Problem Set 3

From "Dummit & Foote"

1. n. 5, 10, 16 p. 48.

2. nn. 10, 20 p. 60.

Further exercises

4. Show that in a commutative group G, the set of elements of finite order determine a subgroup of G.

5. Show that if G is a finite group of even order, there exists at least one element $g \in G$, $g \neq e$ ($e \in G$ denotes the identity of G) such that $g = g^{-1}$.

6. Let G be a group which contains a unique element a of order 2. Show that a belongs to the center $Z(G)$ of G. [Hint: argue by considering the product $(bab^{-1})(bab^{-1})$]

7. Let denote by $Z(G)$ the center of a group G. Let $\varphi : G \to G$ be a homomorphism. Show that if φ is surjective, then $\varphi(Z(G)) \subset Z(G)$.

8. Let $GL(2, \mathbb{Z}/2\mathbb{Z})$ be the group of 2×2 invertible matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$.
   (1) What is the order of $GL(2, \mathbb{Z}/2\mathbb{Z})$?
   (2) Let E be a vector-space of dimension (*i.e.* rank) 2 over (the field) $\mathbb{Z}/2\mathbb{Z}$. Define a non-trivial action of $GL(2, \mathbb{Z}/2\mathbb{Z})$ on E
   $$GL(2, \mathbb{Z}/2\mathbb{Z}) \times E \to E.$$
   (3) Prove that $GL(2, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to S_3.

9. A group G with 35 elements acts on a set M with 19 elements. One also knows that this action does not fix any element of M. How many are the orbits of this action?

10. Let M be a set and G be a group acting on M. Let $x,\ y$ ($x \neq y$) be elements of M belonging to the same orbit. Show that the stabilizers G_x and G_y of the two elements are conjugate subgroups of G *i.e.* there exists $g \in G$ such that $G_y = gG_xg^{-1}$. Deduce that G_x and G_y have the same order.

11. In S_4, determine the centralizer of the permutation $(1\ 2)$. [Hint: Show that in S_n, if $(i\ j)$ is any transposition, then $\sigma(i\ j)\sigma^{-1} = (\sigma(i), \sigma(j))$, $\forall\ \sigma \in S_n$].

12. Let G be a group. We define the *commutator* of a pair of elements $a,\ b \in G$ to be the element
$$[a, b] = a^{-1}b^{-1}ab \in G.$$
Notice that $[a, b] = e$ (identity of G) if and only if a and b commute. Let G' be the subgroup of G generated by the set of the commutators of (pairs of elements of) G. G' is called the *derived* subgroup of G. For a pair $a,\ b$ of elements of G, let denote $a \star b = aba^{-1} \in G$. Show that for any triple $\{g, a, b\}$ of elements of G, one has
$$[g \star a, g \star b] = g \star [a, b].$$
What does it mean for G that $G' = \{e\}$?