Abstract Algebra - Prob Set 4 Solutions Joe Cutrone

1. p. 60 #15

To show $\mathbb{Q} \times \mathbb{Q}$ not cyclic, enough to show that $\mathbb{Q}$ not cyclic. (Any subgroup of cyclic is cyclic)

Assume $\mathbb{Q} = \langle \frac{p}{q} \rangle$, $q \neq 0$. Then $\frac{p}{2q} \in \mathbb{Q}$, so $\exists\, n \in \mathbb{Z}$ s.t. $\frac{np}{q} = \frac{p}{2q}$, ie $n = \frac{1}{2}$ $\rightarrow\leftarrow$

2. p. 65 #6

$\langle (12), (12)(34) \rangle = \{1, (12), (34), (12)(34)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$ by previous hw of classifying groups of order 4

p. 65 #11

$Z(S_4) = 1$, $Z(SL_2(\mathbb{F}_3)) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\} \cong \mathbb{Z}_2$ $\therefore$ not isomorphic

3. Let p prime, $n \geq 1$ an integer. Let $|G| = p^n$, $H \trianglelefteq G$, $H \neq 1$. Let G act on H by conjugation.

By Lagrange's thm, $p \mid |H|$. Any element in the center $Z(G)$ has just one element in its conjugacy class.

Since $H \trianglelefteq G$, $H = \coprod K_i$, where K_i are conjugacy classes. Since $|K_i| = |G : C_G(g_i)|$ for $g_i \in K_i$, $p \mid |K_i|$

Since $1 \in H$, $1 \in Z(G)$, $H = \{1\} \cup K_1 \cup \dots \cup K_n$.

Since p divides both sides, $p \mid |K_i|$, there must exist at least $p-1$ elements α_i whose equivalence class is just $\{\alpha_i\}$

$\therefore$ each $\alpha_i \in Z(G)$ $\therefore 1, \alpha_i \in H \cap Z(G)$ $\therefore H \cap Z(G) \neq 1$

$Z(G) \neq e$ since if we let $H = G \trianglelefteq G$, get $G \cap Z(G) = Z(G) \neq 1$

4. Let G be a group and let $Z(G)$ be center.

a) $N_G(Z(G)) = \{g \in G \mid g Z(G) g^{-1} = Z(G)\} = G$ $\therefore Z(G) \trianglelefteq G$

b) Let $G/Z(G)$ be cyclic with generator $xZ(G)$. Let $g \in G$. Then $g = x^a z$ for $a \in \mathbb{Z}$, $z \in Z(G)$ since $g \in gZ(G)$ partition G. $\therefore g_1 g_2 = (x^{a_1} z_1)(x^{a_2} z_2) = x^{a_1 + a_2} z_1 z_2 = (x^{a_2} z_2)(x^{a_1} z_1) = g_2 g_1$

5. Let $H, K \trianglelefteq G$. Then $gHg^{-1} \subseteq H$, $gKg^{-1} \subseteq K$. Any element of $\langle H \cup K \rangle$ is of the form $h_1 k_1 h_2 k_2 \dots h_n k_n$ for $h_i \in H$, $k_i \in K$. So $g(h_1 k_1 \dots h_n k_n) g^{-1} = g h_1 g^{-1} g k_1 g^{-1} g h_2 g^{-1} g \dots g k_n g^{-1} = h_1' k_1' h_2' k_2' \dots h_n' k_n' \in \langle H \cup K \rangle$

$\therefore g \langle H \cup K \rangle g^{-1} \subseteq \langle H \cup K \rangle$

$\therefore \langle H \cup K \rangle \trianglelefteq G$

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6. See pg 91, Ex 2

7. See p. 171, thm 9

8. Using #7, Get $HK \cong H \times K \trianglelefteq G$, and $|HK| = |H \times K| = |H| \cdot |K|$. See Thm 10 p. 176

9. Let $H = \mathbb{Z}_5$, $K = \mathbb{Z}_{11}$, both normal in G. $H \cap K = 1$, so $HK \cong H \times K \cong \mathbb{Z}_{55}$
(See Prop 6 p 163 for more general statement)

10. See prop 13 p. 133

11. See Def of characteristic subgroup H, denoted H char G on pg 134 and the remarks following the definition. Idea: G acts on H by conjugation, which by #10 is an automorphism, so unique subgroup must get mapped to itself: $gHg^{-1} = H$ $\therefore$ $H \trianglelefteq G$.