Problem Set 9

From "Dummit & Foote"

1. nn. 18, 19 p. 258

2. n. 35, 36 p. 259

Further exercises

3. Let $f : A \to B$ be a ring homomorphism and let $\mathfrak{m}$ be a maximal ideal of B. Let $\mathfrak{n} = f^{-1}(\mathfrak{m})$. Show that
 (1) $\mathfrak{n}$ is not in general a maximal ideal of A
 (2) if f is surjective, then $\mathfrak{n}$ is a maximal ideal of A.

4. Let A be a commutative ring with unit. Let denote by Nil(A) the set of nilpotent elements of A (i.e. $a \in A$ such that $a^n = 0$ for some $n \in \mathbb{N}$). Show that Nil(A) is an ideal of A.
Show that if $a_0 \in A^\times$ and $a_1, \dots, a_n \in$ Nil(A), then the polynomial $p(x) = \sum_{i=0}^n a_i x^i \in A[x]^\times$.
Finally, prove inductively that if $n \geq 1$ and $\sum_{i=0}^n a_i x^i \in A[x]^\times$, then $a_0 \in A^\times$ and $a_n \in$ Nil(A) and $a_{n-1}, \dots, a_1 \in$ Nil(A).

5. Let $a = 3$ and $b = 2 + i\sqrt{5}$ be two elements in $\mathbb{Z}[i\sqrt{5}]$. Show that
 (1) $\mathrm{lcm}(a, b)$ does not exist
 (2) a and b are co-primes i.e. $\gcd(a, b) = 1$
 (3) $\gcd(3a, 3b)$ does not exist.

6. Let k be a field. Study the irreducibility in $k[x, y]$ of the following polynomials:

$$y - x^2, \qquad x^2 + y^2 \pm 1.$$

7. Show that $A = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ and $B = \mathbb{C}[x, y]/(x^2 + y^2 + 1)$ are isomorphic to the ring $C = \mathbb{C}[u, v]/(uv - 1)$. Show that C is a P.I.D.

8. Which among the following gaussian integers are reducible?

$$1 + 3i, \qquad 70 + i, \qquad 201 + 43i.$$