Abstract Algebra - Hw 1 Solns

TA: Joe Cutrone

p.21 1.1

#1 a) No: $2(5-3) \neq (2-5)-3$

b) YES

c) No

D) YES

e) NO

#2 a) NO

b) Yes

c) Yes

d) Yes

e) No

#6 a) Yes

b) No – no identity

c) No – not closed: $\frac{1}{2}+\frac{1}{2}=1 \notin 1$

d) No: $-\frac{3}{2}+1=\frac{1}{2} \notin G$

e) Yes

f) No $\frac{1}{2}+\frac{1}{3}=\frac{5}{6} \notin G$

#8 a) G is a group, called roots of unity

b) $1+1=2 \notin G$

#16 Let $x \in G$; $|x|=n$. Order of x is smallest positive integer s.t. $x^n=1$ $\therefore n \mid 2$ $\therefore n=1$ or 2

Converse obvious

#17 Let $x \in G$, $|x|=n$, ie $x^n=1$

$\therefore x \cdot x^{n-1} = x^n = 1$ so by uniqueness of inverses, $x^{-1}=x^{n-1}$

#20. Let $x^n=1$, where $n=|x|$.

$x^{-n} = x^{-n} \cdot 1 = x^{-n} \cdot x^n = x^0 = 1$

Since n is smallest positive integer s.t. $x^n=1$, $\overbrace{x^{-1} \cdot \dots \cdot x^{-1}}^{n\text{-times}} = x^{-n} = 1$ is smallest $n \in \mathbb{Z}^+$ s.t. $(x^{-1})^n=1$ $\therefore |x^{-1}|=n$

#25 Let $x^2=1$ $\forall x \in G$, ie $x=x^{-1}$

Let $x,y \in G$. $(xy)^2 = xyxy = 1 \Rightarrow xy = y^{-1}x^{-1} = yx$

Ch 0 p. 12 #15 (use Euclidean Alg for these!)

a) $GCD(13,20)=1$ $\quad 13 \cdot \boxed{17} = 221 \equiv 1 \bmod 20$

b) $GCD(69,89)=1$ $\quad 69 \cdot \boxed{40} = 2760 \equiv 1 \bmod 89$

c) $GCD(1891,3797)=1$ $\quad 1891 \cdot \boxed{253} = 4,784,423 = 3797(126)+1$

D) $GCD(a,n)=1$ $\quad a^{-1} =$ [illegible] $\quad -220$

Extra's:

$\varphi(3)=2$

$\varphi(27)=18$

$\varphi(36)=12$

TA: Joe [illegible]

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1.2 #1) a) $D_6 = \{1, r, r^2, s, sr, sr^2\}$

$1, 3, 3, 2, 2, 2$ ← orders

b) $D_8 = \{1, r, r^2, r^3, s, sr, sr^3, sr^4\}$

$1, 4, 2, 4, 2, 2, 2, 2$ ← orders

c) $D_{10} = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$

$1, 5, 5, 5, 5, 2, 2, 2, 2, 2$ ← orders

#8) $\langle r \rangle \leq D_{2n}$ of order n by presentation/def of D_n

#14) $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$

1.3 p. 33

1. $\sigma = (135)(24)$ $\sigma^2 = (153)$ ~~$\tau = (15)(2$~~ $\tau^2\sigma = \tau(\tau\sigma) = (135)(24)$

$\tau = (15)(23)$ $\tau\sigma = (1243)$ $\sigma\tau = (2534)$

4. a) $S_3 = \{1, (12), (13), (23), (123), (132)\}$

$1 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3$ ← orders

b) $S_4 = \{1, \underbrace{(12), (13), (14), (23), (24), (34)}_{2}, \underbrace{(123), (132), (124), (142), (134), (143), (234), (243)}_{3}, \underbrace{(1234), (1243), (1324), (1342), (1423), (1432)}_{4},$

$\underbrace{(12)(34), (13)(24), (14)(23)}_{2}\}$

with 1 of order 1

5. order $= \text{lcm}(5, 2, 3) = \underline{30}$

14. if $|\sigma| = p$, $p = \text{lcm}(n_i)$, where n_i length of cycles. $\therefore n_i = 1$ or p since $n_i \mid \text{lcm}(n_i)$

Converse by def of lcm

If p not prime, $\sigma = (12)(1234)$. This has order 4, but not a product of commuting 4-cycles.

16) # ways of forming an m-cycle in S_n is $\binom{n}{m}$. But there are $(m-1)!$ representations of the same particular m-cycle

(ie # of rotations of numbers ⊨ len $= m$

$\therefore$ total # m-cycles is $= \dfrac{_nP_m}{m} = \dfrac{n(n-1)(n-2)\cdots(n-m+1)}{m}$

where $_nP_m = \dfrac{n!}{(n-m)!}$