

Solutions to Homework III

Ans 6 Let x, y be torsion elements of G . Thus, we can find m, n such that $x^m = 1$ and $y^n = 1$. Then $(xy^{-1})^{mn} = x^{mn}y^{-mn}$ since G is *abelian*. This is clearly 1. Thus, xy^{-1} is also a torsion element. Thus, torsion elements form a subgroup.

When G is nonabelian: Let G be the group of automorphisms of \mathbb{Z} . Then $f(n) = -n$ and $g(n) = 1 - n$ are of order 2, but their composition $g \circ f(n) = n + 1$ is not of finite order. (I owe this example to someone's HW assignment)

Ans 7 The elements of this group are of form (k, \bar{l}) . Then, $m \cdot (k, \bar{l}) = (mk, \bar{ml})$. If we assume that $m > 0$ and that $m \cdot (k, \bar{l}) = 0$, i.e. (k, \bar{l}) is a torsion element, we must have $k = 0$ and $n | ml$. Thus, the torsion subgroup consists of all elements of form $(0, \bar{l})$.

Assume $n \neq 2$. The element $(-1, \bar{1})$ has infinite order and so does $(1, \bar{1})$. But their sum, which is $(0, \bar{2})$ has finite order and is nonzero. Thus the elements of infinite order along with 0 do not form a subgroup.

Ans 8 If $H \supseteq K$ or $K \supseteq H$, then $H \cup K$ is obviously a group. Conversely, assume that $H \cup K$ is a group. Suppose that both $H - K$ and $K - H$ are nonempty. Then, we choose $h \in H - K$ and $k \in K - H$. But h, k are both in $H \cup K$ and hence $hk \in H \cup K$. If $hk \in H$, then $k \in H$, a contradiction. Similarly, $hk \notin K$. Thus we have a contradiction either way.

Ans 10 (a) If $x, y \in H \cap K$, then $xy^{-1} \in H$ and $xy^{-1} \in K$; thus $xy^{-1} \in H \cap K$ and it is a subgroup.

(b) If we consider the arbitrary collection of subgroups H_i , $i \in I$ and take $x, y \in \bigcap_{i \in I} H_i$, we see that $xy^{-1} \in H_i$ for each $i \in I$. Thus, $xy^{-1} \in \bigcap_{i \in I} H_i$ and it is a subgroup.

Ans 6 If H is a subgroup and $h \in H$, then $hHh^{-1} = H$. Thus, $H \subseteq N_G(H)$. If H is not a subgroup at all, it cannot be a subgroup of $N_G(H)$ anyway.

Now suppose that $H \leq C_G(H)$. Thus, if $h, h' \in H$, since $h \in C_G(H)$, $hh'h^{-1} = h'$. Hence, $hh' = h'h$. Thus, H is abelian.

Ans 10 Since H has order 2, H must be of form $\{1, x\}$ where $x^2 = 1$. Then, $y \in C_G(H)$ iff $yx y^{-1} = x$. And $y \in N_G(H)$ if and only if $yHy^{-1} = H \Leftrightarrow yxy^{-1} \in H \Leftrightarrow yxy^{-1} = x$ (if $yxy^{-1} = 1$, then $x = 1$, a contradiction). This shows that $N_G(H) = C_G(H)$.

Ans 10 The order of $\bar{30}$ in $\mathbb{Z}/54\mathbb{Z}$ is 9. The order of $l\bar{30}$ will therefore be

$$\frac{9}{\gcd(l, 9)}$$

Ans 18 The homomorphism ϕ is defined by $x \mapsto h$. We need $\phi(nx) = \phi(0) = 1$ and hence must have $h^n = 1$. This is a homomorphism because

$$\phi(kx)\phi(lx) = h^k h^l = h^{k+l} = \phi((k+l)x)$$

Since x generates all of Z_n , the map is uniquely determined once defined on x .

Ans 3 Let A be an abelian group. Hence the subgroup B is normal and the quotient group A/B is defined. In A/B , we have

$$(a+B) + (a'+B) = (a+a') + B = (a'+a) + B = (a'+B) + (a+B)$$

and hence the quotient group is abelian.

Let $G = D_8$. We know that G is not abelian. Let N be the subgroup generated by the element of order 4. Then $[G : N] = 2$ and hence N is normal in G . The group G/N has order 2 and hence must be abelian.

Ans 4 Recall the definition of multiplication in the quotient group

$$(gN) \cdot (hN) = (ghN)$$

Then

$$(gN)^\alpha = (gN) \cdot (gN) \cdot \dots \cdot (gN) = (g^\alpha N)$$

Ans 5 By definition, the order of gN in G/N is the smallest α such that $(gN)^\alpha = N$. From the preceding exercise, $(gN)^\alpha = g^\alpha N$. Then, α must be the smallest number such that $g^\alpha N = N \Leftrightarrow g^\alpha \in N$.

Consider $G = \mathbb{Z}$ and choose $N = 2\mathbb{Z}$. The element 1 in \mathbb{Z} has infinite order, but its coset $\bar{1}$ has order 2 in $\mathbb{Z}/2\mathbb{Z}$.

Ans 9 Evidently, φ has been defined so that, for any nonzero complex number z , $\varphi(z) = |z|^2$. We know that complex norms are multiplicative, so $\varphi(z)\varphi(z') = \varphi(zz')$. This makes the map a homomorphism.

Then, $\text{Ker}(\varphi) = \{z | z \in \mathbb{C} - \{0\}, |z|^2 = 1\}$, which means that $\text{Ker}(\varphi)$ is the unit circle. The fibres over each $r \in \mathbb{R}^\times$ with $r > 0$ are simply the circles around the origin of radius $\sqrt{|r|}$.

Ans 5(a) Let the orbits $\mathcal{O}_1, \dots, \mathcal{O}_r$ be given by $\mathcal{O}_i = Ha_i$ for some $a_i \in A$. Then, for any $g \in G$, $g\mathcal{O}_i = gHa_i = (gHg^{-1})ga_i = H(ga_i)$. Thus, g sends the orbit of $a \in A$ to the orbit of ga . This mapping is surjective because G acts *transitively* on A . This is a surjective mapping between two finite sets of the same size and hence must be an injection. Thus, this is a permutation of $\{\mathcal{O}_1, \dots, \mathcal{O}_r\}$.

(b) If $a \in \mathcal{O}_i$, then $Ha = \mathcal{O}_i$. The group H acts on the set A . the stabilizer of a under this map is $G_a \cap H$. Thus, $|H| = |G_a \cap H| |\mathcal{O}_i|$ and hence, dividing out, we get $|\mathcal{O}_i| = [H : H \cap G_a]$.

Ans 6 G is a group such that $G/Z(G)$ is cyclic. Suppose that $x \in G$ is such that the coset $xZ(G)$ generates $G/Z(G)$. Then, every element in G , say g can be written as $x^k z$ for some $z \in Z(G)$. But then, if $g = x^k z$ and $g' = x^l z'$, with $z, z' \in Z(G)$, we get

$$gg' = (x^k z)(x^l z') = x^{k+l} z z' = x^{k+l} z z' = (x^l z')(x^k z) = g'g$$

Ans 7 Consider the left coset gH . Associate:

$$\phi : gH \mapsto Hg^{-1}$$

Surjective: $Hg = \phi(g^{-1}H)$

Injective: Say $\phi(gH) = \phi(g'H)$. Then, by definition, $Hg^{-1} = Hg'^{-1}$. Thus, there exists $h \in H$ such that $hg^{-1} = g'^{-1}$. Taking inverses, $g' = gh^{-1}$. Thus, $g'H = gH$.

NB: The association $gH \mapsto Hg$ is not correct.