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Abstract Algebra Hw #3 Solns

TA: Joe Cutrone

1 **p. 60-61**

10. use the general formula $|x| = \frac{n}{(x,n)}$ to find $|30| = \frac{54}{6} = 9$

*this formula only works for cyclic groups

$\langle 30 \rangle = \{0, 6, 12, 18, 24, 30, 36, 42, 48\}$

11. Cyclic Subgroups of D_8: $\langle 1 \rangle = \{1\}$ $\langle r^2 \rangle = \{1, r^2\}$ $\langle sr^i \rangle = \{1, sr^i\}$

$\langle r \rangle = \{1, r, r^2, r^3\}$ $\langle s \rangle = \{1, s\}$

A proper subgroup that is not cyclic: $\langle s, r^2 \rangle = \{1, s, r^2, sr^2\}$

13 a) $\mathbb{Z} \times \mathbb{Z}_2$ has an element of order 2, namely $(0,1)$, and $\mathbb{Z}$ does not.

b) $\mathbb{Q} \times \mathbb{Z}_2 \not\cong \mathbb{Q}$ by same reason as a)

15. To show $\mathbb{Q} \times \mathbb{Q}$ not cyclic, it suffices to show $\mathbb{Q}$ not cyclic.

Assume $\mathbb{Q} = \langle \frac{p}{q} \rangle$ where $(p,q) = 1$, $q \neq 0$

Then since $\frac{p}{2q} \in \mathbb{Q}$, $\exists\, n \in \mathbb{Z}$ s.t. $n \cdot (\frac{p}{q}) = \frac{p}{2q} \Rightarrow n = \frac{1}{2} \rightarrow\leftarrow$

$\therefore \mathbb{Q}$ not cyclic.

18. Let $h \in H$, $h^n = 1$

The map $\varphi: \mathbb{Z}_n \to H$ defined by $1 \mapsto h$ is a homomorphism (easy to check)

As $\mathbb{Z}_n$ is cyclic, any homomorphism from $\mathbb{Z}_n$ is completely determined by where 1 gets sent,

and since h is fixed, φ is unique.

26. See p. 135 prop 16.

2 **p. 85-86**

3. Let A abelian, $B \leq A$. Let $\alpha B, \beta B \in A/B$. Then $\alpha B + \beta B = (\alpha + \beta)B = (\beta + \alpha)B = \beta B + \alpha B$ $\therefore A/B$ abelian

Let $G = D_8$, $\langle r \rangle \trianglelefteq D_8$ and $D_8/\langle r \rangle \cong \mathbb{Z}_2$, abelian.

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4. In G/N, $(gN)^\alpha = \overbrace{(gN)(gN)\cdots(gN)}^{\alpha \text{ times}} = g^\alpha N \quad \forall \alpha \in \mathbb{Z}$

5. $|gN|$ in G/N is the smallest positive integer n s.t. $(gN)^n = 1$, ie $g^n N = 1$ by #4, ie $g^n \in N$

in $Q_8/\{1,-1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $\{i,-i\}$ has order 2, but $|i| = 4$ in Q_8

9. This is just the norm squared! Easy to check this is a homomorphism, which follows since the norm is multiplicative. The image $\varphi(\mathbb{C}^\times) = \mathbb{R}_{>0}$, and the kernel is the unit circle in $\mathbb{C}^\times$. Fibers of φ are circles of radius $\sqrt{a^2+b^2}$

14. $\mathbb{Q}/\mathbb{Z}$ w.r.t. $+$

a) $\bar{q}$ has one representative $q \in \mathbb{Q}$ with $0 \le q < 1$, since if $q_1 = q_2$ in the coset, $q_1 - q_2 = 0$, ie $q_1 - q_2 \in \mathbb{Z}$

$\therefore \frac{p_1}{q_1} - \frac{p_2}{q_2} = 0$ since $0 \le q < 1$ $\therefore q_1 = q_2$

b) Let $q \in \mathbb{Q}$, $q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, where $q = \frac{a}{b}$

then $|q + \mathbb{Z}|$ is smallest $n \in \mathbb{Z}^+$ s.t. $nq \in \mathbb{Z}$.

every element has order at least b, so $|q + \mathbb{Z}| < \infty$

since b can be arbitrarily large, $\exists$ elements of arbitrarily large order.

c) $\mathbb{Q}/\mathbb{Z} \le \mathbb{R}/\mathbb{Z}$ by subgroup criterion and by b), every ~~ele~~ element has finite order

d) Let $\varphi: \mathbb{Q}/\mathbb{Z} \to \mu_n = \{\zeta \in \mathbb{C}^\times \mid \zeta^n = 1 \ n \in \mathbb{Z}^+\}$ be defined by $\frac{a}{b} \mapsto \zeta_b$

Easy to check this map is a homomorphism with kernel 1 $\therefore$ φ surjective

$\therefore$ by 1st Iso Thm, $\mathbb{Q}/\mathbb{Z} \cong \mu_n$

Further Exercises

3. Let G act transitively on a finite set A. Let $H \trianglelefteq G$. Let $\{\mathcal{O}_i\}_i^r$ be distinct orbits of H on A

a) $\mathcal{O}_i = H \cdot a_i$, $a_i \in A$. Let $g \in G$. Then $\exists j$ s.t. $ga_i \in \mathcal{O}_j = Ha_j$ $\therefore g\mathcal{O}_i = gHa_i = Hga_i = Ha_j = \mathcal{O}_j$

Since G acts transitively on A, G acts transitively on $\{\mathcal{O}_i\}^r$

in particular, for $i \ne j$, $\exists g \in G$ s.t. $g\mathcal{O}_i = \mathcal{O}_j$ $\therefore |\mathcal{O}_i| = |g\mathcal{O}_i| = |\mathcal{O}_j|$

$\therefore$ all orbits have same cardinality.

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b) Let $a \in \mathcal{O}_1$. Then $\mathcal{O}_1 = Ha$. H acts transitively on $\mathcal{O}_1$, so $\mathcal{O}_1$ is an H-orbit of a.

The stabilizer of a in H is $H_a = G_a \cap H$

$\therefore$ have natural bijection $H/H_a \cong Ha = \mathcal{O}_1$

$\therefore |\mathcal{O}_1| = |H : H \cap G_a|$

$$\therefore r = \frac{|A|}{|\mathcal{O}_1|} = \frac{|G|/|G_a|}{|H|/|H \cap G_a|} = \frac{|G|/|G_a|}{|G_aH|/|G_a|} = \frac{|G|}{|G_aH|} = |G : HG_a|$$

4. Let $G/Z(G)$ be cyclic with generator $xZ(G)$

Let $g \in G$. Then $g = x^a z$ where $z \in Z(G)$ since $g \in gZ(G)$ partition G

$$\therefore g_1 g_2 = x^{a_1} z_1 \cdot x^{a_2} z_2 = x_1^{a_1 + a_2} z_1 z_2 = x^{a_2} z_2 x^{a_1} z_2 = g_2 g_1$$

$\therefore G$ abelian

5. Let G group $H < G$.

Then there is a 1:1 correspondence: $\{\text{left cosets of } H \text{ in } G\} \xleftrightarrow{1:1} \{\text{right cosets of } H \text{ in } G\}$

$$gH \longleftrightarrow Hg^{-1}$$