Problem Set 7

From "Dummit & Foote"

1. nn. 2, 5, 11, 12, 18, p. 137-138

Further exercises

2. A Boolean ring is a ring with (multiplicative) unit, whose elements are idempotent *i.e.* $x^2 = x$. Show then that in any such a ring R an element $x \in R$ verifies the identity $x + x = 0$ and that the ring is commutative.

3. Let A be a ring with unit and let $a \in A$ be a nilpotent element *i.e.* $a^n = 0$ for some $n \in \mathbb{N}$. Show that if $x \in A$ is a unit and commutes with a, then $x - a$ is also a unit and determine its inverse.

4. What is the smallest sub-ring of $\mathbb{Q}$ containing $1/5$?

5. Show that a finite integral ring (*i.e.* a finite domain) is a field.

6. Let $\mathbb{Z}[\sqrt{2}]$ be the subset of $\mathbb{R}$ defined by the real numbers $a + b\sqrt{2}$, with $a, b \in \mathbb{Z}$. Show that $\mathbb{Z}[\sqrt{2}]$ is a sub-ring of $\mathbb{R}$.

7. Determine the units in $\mathrm{M}_2(\mathbb{Z})$: the set of 2×2 matrices with coefficients in $\mathbb{Z}$.

8. Let $A = C([0,1], \mathbb{R})$ be the ring of continuous functions $f : [0,1] \to \mathbb{R}$. Let I be the subset of A of those functions such that $f(1/2) = 0$. Show that I is a prime ideal of A. Is it also a maximal ideal?

9. Determine all the proper ideals of the ring $A = \mathbb{Z}/(180)\mathbb{Z}$. Determine the prime ideals and the maximal ideals of A.