1. p. 95–96

48. Let \( |G| = pq \) for primes \( p \) and \( q \).

By Lagrange, \( |Z(G)| = 1, p, q, \) or \( pq \).

\[ \frac{|G|}{|Z(G)|} = \frac{16}{1} = \frac{16}{16} = \frac{p^4}{p} = \frac{q^2}{q} = q \text{ or } p \]

:. by cor. 10 p. 90, \( Z(G) \) is cyclic.

:. by group law, \( G \) is a kubian.

\[ |(Z(G))| = pq, \quad G = Z(G) \text{ so } G \text{ abelian.} \]

49. Let \( H < G, \ g \in G \). Let left coset \( gH = Hg \) for some \( g \in G \). Want to show \( gH = Hg \).

\[ \text{Since cosets partition } G, \ gH = gH^{g} \text{ iff } gH \cap Hg \neq \emptyset \]

but \( gH = Hg \), and clearly \( g \in Hg \) and \( g \in H \).

\[ \text{Since } Hg = gH, \ ie \ H = gH^{-1}, \ g \in \ker(H). \]

50. Let \( H, K \) finite subgroups of \( G \), with \( |H|=n, \ |K|=n, \) and \( (m,n) = 1 \).

\[ \text{Let } \sigma \in HnK. \ \text{Then by Lagrange, } |\sigma| |H|, \ |\sigma| |K| : |\sigma| = (m,n)=1 : |\sigma|=1 : \sigma=1 \]

51. \( \text{Let } |G:H| = m, \ |G:K| = n. \)

\[ \text{Then } |G:H| |K| = m \cdot n, \text{ and } |G:H| |K| = |G:HnK| \text{ by remaining together for } HnK. \]

\[ \therefore |G:HnK| = m \text{ as well } |G:H| \leq m \cdot n \]

\[ \text{Since } m | \lceil G: |HnK| \rceil, \ n | \lceil G: |HnK| \rceil, \ \text{clearly } \lceil G: |HnK| \rceil = |G:HnK|, \text{ so } \text{lcm}(m,n) = |G:HnK| \]

if \( \gcd(m,n) = 1 \), then \( \text{lcm}(m,n) = mn \), so result follows.

(\text{Note: here I used } 2^\text{nd} \text{ inv thm and result from } #41 \text{ on same pg})

52. \( |H| = |G| = |H|^{-1}, \) and \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a bijection as shown in previous pg.
2. p. 101 #21

Let \( p \) prime, \( G \) group of \( p \)-th power roots of 1 in \( C \), i.e. \( G = \{ z \in C \mid z^p = 1 \} \).

Let \( \phi : G \to G \) defined by \( z \to z^p \).

\( \phi \) is a homomorphism since \( \phi(xy) = (xy)^p = x^p y^p = \phi(x) \phi(y) \).

\( \phi \) surjective since for any \( z \in G \), \( \phi(z^{p^{-1}}) = (z^{p^{-1}})^p = z^p = 1 \).

\( \ker \phi \neq 1 \) since any \( z^{-p^{-1}} \to z^p = 1 \), so \( \ker \phi \) proper subgroup.

By 1st Iso Thm: \( \frac{G}{\ker \phi} \cong \phi[G] = G \).

#21 Let \( p \) prime, \( |G| = p^m \) for \( p \) a prime factor of \( n \). Let \( |G| = p^m \), \( N \leq G \), \( |N| = p^k \), \( p \nmid k \).

Then \( PN \leq G \), \( |PN| = \frac{|P| \cdot |N|}{\gcd(|P|, |N|)} \).

\( \gcd(|P|, |N|) = 1 \) \( \implies \frac{|P| \cdot |N|}{\gcd(|P|, |N|)} = |P| \cdot |N| \).

If \( N \neq P \), \( a = b \) and result clearly follows.

If \( N \neq P \), \( a \neq b \).

Since \( N \leq G \), \( a \neq b \) and all powers of \( b \) in \( \text{PN} \).

\( \implies |\text{PN}| = p^k \).

By 2nd Iso Thm, \( |\text{PN}| = |P| \cdot |N| = |P| \cdot |\text{PN}| = |P| \cdot |P| = p^k = p^k = p^k \).

2. Let \( H, K \leq G \), \( 16 : H \leq p \), \( 16 : K \leq p \), \( H \cap K = 1 \).

By Cor 15, \( HK \leq G \), so by 2nd Iso Thm, \( |HK| = |P| \cdot |HK| = |HK| = |H| \cdot |K| = 1 \).

By Lagrange, \( |HK| = 16 \), so since \( 16 = 16 : H \leq HK \), \( |H| = p \), \( |G : HK| = 1 \).

\( \implies G = HK \).

By similar reasoning, \( |H| = 16 : K \leq p \), \( |G : HK| = p^2 \)

\( \implies G = HK \).

Since \( H = C_p \), \( K = C_p \), hence \( HK = C_p \times C_p \). Since \( H \leq G \), \( K \leq G \), and \( H \cap K = 1 \), no element has order \( p^2 \) in \( G \).

Easy to see \( G \) as \( Z_p \times Z_p \).

4. Let \( H = \langle g \rangle \) which is abelian since each element pairwise commute.

Since every conjugacy class contains \( H \) non-trivially, \( G = U g H g^{-1} \).

By result of 35 (not proven), \( H \) cannot be proper, so \( G = H \) which is abelian.
5. Let $H$ proper subgroup of finite group $G$, $|G| = n$, $|H| = m$
   
   If $H < G$, $gHg^{-1} = H$, so clearly $G \neq U gHg^{-1}$

   If $H \nsubseteq G$, put $H$ in some maximal subgroup $M$ not normal in $G$

   $N_G(M) = M$, so number of nonidentity elements of $G$ contained in conjugates of $M \leq (|M|-1)|G:M|$

   $U gHg^{-1} \leq U gHg^{-1} \leq (|M|-1)|G:M| < |M||G:M| = |G|$