

Solutions to Homework II

Ans 1 (7) Let $a = s$ and $b = sr$.

If $r^n = 1$, $s^2 = 1$, $srs = r^{-1}$; then $a^2 = s^2 = 1$, $b^2 = sr sr = r^{-1}r = 1$ and $(ab)^n = (s^2r)^n = r^n = 1$.

If $a^2 = b^2 = (ab)^n = 1$, then $s^2 = a^2 = 1$, $r^n = (s^2r)^n = (a)^n = 1$ and $srs = (sr)(sr)r^{-1} = b^2r^{-1} = r^{-1}$.

(15)

$$\mathbb{Z}/n\mathbb{Z} = \langle \bar{1} \mid n \cdot \bar{1} = 0 \rangle$$

Ans 2 (2) For instance:

$$\begin{aligned} \sigma &= (1, 13, 5, 10)(2)(3, 15, 8)(4, 14, 11, 7, 12, 9)(6) \\ \tau &= (1, 14)(2, 9, 15, 13, 4)(3, 10)(5, 12, 7)(6)(8, 11) \end{aligned}$$

(9a) The positive integers are those which are relatively prime to 12. This is because order of σ^k is $12/\gcd(k, 12)$.

(10) Clear from the definitions.

(11) As in 9a, the order of σ^i equals $m/\gcd(m, i)$. This equals m if and only if i and m are relatively prime.

Ans 3 (4) If n is not prime, we can get $1 < p, q < n$ such that $pq = n$. Then, $\bar{p}, \bar{q} \neq 0$ but $\bar{p}\bar{q} = 0$ in $\mathbb{Z}/n\mathbb{Z}$ and hence it cannot be a field.

(5) If $q \in F$ and $q \neq 0$, then q is invertible. Then, the $n \times n$ matrix $\text{diag}(q, 1, \dots, 1)$ lies in $GL_n(F)$. Thus, $GL_n(F)$ is finite if and only if $F - \{0\}$ is; i.e. iff F is finite.

(3) The required set is

$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = 1, ijk = -1 \rangle$$

Ans 4 (2) If $\phi : G \rightarrow H$ is an isomorphism, then $\phi(x)^k = 1$ iff $x^k = 1$. Hence they have the same order.

If the mapping $\phi : G \rightarrow H$ is not injective, this is not true. Take for instance the order to 1 in \mathbb{Z} and that of $\bar{1}$ in $\mathbb{Z}/2\mathbb{Z}$.

(7) All elements of Q_8 (except identity) have order 4. But D_8 has an element of order 2; they cannot be isomorphic.

(11) Simply map $(a, b) \mapsto (b, a)$.

(14) If $g_1, g_2 \in \text{Ker}(\phi)$, then $\phi(g_1g_2^{-1}) = 1_H$. Thus, $g_1g_2^{-1} \in \text{Ker}(\phi)$. Therefore, $\text{Ker}(\phi)$ is a subgroup. Now, ϕ is injective $\Leftrightarrow \phi(g_1) = \phi(g_2) \Rightarrow g_1 = g_2$. Since $\phi(g_1) = \phi(g_2) \Leftrightarrow g_1^{-1}g_2 \in \text{Ker}(\phi)$, we are through.

(17) Let $\phi(g) = g^{-1}$. We need, for any $g, h \in G$, $\phi(gh) = \phi(g)\phi(h)$. Thus, $h^{-1}g^{-1} = (gh)^{-1} = g^{-1}h^{-1}$. This is possible iff G is abelian.

Ans 5 Let G be a group of prime order p . Take any nonidentity element $x \in G$ and consider the cyclic subgroup H generated by x . Since $x \neq 1$, $|H| > 1$. But $|H| \mid |G|$. Since p is prime, $|H| = p$; hence $G = H$ and G is cyclic. This works for the cases $p = 2, 3, 5$.

Ans 6 Let $g \in G$ and $g' \in G'$. Let $l = \text{lcm}(|g|, |g'|)$. Since $|g| \mid l$, $g^l = 1$. Similarly, $g'^l = 1$. Thus, $(g, g')^l = 1$. Further, if $(g, g')^k = 1$, $g^k = 1$ and $g'^k = 1$. Thus, $|g| \mid k$ and $|g'| \mid k$. Thus $l \mid k$. We therefore see that $l = k$.

Now, let $|G| = m$ and $|G'| = n$ and let both be cyclic, generated by g and g' respectively. If m, n are relatively prime, $|(g, g')| = \text{lcm}(m, n) = mn = |G \times G'|$. Hence, $G \times G'$ must be cyclic. Now suppose that $G \times G'$ is cyclic. Let $l = \text{lcm}(m, n)$. So the group contains an element of order mn . But, if $x \in G$, then $x^l = 1$. Hence, $mn \leq l$ and thus $l = mn$. Thus, m, n are relatively prime.

Ans 7 The map ψ takes $0 = \bar{6} \in \mathbb{Z}/6\mathbb{Z}$ to $0 \neq \bar{6} \in \mathbb{Z}/8\mathbb{Z}$. Hence the map cannot be a homomorphism. A homomorphism ψ between the groups is completely determined by $\psi[1]$. Hence, we can choose $\psi[1]$ to be any $\bar{b} \in \mathbb{Z}/8\mathbb{Z}$ such that $8 \mid 6b$.