Abstract Alg Hw #5 Solutions TA: Joe Cutrone

1. p. 116-117

#2| To show $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$, need to show $(\sigma G_a \sigma^{-1})(\sigma(a)) = \sigma(a)$

But $(\sigma G_a \sigma^{-1})(\sigma(a)) = (\sigma G_a)(\sigma^{-1}\sigma)(a) = \sigma(G_a(a)) = \sigma(a)$ ✓

Now let G act transitively on A. Assume $\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} \neq 1$

$\therefore$ for $\sigma \in G$, $\sigma \neq 1$, if $\sigma(a) = a$, $\sigma \in G_a$, and by previous part, $\bigcap \sigma G_a \sigma^{-1} = \bigcap G_{\sigma(a)}$

If this intersection is not $= 1$, $\exists \sigma \neq 1$ which is in the kernel of the permutation representation $\varphi: G \to S_n$

but this contradicts G acting transitively on A, which gives that $\ker \varphi = 1$

#10| Let $H, K \leq G$, $HxK = \{hxk \mid h \in H, k \in K\}$

a) $HxK = H \cdot (xK)$ where xK are sets of cosets acted on by H w.r.t left multiplication

$\therefore$ by prop 4 pg 80, $HxK = \bigcup_{h \in H} h(xk)$

b) Similar to a)

c) Let $HxK \cap HyK \neq \emptyset$. Show $HxK = HyK$:

Let $\alpha \in HxK \cap HyK$, so $\alpha = h_1 x k_1 = h_2 y k_2$ for $h_1, h_2 \in H$, $k_1, k_2 \in K$

$\therefore x k_1 = h_1^{-1} h_2 y k_2 = h_3 y k_2$ $\quad \therefore x = h_3 y k_3 \in HyK$

$\therefore h_1 x k_1 = h_1 h_3 y k_3 k_1$ $\quad \therefore HxK \subseteq HyK$.

Similar for reverse inclusion $\therefore$ 2 cosets w/ nonempty intersections coincide

Since $1 \in HxK$, $g = g \cdot 1 \in HgK$ $\forall g \in G$ $\quad \therefore G = \bigcup_{g \in G} gHK$

d) HK = union of left cosets of K, so $HxK = \bigcup_{h \in H} hxK$

since each coset of xK has $|K|$ elements, need # of distinct left cosets of form hxK

but by c), $h_1 x K = h_2 x K$ for $h_1, h_2 \in H$ iff $h_2^{-1} h_1 x^{-1} \in xK$

$\therefore h_1 x K = h_2 x K \leftrightarrow h_2^{-1} h_1 \in H \cap xKx^{-1} \to h_1(H \cap xKx^{-1}) = h_2(H \cap xKx^{-1})$

$$\therefore |HxK| = \frac{|H||K|}{|H \cap xKx^{-1}|} = |K| \cdot |H : H \cap xKx^{-1}|$$

e) Similar to d)

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2. p. 130-132

#6 | Let G be non-abelian group of order 15

By Lagrange's Thm, $|Z(G)| = 1, 3, 5, 15$

if $|Z(G)| = 15$, G abelian $\times$

if $|Z(G)| = 3$ or 5, $|G/Z(G)|$ has prime order, thus cyclic, so G abelian by hw ex $\rightarrow\leftarrow$

$\therefore |Z(G)| = 1$

#28 | Let $|G|$ be odd, $x \neq 1$

Assume x conjugate to x^{-1}, ie $\exists g \in G$ s.t. $gxg^{-1} = x^{-1}$

Let K be the conjugacy class containing x, x^{-1}. Let $y \in K$

Then $\exists g_1 \in G$ s.t $g_1 y g_1^{-1} = x$.

By taking inverses, we see that $g_1 y^{-1} g_1^{-1} = x^{-1}$, so y^{-1} conjugate to x^{-1}

$\therefore y^{-1} \in K$

$\therefore$ for any $y \in K$, $y^{-1} \in K$, so $|K| = \text{even} = |G : C_G(a)|$ which has to divide $|G|$ by Lagrange $\rightarrow\leftarrow$

Further Exercises:

3. The map $\varphi : \langle \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \rangle \longrightarrow Q_8$ defined by $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \mapsto i$, $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \mapsto j$ is easily checked to be an isomorphism satisfying the relations for $Q_8 = \langle i, j \mid i^4 = j^4 = 1, i^2 = j^2, iji^{-1} = j^{-1} \rangle$ (work this out!)

Then all results follow.

4. Let $|G| < \infty$, $H \leq G$, $N \trianglelefteq G$, $(|H|, |G:N|) = 1$. Let $\pi : G \to G/N$

Then $\pi(H) \leq G/N$, and $|\pi(H)| = \frac{|H|}{|\ker \pi|_H|}$

$\therefore |\pi(H)| \mid |G/N| = |G:N|$ and $|\pi(H)| \mid |H| \implies |\pi(H)| = 1$

$\therefore H \leq N$

Abstract Alg Hw #5 Solns

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5. $|S_4| = 24 = 2^3 \cdot 3$

Since $n_2 | 3$, $n_2 = 1$ or 3

Since both $\langle (12), (13)(24) \rangle$ and $\langle (13), (14)(12) \rangle \cong D_8$, can't have $n_2 = 1$ $\therefore n_2 = 3$

↑ by #7 p. 65

Since S_4 contains one subgroup isomorphic to D_8, every Sylow 2-subgroup of S_4 is isomorphic to D_8