Abstract Algebra – Hw #4 Solutions   TA: Joe Cutrone

1\. p. 95-96 #4) Let $|G| = pq$ for primes p and q.

By Lagrange, $|Z(G)| = 1, p, q$, or pq

if $|Z(G)| = 1$, done

if $|Z(G)| = p$ or q, $|G/Z(G)| = |G|/|Z(G)| = pq/p$ or $pq/q = q$ or p

$\therefore$ by Corr 10 p.90, $G/Z(G)$ is cyclic

$\therefore$ by prev. hw, G is abelian

if $|Z(G)| = pq$, $G = Z(G)$ so G abelian.

#6) Let $H < G$, $g \in G$. Let left coset $g'H = Hg$ for some $g' \in G$. Want to show $g'H = gH$.

$\therefore$ Since cosets partition G, $g'H = gH$ iff $g'H \cap gH \neq \emptyset$

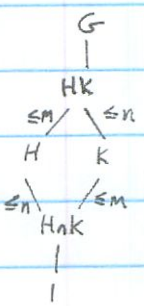
but $g'H = Hg$, and clearly $g \in Hg$ and $g \in gH$ since $1 \in H$.

Since $Hg = gH$, ie $H = gHg^{-1}$, $g \in N_G(H)$

#8) Let H, K finite subgroups of G, with $|H| = n$, $|K| = n$, and $(n,m) = 1$.

Let $\alpha \in H \cap K$. Then by Lagrange, $|\langle\alpha\rangle| \mid |H|$, $|\langle\alpha\rangle| \mid |K|$ $\therefore |\langle\alpha\rangle| \mid (n,m) = 1$ $\therefore |\langle\alpha\rangle| = 1$ $\therefore \alpha = 1$

#10) Let $|G:H| = m$, $|G:K| = n$.



Then $|HK:K| \le n$, and $\dfrac{|HK|}{|K|} = \dfrac{|H|}{|H \cap K|}$ by rearranging formula for $|HK|$.

$\therefore |H : H\cap K| \le n$ as well $\therefore |G : H \cap K| \le m \cdot n$

Since $m \mid |G : H\cap K|$, $n \mid |G: H\cap K|$, clearly $\operatorname{lcm}(m,n) \mid |G : H\cap K|$, so $\operatorname{lcm}(m,n) \le |G : H \cap K|$

if $\gcd(n,m) = 1$, then $\operatorname{lcm}(n,m) = nm$, so result follows

(Note here I used 2nd iso thm and result from #11 on same pg)

#12) $|H| = |gH| = |Hg^{-1}|$, and $\varphi : \{gH\} \longrightarrow \{Hg\}$ is a bijection as shown in previous hw.

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2\. p101 #81

Let p prime, G group of p-power roots of 1 in $\mathbb{C}$, ie $G = \{z \in \mathbb{C} \mid z^{p^\alpha} = 1\}$

Let $\varphi: G \to G$ defined by $z \mapsto z^p$

φ is a homomorphism since $\varphi(z_1 z_2) = (z_1 z_2)^p = z_1^p z_2^p = \varphi(z_1)\varphi(z_2)$

φ surjective since for any $z^{p^\alpha} \in G$, $\varphi(z^{p^{\alpha-1}}) = (z^{p^{\alpha-1}})^p = z^{p^\alpha}$

$\ker \varphi \neq \{1\}$ since any $z^{p^{\alpha-1}} \mapsto z^{p^\alpha} = 1$, so $\ker\varphi$ proper subgroup

$\therefore$ by 1st Iso Thm: $G/\ker\varphi \cong G$

#91 Let p prime, $|G| = p^\alpha m$ for $p \nmid m$. Let $|P| = p^\alpha$, $N \trianglelefteq G$, $|N| = p^b n$, $p \nmid n$

Then $PN \leq G$, $|PN| = \frac{|P|\cdot|N|}{|P\cap N|}$ $\therefore |P\cap N| = \frac{p^\alpha \cdot p^b n}{|PN|}$

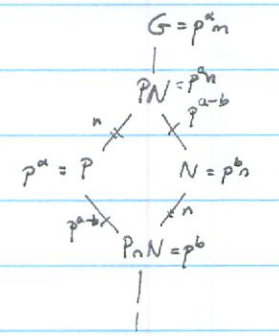
If $N = P$, $a = b$ and result clearly follows

If $N \neq P$, $a \neq b$.

Since $N \trianglelefteq G$, $a > b$ and all powers of $p^b \in P\cap N$

$\therefore |P\cap N| = p^b$

By 2nd Iso Thm, $\frac{|PN|}{|N|} = |PN:N| = |P:P\cap N| = \frac{|P|}{|P\cap N|} = \frac{p^a}{p^b} = p^{a-b}$



3\. Let $H, K \trianglelefteq G$, $|G:H| = p$, $|G:K| = p$, $H \cap K = 1$.

By Cor 15, $HK \leq G$, so by 2nd Iso Thm, $|HK:K| = p = |H : H\cap K| = |H|$

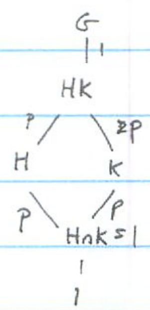
By Lagrange, $|HK| \mid |G|$, so since $|G:H| = |G:HK||HK:H| = p$, $|G:HK| = 1$

ie $G = HK$

By similar reasoning, $|H| = |K| = p$, $|G| = |HK| = p^2$

Since $H = \langle h \rangle$, $K = \langle k \rangle$, where $|h| = |k| = p$, no element has order p^2 $\therefore G$ not cyclic

Easy to see $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$



4\. Let $H = \langle g_1, \ldots, g_r \rangle$ which is abelian since each element pairwise commute.

Since every conjugacy class intersects H non-trivially, $G = \bigcup gHg^{-1}$

by result of #5 (next exercise), H cannot be proper, so $G = H$ which is abelian.

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5. Let H proper subgroup of finite group G, $|G| = n$, $|H| = m$

if $H \trianglelefteq G$, $gHg^{-1} = H$, so clearly $G \neq \bigcup_g gHg^{-1}$

if $H \ntrianglelefteq G$, put H in some maximal subgroup M not normal in G

$\therefore N_G(M) = M$, so number of nonidentity elements of G contained in conjugates of $M \leq (|M|-1)|G:M|$

$\therefore \bigcup_{g \in G} gHg^{-1} \leq \bigcup_{g \in G} gMg^{-1} \leq (|M|-1)|G:M| < |M||G:M| = |G|$