_{xx}(x,y) = \frac{16}{x^3} \Rightarrow f_{xx}(4,\frac{1}{2}) = 2$$

$$f_{xy}(x,y) = 1$$

$$f_{yy}(x,y) = \frac{2}{y^3} \Rightarrow f_{yy}(4,\frac{1}{2}) = 16$$

$$d_1 = f_{xx}(4,\frac{1}{2}) = 2 > 0$$

$$d_2 = f_{xx}(4,\frac{1}{2})f_{yy}(4,\frac{1}{2}) - f_{xy}^2(4,\frac{1}{2}) = 31 > 0$$

So f has a minimum at point $(4,\frac{1}{2})$

16.

$$f(x,y,z) = (x^2+2y^2+1)\cos z$$

$$f_x = 2x\cos z \ , \ f_y = 4y\cos z \ , \ f_z = -(x^2+2y^2+1)\sin z$$

$$f_x = f_y = f_z = 0 \Rightarrow (x,y,z) = (0,0,0)$$

$$f_{xx} = 2\cos z \ , \ f_{yy} = 4\cos z \ , \ f_{zz} = -(x^2+2y^2+1)\cos z$$

$$f_{xy} = f_{yx} = 0 \ , \ f_{xz} = f_{zx} = -2x\sin z \ , \ f_{yz} = f_{zy} = -4y\sin z$$

The Hessian matrix of f at $\mathbf{0}$ is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$d_1 = 2 \quad , \quad d_2 = 8 \quad , \quad d_3 = -8$$

so $\mathbf{0}$ is a saddle point for f .

21. (a)

$$f(x,y) = \frac{2y^3 - 3y^2 - 36y + 2}{1+3x^2}$$

$$f_x(x,y) = -\frac{6x(2y^3-3y^2-36y+2)}{(1+3x^2)^2}$$
$$f_y(x,y) = \frac{6y^2-6y-36}{1+3x^2}$$
$$f_x(x,y) = f_y(x,y) = 0 \Rightarrow (x,y) = (0,3), (0,-2)$$
$$f_{xx}(x,y) = \frac{(54x^2-6)(2y^3-3y^2-36y+2)}{(1+3x^2)^2}$$
$$f_{xy}(x,y) = -\frac{36x(y^2-y-6)}{(1+3x^2)^2}$$
$$f_{yy}(x,y) = \frac{12y-6}{1+3x^2}$$

The Hessian matrix of f at (x,y) is

$$\begin{pmatrix} \frac{(54x^2-6)(2y^3-3y^2-36y+2)}{(1+3x^2)^2} & -\frac{36x(y^2-y-6)}{(1+3x^2)^2} \\ -\frac{36x(y^2-y-6)}{(1+3x^2)^2} & \frac{12y-6}{1+3x^2} \end{pmatrix}$$

At point $(0,3)$ we have :

$$d_1 = 474 \quad , \quad d_2 = 14220$$

so f has a minimum at $(0,3)$.

At point $(0,-2)$ we have :

$$d_1 = -276 \quad , \quad d_2 = 8280$$

so f has a maximum at $(0,-2)$.

23. (a)

$$f(x,y) = ax^2 + by^2$$
$$f_x(x,y) = 2ax \ , \ f_y(x,y) = 2by$$
$$f_{xx}(x,y) = 2a \ , \ f_{xy}(x,y) = f_{yx}(x,y) = 0, \ , f_{yy}(x,y) = 2b$$
$$f_x(x,y) = f_y(x,y) = 0 \Rightarrow (x,y) = (0,0)$$
$$d_1 = f_{xx}(x,y) = 2a \ , \ d_2 = 2ab$$

so f has a maximum at $(0,0)$ iff a,b are negative and has a minimum at $(0,0)$ iff a,b are positive. Otherwise $(0,0,0)$ is a saddle point.

(b)

$$f(x,y,z) = ax^2 + by^2 + cz^2$$
$$f_x(x,y,z) = 2ax \ , \ f_y(x,y,z) = 2by \ , \ f_z(x,y,z) = 2cz$$
$$f_x(x,y,z) = f_y(x,y,z) = f_z(x,y,z) = 0 \Rightarrow (x,y,z) = (0,0,0)$$

$$f_{xx}(x,y,z) = 2a \ , \ f_{yy}(x,y,z) = 2b \ , \ f_{zz}(x,y,z) = 2c$$

$$f_{xy} = f_{yx} = f_{xz} = f_{zx} = f_{yz} = f_{zy} = 0$$

$$d_1 = 2a \ , \ d_2 = 4ab \ , \ d_3 = 8abc$$

so f has a maximum at $(0,0,0)$ iff a, b, c are negative and has a minimum at $(0,0,0)$ iff a, b, c are positive. Otherwise $(0,0,0)$ is a saddle point.

(c)

$$f(x_1, x_2, ..., x_n) = a_1 x_1^2 + a_2 x_2^2 + ... + a_n x_n^2$$

$$f_{x_i} = 2a_i x_i \qquad i = 1, 2, ..., n$$

$$f_{x_1} = f_{x_2} = ... = f_{x_n} = 0 \ \Rightarrow \ (x_1, x_2, ..., x_n) = (0, 0, ..., 0)$$

$$f_{x_i x_i} = 2a_i \qquad i = 1, 2, ..., n$$

$$f_{x_i x_j} = 0 \qquad i \neq j$$

$$d_1 = 2a_1 \qquad d_2 = 4a_1 a_2 \quad ... \quad d_n = 2^n a_1 a_2 ... a_n$$

so f has a maximum at $(0, 0, ..., 0)$ iff $a_1, a_2, ..., a_n$ are negative and has a minimum at $(0, 0, ..., 0)$ iff $a_1, a_2, ..., a_n$ are positive. Otherwise $(0, 0, ..., 0)$ is a saddle point.

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19.

$$f(x, y) = x^2 + xy + y^2$$

first we find the critical point of f on the region $\{(x, y) : x^2 + y^2 < 4 \ \}$

$$f_x(x, y) = 2x + y \quad , \quad f_y(x, y) = x + 2y$$

$$f_x(x, y) = f_y(x, y) = 0 \quad \Rightarrow \quad (x, y) = (0, 0)$$

we use the Lagrange method for finding the critical points of f on the boundary $x^2 + y^2 = 4$. If we define $g(x, y) = x^2 + y^2 - 4$ then we have :

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad g(x, y) = 0$$

$$\begin{cases} 2x + 1 = 2\lambda x \\ 2y + 1 = 2\lambda y \\ x^2 + y^2 - 4 = 0 \end{cases}$$

which has 2 solutions : $(x, y) = (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$. Now

$$f(0, 0) = 0 \qquad f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) = 6$$

so the maximum of f on D is 6 and its minimum is 0.

26. If we define

$$g(x,y,z) = x+y+z-4 \quad , \quad h(x,y,z) = x^2+y^2-z$$

then we want to find a minimum of the function $f(x,y,z) = x^2+y^2+z^2$ when $g(x,y,z) = h(x,y,z) = 0$ (note that when the distance is minimum f is also minimum). Using the Lagrange method we have :

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla g \qquad g(x,y,z) = h(x,y,z) = 0$$

$$\begin{cases} 2x = \lambda_1 + 2\lambda_2 x \\ 2y = \lambda_1 + 2\lambda_2 y \\ 2z = \lambda_1 - \lambda_2 \\ x+y+z-4 = 0 \\ x^2+y^2-z = 0 \end{cases}$$

which has 2 solutions : $(x,y,z) = (1,1,2), (-2,-2,8)$. Now

$$f(1,1,2) = 6 \qquad f(-2,-2,8) = 72$$

so the closest point of the intersection of $g(x,y,z) = h(x,y,z) = 0$ to the origin is $(1,1,2)$.

Online Home works :

1.(a) The domain of f is the set $\{(x,y) : |x| \geq |y|\}$.

(b)

$$f_x(x,y) = 2x - \frac{x}{\sqrt{x^2-y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2-y^2}}$$

$$f_x(x,y) = f_y(x,y) = 0 \quad \Rightarrow \quad (x,y) = (\frac{1}{2}, 0) \ , \ (-\frac{1}{2}, 0)$$

The other critical points of f are points $\{(x,y) : |x| = |y|\}$ where f_x , f_y are not defined.

(c) When $y = x$ then $f(x,y) = x^2$ which has no absolute maximum in the domain of f.

2. When $g(x,y) = y^4 - x^2 = 0$ we have

$$y^2 = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

so :

$$f(x,y) = \begin{cases} 2x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

So f has no absolute maximum and its absolute minimum is 0 for the set $\{(x,y) : x \leq 0\}$. If we want to use the Lagrange method we have :

$$\nabla f(x,y) = \lambda \nabla g(x,y) \qquad g(x,y) = 0$$

$$\begin{cases} 1 = -2\lambda x \\ 2y = 4\lambda y^3 \\ y^4 - x^2 = 0 \end{cases}$$

the solutions of these system of equations is the set $\{(x,y) : x < 0\}$. We can't get the point $(0,0)$ by the Lagrange method because $\nabla g(0,0) = 0$.