<u>Problem Set 8</u>

From "Dummit & Foote"

1. nn. 8, 10 p. 248

2. n. 9, 14 p. 256-7

<u>Further exercises</u>

3. Consider the ring $A = \mathbb{R}^{\mathbb{Z}/2\mathbb{Z}}$ of maps from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{R}$. Find the zero divisors of A. Show that if I is a maximal ideal of A, then $A/I \simeq \mathbb{R}$. Prove that if I and J are two maximal ideals of A, then $A = I \oplus J$ as $\mathbb{R}$-vector spaces. Finally, find the idempotent elements (i.e. the elements $a \in A$ such that $a^2 = a$) of A.

4. Let A be a commutative ring with unit. Let I and J be two ideal of A satisfying the property that $I + J = A$; show that the homomorphism

$$\varphi : A \to A/I \times A/J, \quad \varphi(a) = ([a]_I, [a]_J)$$

is surjective, with kernel IJ.

5. Let A be a commutative ring with no proper ideal. Show that if A has unit, then A is a field and that if A has no unit, then the product of two elements is zero in A and that A is finite and that its cardinality is a prime number.

6. Show that in a finite commutative ring with unit every prime ideal is maximal.

7. Let A be a commutative ring with unit which is not a field. Show that the following conditions are equivalent:

i. the sum of two non-invertible elements is non-invertible
ii. the non-invertible elements determine a proper ideal
iii. the ring A has a unique maximal ideal.

If either one of these conditions are satisfied, A is said a local ring. Show that the set of fractions such as $P(X)/Q(X)$, where $P(X)$ and $Q(X)$ are polynomials with coefficients in A and $Q(a) \neq 0$ is a local ring.

8. Let A be a local ring. Show that for every element $x \in A$, either x or $1-x$ is invertible. Show that the only idempotent elements of A are 1 and 0.