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Abstract Alg - Hw #2 Solutions

TA: Joe Cutrone

1. p. 35 #4

If n is not prime, then $\exists$ some $d \in \mathbb{Z}$ s.t $d \mid n$, say $ds = n$

$\therefore (x^d)^s = x^{ds} = x^n = 0$ in $\mathbb{Z}/n\mathbb{Z}$ so x^d is a zero-divisor

$\therefore \mathbb{Z}/n\mathbb{Z}$ not a field.

p. 35 #5

This follows from statement 2) in the last paragraph of the section, which says $|GL_n(F)| = \prod_{i=0}^{n-1}(q^n - q^i)$

p. 36 #3

$Q_8 = \{ i, j \mid i^4 = j^4 = 1,\ i^2 = j^2,\ i^{-1} j i = j^{-1} \}$

2. p. 40 #2

if $\varphi: G \to H$ is any homomorphism, we always have that $|\varphi(x)| \mid |x|$. Since φ is an isomorphism, $\exists$ an inverse map $\varphi^{-1}: H \to G$, which is also a homomorphism, so $|x| \mid |\varphi(x)|$, thus $|x| = |\varphi(x)|$

p. 40 #7

D_8 has several elements of order 2, namely sr^i for $0 \le i \le 3$, whereas Q_8 only has 1, namely -1

p. 40 #11

The map $\varphi: A \times B \to B \times A$ defined by $(a,b) \mapsto (b,a)$ is easily seen to be an isomorphism

p. 40 #14

By the Subgroup Criterion: $1_G \in \ker\varphi$ so $\ker\varphi \neq \emptyset$.

Now let $x, y \in \ker\varphi$. Then $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} = 1_H \cdot 1_H^{-1} = 1_H \therefore xy^{-1} \in \ker\varphi$ so $\ker\varphi$ is subgroup.

φ injective iff $a_1 \neq a_2 \Rightarrow \varphi(a_1) \neq \varphi(a_2)$ iff $\varphi(a_1) = \varphi(a_2) \Rightarrow a_1 = a_2$ iff $\varphi(a_1 a_2^{-1}) = 1_H \Rightarrow a_1 = a_2$ iff $\ker\varphi = 1_G$

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p. 40 #17

($\Rightarrow$) $\varphi: G \to G$ def by $g \mapsto g^{-1}$ is a homomorphism.

Then for $x,y \in G$, $\varphi(xy) = \varphi(x)\varphi(y)$, ie $(xy)^{-1} = x^{-1}y^{-1}$, ie $y^{-1}x^{-1} = x^{-1}y^{-1}$, ie $xy = yx$ so G abelian.

($\Leftarrow$) if G abelian, then $\forall x,y \in G$, $xy = yx$. In particular, $y^{-1}x^{-1} = x^{-1}y^{-1}$ so $\varphi(xy) = \varphi(x)\varphi(y)$ is a homomorphism

p. 40 #25

a) Follows from linear algebra

b) Using $D_{2n} = \langle r, s \mid r^n = s^2 = 1, r^i s = s r^{-i} \rangle$, easy to check that $|\varphi(s)| = 2$, $|\varphi(r)| = 4$ and $\varphi(r^i s) = \varphi(s r^{-i})$

c) As the matrix is a rotation from part a), and both these matrices have non-zero determinant, thus are invertible,

we see that the ker $\varphi = 0$, thus φ injective by p. 40 #14

3. p. 49 #6

$1_G \in \text{Tor } G$ so $\text{Tor } G \neq \emptyset$.

Let $x,y \in \text{Tor } G$. Then $|xy^{-1}| = |x||y^{-1}| = |x||y| < \infty$ $\therefore xy^{-1} \in \text{Tor } G$ $\therefore$ Tor G subgroup of G by Subgroup Criterion

since G abelian

In $GL_2(\mathbb{Z})$, let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Then $|A| = 4$, $|B| = 3$, but $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

so $AB^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ which has infinite order

p. 49 # 7.

$\text{Tor}(\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) = (0, a)$ for $a \in \mathbb{Z}/n\mathbb{Z}$

(b, a), $b \neq 0$; $(-b, a)$ is a set of elements of infinite order

but $(b,a) + (-b,a) = (0, 2a)$ has finite order.

p. 49 #8

($\Rightarrow$) Let $H \cup K$ be subgroup. Suppose $H \not\subseteq K$. Let $x \in K - H$, $y \in H$. Then $xy \in G$, but $x, y \in H \cup K$

$\therefore xy \in H$ or $xy \in K$. if $xy \in H$, so is $xyy^{-1} = x$ ~~$\Rightarrow\Leftarrow$~~

$\therefore xy \in K$ $\therefore x x^{-1} y = y \in K$ $\therefore K \subseteq H$.

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p 49 # 10

a) $1 \in H \cap K \therefore H \cap K \neq \emptyset$

if $x, y \in H \cap K$, $y^{-1} \in H \cap K \therefore xy^{-1} \in H \cap K$ by properties of groups H and K

$\therefore$ by Subgroup Criterion, $H \cap K \leq G$

b) Similar

4. p. 53 # 6

a) let $H \leq G$. $H \subseteq N_G(H)$ clearly, so just need to show H is a subgroup of $N_G(H)$

$1 \in H$ so $H \neq \emptyset$ and if $x, y \in H$, $xy^{-1} \in H$ since $H \leq G$ $\therefore$ H subgroup.

Counter Example: $G = GL_2(\mathbb{R})$ $H = \{ \begin{pmatrix} a & t \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}^{\times}, t \geq 1 \}$

H not subgroup since $1 \notin H$

b) $H \leq C_G(H) \leftrightarrow \forall h \in H, hah^{-1} = a \ \forall a \in G \leftrightarrow \forall h \in H, ha = ah \leftrightarrow H$ is abelian

p. 53 # 10

let $|H| = 2$, $H \leq G$

$N_G(H) = \{ g \in G \mid gHg^{-1} = H \}$, ie $g\{1, a\}g^{-1} = \{1, a\} \Rightarrow \{1, gag^{-1}\} = \{1, a\} \ \forall g \in G$

ie $g \in C_G(H)$

if $N_G(H) = G$, ie $H \trianglelefteq G$, $C_G(H) = G$ ie ~~[illegible]~~, so $H \leq Z(G)$

Further Ex

5. This is Cor 10, p. 90

6. $|(g, g')| = \text{lcm}(|g|, |g'|)$ follows by definition of order.

$\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{(n,m)}$ iff $(m,n) = 1$ by Prop 6, p. 163

7. $\psi(\bar{a}) = [a]$ not homomorphism since $\psi(8) = \psi(2) = 2 \mod 8$ but $\psi(8) = \psi(4) + \psi(4) = 0 \mod 8$ $\rightarrow\!\!\leftarrow$

As shown in section, $\text{Hom}(\mathbb{Z}_6, \mathbb{Z}_8) \cong \mathbb{Z}_2$, where elements are zero map & $1 \mapsto 4$