Johns Hopkins University, Department of Mathematics
110.401 Abstract Algebra - Spring 2009
Midterm

Instructions: This exam has 8 pages. No calculators, books or notes allowed. **You must answer the first 4 questions, and then answer one of question 5 or 6.** Do not answer both. No extra points will be rewarded. Place an "X" through the question you are not going to answer. Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper. Be sure to clearly label each problem and attach them to the exam. You have 75 MINUTES.

Academic Honesty Certification

I certify that I have taken this exam with out the aid of unauthorized people or objects.

Signature: ______________________________ Date: ____________________

Name: ____________________________

Problem	Score
1	
2	
3	
4	
5 or 6	
Total	

1. (10 points) Show that for $n \geqslant 3$, the order $|\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})|$ of the group of automorphisms of $\mathbb{Z}/n\mathbb{Z}$ is even.

   Solution: It is a well-known fact that the order of $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ is equal to $|(\mathbb{Z}/n\mathbb{Z})^{\times}| = \varphi(n)$ (Euler's phi-function). Since $n \geqslant 3$, $-1 \neq 1$ in $\mathbb{Z}/n\mathbb{Z}$, hence -1 has order 2 in $(\mathbb{Z}/n\mathbb{Z})^{\times}$. It follows that 2 divides $|(\mathbb{Z}/n\mathbb{Z})^{\times}| = \varphi(n)$.

2. (20 points) Show that if the center of a group G (not necessarily finite) is of index n in G, then every conjugacy class $\mathcal{O}_g = [g]$ of G ($g \in G$) has at most n elements.

   Solution: For a given $g \in G$, the cardinality of the conjugacy class of g coincides with the index of the centralizer $C_G(g)$: $|\mathcal{O}_g| = |G : C_G(g)|$. Since the center of a group $Z(G)$ is contained in the centralizer of any element of the group, then $n = |G : Z(G)| = |G : C_G(g)| \cdot |C_G(g) : Z(G)|$ from which the result follows.

3. (20 points) Compute $\mathrm{Hom}(D_8, \mathbb{Z}/8\mathbb{Z})$. Explicitly state what group this is isomorphic to.

Solution: In a group homomorphism $\varphi : G \to G_1$, the order (period) of $\varphi(g) \in G_1$ must divide the order of g (and the order of G_1). It is enough to define $\varphi \in \mathrm{Hom}(D_8, \mathbb{Z}/8\mathbb{Z})$ on the generators r ($r^4 = 1$) and s ($s^2 = 1$) of D_8. It follows that $\varphi(r) = \bar{0}, \bar{2}, \bar{4}, \bar{6}$ and that $\varphi(s) = \bar{0}, \bar{4}$. Also, the relation $rs = sr^{-1}$ has to be satisfied which implies that $\varphi(rs) = \varphi(sr^{-1})$. This in turns produces the condition $\varphi(r) + \varphi(s) = \varphi(s) - \varphi(r)$, i.e. $2\varphi(r) = \bar{0}$. So r can not be mapped to $\bar{2}$ or $\bar{6}$. It follows that both r and s need to be mapped to either $\bar{0}$ or $\bar{4}$. This is a group of order 4 with every element having order at most 2, so $\mathrm{Hom}(D_8, \mathbb{Z}/8\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4. (20 points) Let $Z(G)$ be the center of a group G.

(a) (10 points) Define the set $Z(G)$, show that $Z(G)$ is a subgroup of G and in particular show that $Z(G)$ is a normal subgroup of G.

Solution: The center is the set of elements which commute with every element of G. The center is a subgroup by direct application of the Subgroup Criterion, and since action by conjugation is trivial with every element in the center, clearly $Z(G) \trianglelefteq G$.

(b) (10 points) Assume that G is a finite non-abelian group. Show that $|Z(G)| \leqslant \frac{1}{4}|G|$.

Solution: By rearranging, this inequality is $4 \leqslant |G|/|Z(G)| = |G/Z(G)|$ since G is finite. Since G is non-abelian, $|G/Z(G)|$ can not have prime order, else the quotient group would be cyclic and G would be abelian, so $|G/Z(G)| \geqslant 4$ as desired.

5. (30 points) (**ANSWER THIS QUESTION OR NUMBER 6**)

Recall that $GL_n(\mathbb{F}_q)$ denotes the general linear group of invertible $n \times n$ matrices ($n \in \mathbb{N}$) with entries in the finite field $\mathbb{F}_q$, where and $q = p^m$ for some prime p.

(a) (10 points) Show that the order of $GL_n(\mathbb{F}_q)$ is $(q^n - 1)(q^n - q)(q^n - q^2)\ldots(q^n - q^{n-1})$

Solution: We will use the fact from linear algebra that the determinant is nonzero if the columns are linearly independent. Pick a nonzero-vector w_1, of which there are $q^n - 1$ choices. Any vector not in the span of this vector can be chosen for the second column. A 1 dimensional space is isomorphic to $\mathbb{F}_q$, so there are $q^n - q$ choices for w_2. Continuing this way we have $q^n - q^{i-1}$ for w_i.

(b) (10 points) Let F be a field. Consider the special linear group $SL_n(F) = \{A \in GL_n(F) \mid \det(A) = 1\}$. Prove that $SL_n(F) \trianglelefteq GL_n(F)$. (Note: you have to show it's a subgroup first, and then show it is normal!)

Solution: The fact that $SL_n(F)$ is a subgroup follows immediately from the Subgroup Criterion. It is normal since it is the kernel of the determinant homomorphism $\det : \mathrm{GL_n(F)} \to \mathrm{F}^\times$

(c) (10 points) Find an expression for the order of $SL_n(\mathbb{F}_q)$ in terms of the order of $GL_n(\mathbb{F}_q)$.

Solution: Using the first isomorphism theorem on the homomorphism above, which is clearly surjective, we get that $|SL_n(\mathbb{F}_q)| = |GL_n(\mathbb{F}_q)|/|\mathbb{F}_q^\times| = |GL_n(\mathbb{F}_q)|/(q-1)$.

6. (30 points) **(ANSWER THIS QUESTION OR NUMBER 5)**

Let G be a finite group of order pqr, where p, q and r are distinct primes with $2 \leqslant p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

Solution: Apply Sylow's theorem. Assume $n_q \neq 1, n_r \neq 1, n_p \neq 1$.

By Sylow's theorem, $n_r \equiv 1 \bmod r$ and $n_r | pq$. So $n_r = p, q$ or pq. Since $p < q < r$, $p \not\equiv 1 \bmod r$ and $q \not\equiv 1 \bmod r$, so $n_r = pq$.

Similarly, $n_q | pr$, and $n_1 \equiv 1 \bmod q$. Since $p < q, n_q = pr$ or r. Therefore $n_q \geqslant r$

Lastly, $n_p | qr$, so $n_p = q, r$, or qr, so $n_p \geqslant q$.

Counting elements: $|G| = 1 + (r-1)pq + (q-1)n_q + (p-1)n_p \geqslant 1 + (r-1)pq + (q-1)r + (p-1)q$
$= (pqr) + (q-1)(r-1) > pqr = |G|$.

Contradiction, so some Sylow number is 1 and G has a normal Sylow subgroup.