pg 1

Abstract Algebra Hw #3 Soln's — Joe Cutrone

1. p. 48 #5

Let G be a group, $|G| = n > 2$. Then by Lagrange's Thm, (p 45 #19), $|H| / |G|$ for any $H \leq G$ $\therefore |H| \neq n-1$

p. 48 #6

Let G be abelian, $H = \{g \in G \,/\, |g| < \infty\}$

Then since $1 \in H$, $H \neq \phi$ and if $x, y \in H$, say $|x| = n$, $|y| = m$, then since $|xy| \leq |x| \cdot |y|$ and $|y| = |y^{-1}|$, $xy^{-1} \in H$

$\therefore H$ is a subgroup by Subgroup Criterion

As in Hw #1, problem #8 gives example of $|A| = 3$, $|B| = 4$, but $|AB| = \infty$ for $A, B \in GL_2(\mathbb{Z})$

p. 48 #10

a) Let $H, K \leq G$. Then $1 \in H \cap K$ and if $x, y \in H \cap K$, then by closure prop of each group, $xy^{-1} \in H \cap K$

$\therefore H \cap K$ is a subgroup by Subgroup Criterion

b) Same argument as in a)

p. 48 #16

By properties of linear algebra, if A is upper-triangular (UT), so is A^{-1}. The product of two UT's is again UT, and since $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ clearly UT, the set of all UT-matrices is a subgroup by the subgroup criterion.

2. p. 60 #10

Use general formula $|x| = \dfrac{|G|}{(|G|, x)}$ to get $|30| = \dfrac{54}{(30, 54)} = \dfrac{54}{6} = 9$

$\langle 30 \rangle = \{0, 6, 12, 18, 24, 30, 36, 42, 48\}$

orders: 1 9 9 3 9 9 3 9 9

p. 60 #20

for prime p, $n \in \mathbb{Z}^+$, $x \in G$ s.t. $x^{p^n} = 1$.

Then $|x| = p^m$ since if $x^d = 1$, $|x| / d$ and only divisors of p^n are p^m for $m \leq n$.

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4. This is #1 ex 6

5. This is Cauchy's Thm for $p=2$ (p. 93 Thm 11)

6. Let G be a group with a unique element a of order 2.
Then conjugation by $b \in G$ is an automorphism, so $bab^{-1} = a$, ie $ba = ab$ ie $a \in Z(G)$

7. Let $\varphi: G \twoheadrightarrow G$ be a surjective homomorphism. Let $x \in Z(G)$. Want to show $\varphi(x) \in Z(G)$
ie for any $\alpha \in G$, $\varphi(x)\alpha = \alpha\varphi(x)$
Since φ is surjective, $\exists \beta \in G$ s.t. $\varphi(\beta) = \alpha$
so $\varphi(x)\alpha = \alpha\varphi(x) \longleftrightarrow \varphi(x)\varphi(\beta) = \varphi(\beta)\varphi(x) \longleftrightarrow \varphi(x\beta) = \varphi(\beta x)$ which is true since $x \in Z(G)$
$\therefore \varphi(Z(G)) \subset Z(G)$

8. a) $|GL_2(\mathbb{Z}_2)| = (2^2-1)(2^2-2) = 3\cdot 2 = 6$
b) Define action $GL_2(\mathbb{Z}_2) \times E \to E$ by matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \to \begin{pmatrix} ae_1 + be_2 \\ ce_1 + de_2 \end{pmatrix}$
c) $GL_2(\mathbb{Z}_2) \cong S_3$ by comment p. 38.

9. Let $|G| = 35$, $|M| = 19$, and let G act on M: $G \times M \to M$ s.t no element of M is fixed by all non-identity elements of G. The stabilizer of an element $a \in M$, G_a, is a subgroup of G, so $|G_a| \mid |G| = 35$ by Lagrange's Thm. $\therefore |G_a| = 1, 5, 7, 35$. $|G_a| \neq 1$ since this action doesn't fix any element of M. Also, $|G_a| \neq 35$ since $|M| = 19$. The size of the orbits is equal to $|G|/|G_a|$ by Prop 2 p. 114, and must partition M. Thus the only possible combination is orbits of size $5+7+7$. Thus there are $\underline{3}$ orbits.

10. Let $G \times M \to M$ be an action on M. Let $x \neq y \in M$ belonging to the same orbit. ie $\exists g \in G$ s.t $g\cdot x = y$
Let $\alpha \in G_x$, ie $\alpha \cdot x = x$. Want to show that $g\alpha g^{-1} \in G_y$. But $g\alpha g^{-1}\cdot y = (g\alpha)g^{-1}y = g(\alpha\cdot x) = g\cdot x = y$
So $gG_xg^{-1} \leq G_y$ and similarly, $g^{-1}G_yg \leq G_x$, so $G_y = gG_xg^{-1}$. Since action by conjugation is an isomorphism, G_x and G_y have the same order.

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11. By comment in middle of pg 127, $C_{S_4}((12)) = \{(12)^i \tau \mid 0 \le i \le 1, \tau \in S_2\}$

This is a group of order 4 : $\{1, (12), (34), (12)(34)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

12. $[g*a, g*b] = [gag^{-1}, gbg^{-1}] = (gag^{-1})^{-1}(gbg^{-1})^{-1}(gag^{-1})(gbg^{-1})$

$= (ga^{-1}\cancel{g^{-1}})(\cancel{g}b^{-1}\cancel{g^{-1}})(\cancel{g}a\cancel{g^{-1}})(\cancel{g}bg^{-1})$

$= ga^{-1}b^{-1}ab\,g^{-1}$

$= g * [a,b]$

If $G' = \{e\}$, G is abelian (See p. 171 ex 1, cf p. 169)

Problem Set 3

From "Dummit & Foote"

1. n. 5, 6, 10, 16 p. 48.

2. nn. 10, 20 p. 60.

Further exercises

4. Show that in a commutative group G, the set of elements of finite order determine a subgroup of G.

5. Show that if G is a finite group of even order, there exists at least one element $g \in G$, $g \neq e$ ($e \in G$ denotes the identity of G) such that $g = g^{-1}$.

6. Let G be a group which contains a unique element a of order 2. Show that a belongs to the center $Z(G)$ of G. [Hint: argue by considering the product $(bab^{-1})(bab^{-1})$]

7. Let denote by $Z(G)$ the center of a group G. Let $\varphi : G \longrightarrow G$ be a homomorphism. Show that if φ is surjective, then $\varphi(Z(G)) \subset Z(G)$.

8. Let $GL(2, \mathbb{Z}/2\mathbb{Z})$ be the group of 2×2 invertible matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$.
   (1) What is the order of $GL(2, \mathbb{Z}/2\mathbb{Z})$?
   (2) Let E be a vector-space of dimension (*i.e.* rank) 2 over (the field) $\mathbb{Z}/2\mathbb{Z}$. Define a non-trivial action of $GL(2, \mathbb{Z}/2\mathbb{Z})$ on E
   $$GL(2, \mathbb{Z}/2\mathbb{Z}) \times E \longrightarrow E.$$
   (3) Prove that $GL(2, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to S_3.

9. A group G with 35 elements acts on a set M with 19 elements. One also knows that this action does not fix any element of M. How many are the orbits of this action?

10. Let M be a set and G be a group acting on M. Let x, y ($x \neq y$) be elements of M belonging to the same orbit. Show that the stabilizers G_x and G_y of the two elements are conjugate subgroups of G *i.e.* there exists $g \in G$ such that $G_y = gG_xg^{-1}$. Deduce that G_x and G_y have the same order.

11. In S_4, determine the centralizer of the permutation $(1\ 2)$. [Hint: Show that in S_n, if $(i\ j)$ is any transposition, then $\sigma(i\ j)\sigma^{-1} = (\sigma(i), \sigma(j))$, $\forall\ \sigma \in S_n$].

12. Let G be a group. We define the *commutator* of a pair of elements a, $b \in G$ to be the element
$$[a, b] = a^{-1}b^{-1}ab \in G.$$
Notice that $[a, b] = e$ (identity of G) if and only if a and b commute. Let G' be the subgroup of G generated by the set of the commutators of (pairs of elements of) G. G' is called the *derived* subgroup of G. For a pair a, b of elements of G, let denote $a \star b = aba^{-1} \in G$. Show that for any triple $\{g, a, b\}$ of elements of G, one has
$$[g \star a, g \star b] = g \star [a, b].$$
What does it mean for G that $G' = \{e\}$?