Abstract Algebra - Hw #6 Solutions

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1. p 117 #9

Let G act transitively on the finite set A, $H \trianglelefteq G$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be distinct orbits of H on A

a) $\mathcal{O}_i = H \cdot a_i$ for $a_i \in A$. Let $g \in G$. $\therefore \exists j$ s.t $g \cdot a_i \in \mathcal{O}_j = Ha_j$

$\therefore g\mathcal{O}_i = gHa_i = Hga_i = Ha_j = \mathcal{O}_j$

Since G acts transitively on A, G acts transitively on $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$

In particular, for $i \neq j$, $\exists g \in G$ s.t $g\mathcal{O}_i = \mathcal{O}_j$ $\therefore |\mathcal{O}_i| = |g\mathcal{O}_i| = |\mathcal{O}_j|$ so all orbits have same cardinality

b) Let $a \in \mathcal{O}_1$. Then $\mathcal{O}_1 = H \cdot a$. H acts transitively on $\mathcal{O}_1$, so $\mathcal{O}_1$ is an H-orbit of a

The stabilizer of a in H is $H_a = G_a \cap H$

$\therefore$ have natural bijection $H/H_a \cong H \cdot a = \mathcal{O}_1$

$\therefore |\mathcal{O}_1| = |H : H \cap G_a|$

$$\therefore r = \frac{|A|}{|\mathcal{O}_1|} = \frac{|G|/|G_a|}{|H|/|H \cap G_a|} = \frac{|G|/|G_a|}{|G_a H|/|G_a|} = \frac{|G|}{|HG_a|} = |G : HG_a|$$

p. 117 #10

Let $H, K \leq G$. For each $x \in G$, define $HxK = \{hxk \mid h \in H, k \in K\}$

a) $HxK = H \cdot (xK)$ where xK are set of cosets acted on by H w.r.t left multiplication

by prop 4, pg 80, $HxK = \bigcup_{h \in H} h(xk)$

b) Similar to a)

c) Let $HxK \cap HyK \neq \emptyset$. Show $HxK = HyK$. Let $\alpha \in HxK \cap HyK$

$\therefore \alpha = h_1 x k_1 = h_2 y k_2$ for $h_1, h_2 \in H$, $k_1, k_2 \in K$

$\therefore xk_1 = h_1^{-1} h_2 y k_2 = h_3 y k_2$ $\therefore x = h_3 y k_3 \in HyK$

$\therefore h_1 x k = h_1 h_3 y k_3 k_1$ $\therefore HxK \subseteq HyK$

Reverse inclusion is similar $\therefore HxK = HyK$ if $HxK \cap HyK \neq \emptyset$

Since $1 \in HxK$, $g = g \cdot 1 \in HgK$ $\forall g$ $\therefore G = \bigcup_{g \in G} gHK$

d) HxK is the union of left cosets of K, so $HxK = \bigcup_{h \in H} hxK$

Since each coset of xK has $|K|$ elements, need # of distinct left cosets of form hxK.

by part c), $h_1xK = h_2xK$ for $h_1, h_2 \in H$ iff $h_2^{-1} h_1 x^{-1} \in xK$

$\therefore h_1xK = h_2xK \Longleftrightarrow h_2^{-1}h_1 \in H \cap xKx^{-1} \Longleftrightarrow h_1(H \cap xKx^{-1}) = h_2(H \cap xKx^{-1})$

$$\therefore |HxK| = \frac{|H||K|}{|H \cap xKx^{-1}|} = |K| \cdot |H : H \cap xKx^{-1}|.$$

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e) Similar to d)

2. p. 122 #10

Let G be a non-abelian group of order $6 = 2\cdot 3$

$\therefore$ by Cauchy's Thm, $\exists\, H_1, H_2 \leq G$ s.t. $|H_1| = 2$, $|H_2| = 3$, $\therefore H_1 \cong \mathbb{Z}_2$, $H_2 \cong \mathbb{Z}_3$

H_2 has index 2 $\therefore H_2 \trianglelefteq G$

if $H_1 \trianglelefteq G$, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, which is abelian $\rightarrow\leftarrow$ $\therefore H_1 \ntrianglelefteq G$

$\therefore |G/H_1| = 6/2 = 3$, so there are 3 left cosets of H_1, say $\{H_1, gH_1, g_2H\}$

Let G act on this set by left multiplication, which gives rise to the permutation representation $\pi: G \to S_3$

By Thm 3, part 3, p. 119, $\ker\pi = \bigcap_{x\in G} xHx^{-1}$, which is the largest normal subgroup of G contained in H.

Since $H_1 \ntrianglelefteq G$, $\ker\pi = 1$ $\therefore \pi$ injective

Since have an injective map ~~between~~ finite sets of the same order, π surjective $\therefore G \cong S_3$

p. 130 #2

a) $D_8 = \{1\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}$

b) $Q_8 = \{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$

c) $A_4 = \{1\}$, {4 3-cycles}, {3 conjugates to $(12)(34)$}, {4 3-cycles}

3. $|D_8| = 2^3$ $\therefore Syl_2(D_8) = \{D_8\}$ $(n_2 = 1)$

$|D_{10}| = 2\cdot 5$ $\therefore Syl_2(D_8) = \{\{1,s\}, \{1, rs\}, \{1, r^2s\}, \{1, r^3s\}, \{1, r^4s\}\}$ $(n_2 = 5)$

$Syl_5(D_8) = \{\langle r\rangle\}$ $(n_5 = 1)$

4. $|S_4| = 4! = 24 = 2^3\cdot 3$

Only choices for n_2 are 1 and 3 by Thm 18(3).

Since both $\langle (12), (13)(24)\rangle$ and $\langle (13), (14)(12)\rangle \cong D_8$, and both are different set-wise, $n_2 \neq 1$ $\therefore n_2 = 3$

by #7 p. 65

Since S_4 contains one subgroup isomorphic to D_8, every Sylow 2-subgroup of S_4 is isomorphic to D_8 by Thm 18(2)

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$|A_5| = 60 = 2^2 \cdot 3 \cdot 5$

$n_5 \equiv 1 \bmod 5$, $n_5 \mid 2^2 \cdot 3 = 12$ $\therefore n_5$ either 1 or 6, but A_5 simple $\therefore n_5 = 6$

$\therefore$ 6 copies of $P \in Syl_5(A_5)$, where $|P| = 5$ $\therefore P \cong \mathbb{Z}_5$

$n_3 \equiv 1 \bmod 3$, $n_3 \mid 2^2 \cdot 5 = 20$ $\therefore n_3$ either 4 or 10

if $P \in Syl_3(A_5)$, $|P| = 3$ $\therefore P \cong \mathbb{Z}_3$,

By #16, pg 33, there are $\frac{5 \cdot 4 \cdot \cancel{3}}{\cancel{3}} = 20$ distinct 3 cycles in S_5, all contained in A_5

$\therefore n_3 = 10$ and thus 10 copies of $\mathbb{Z}_3$ in A_5

$n_2 \mid 15$ $\therefore n_2 = 3, 5$ or 15

There are at least 5 copies of V_4 in A_5, since can have $\{1, (12), (34), (12)(34)\}$ where in this group, "5" is omitted. Can do the same and just omit each of 1, 2, 3, 4 & 5

But we also know by ex 4 p 142 that $V_4 \trianglelefteq A_4$ (viewed as a copy in A_5), so $A_4 \le N_{A_5}(V_4)$. $\therefore |A_5 : N_{A_5}(V_4)| \le 5$, and since $n_2 = |A_5 : N_{A_5}(V_4)|$, $n_2 = 5$. So there are 5 copies of V_4 in A_5

6. This will follow from #9. Ans: $\mathbb{Z}_{21}$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$

7. let $|G| = 56 = 2^3 \cdot 7$. Assume G ~~is~~ simple

$n_7 \equiv 1 \bmod 7$ and $n_7 \mid 8$ $\therefore n_7 = 1$ or 8, but since we are assuming G is simple, $n_7 = 8$

$\therefore$ we have 48 elements of order 7. Including the identity, that gives $56 - 48 - 1 = 7$ elements, which gives that they must lie in $P \in Syl_2(G)$, so $n_2 = 1$ $\rightarrow\leftarrow$

$\therefore G$ is not simple, with either $n_2 = 1$ or $n_7 = 1$

8. If $|G| = pq$, $p < q$, by Cauchy's Thm, $\exists H \le G$ of order q and index p, which is the smallest prime dividing $|G|$, $\therefore H \trianglelefteq G$

9. This is best done with semi-direct products: See p. 181, example.