

Solutions to Homework V

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Ans 5: To show that the action is faithful, we must check that

$$\text{Ker}(\pi_H) = \{e\}$$

We know that $\text{Ker}(\pi_H) = \bigcap_{x \in D_s} xHx^{-1}$. We know that $rHr^{-1} \cap H = \{e\}$ and hence $\text{Ker}(\pi_H) = \{e\}$.

Ans 5: Consider the group acting on itself by conjugation. We know that if the orbit of $x \in G$ is \mathcal{O}_x and the stabilizer is G_x , then

$$|\mathcal{O}_x| = [G : G_x]$$

But the center $Z(G) \subseteq G_x$ and hence

$$|\mathcal{O}_x| = [G : G_x] \leq [G : Z(G)] = n$$

Ans 8: Let $\sigma \in Z(S_n)$ where $n \geq 3$. Let $\sigma \neq 1$. Then $\exists i$ such that $\sigma(i) = j \neq i$. As $n \geq 3$, we can choose k such that $k \neq i, k \neq j$. Then, we know that

$$(i, k)\sigma(i, k) = \sigma$$

Hence,

$$(i, k)\sigma(i, k)(k) = (i, k)\sigma(i) = \sigma(k)$$

But, $\sigma(i) \neq \sigma(k)$ and hence $j = \sigma(i) \in \{i, k\}$, a contradiction. Thus, $Z(S_n) = 1$.

Ans 11 (a) Conjugate $\tau : \{1, 2, 3, 4, 5\} \rightarrow \{4, 5, 1, 2, 3\}$ (b) Conjugate $\tau : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} \rightarrow \{4, 12, 13, 1, 9, 7, 11, 5, 2, 3, 10, 6, 8\}$ (c) Not conjugate (d) Not conjugate.

Ans 1: Clearly,

$$\sigma\varphi_g\sigma^{-1}(h) = \sigma(g\sigma^{-1}(h)g^{-1}) = \sigma(g)h\sigma(g)^{-1} = \varphi_{\sigma(g)}(h)$$

This immediately means that the inner automorphisms form a normal subgroup.

Ans 15: The generators are 2, 2 and 5.

Hence ab is nilpotent in \mathbb{Z}/\mathbb{Z} .

(b) Let \bar{a} be nilpotent in \mathbb{Z}/\mathbb{Z} . Then $n|a^k$ for some k . Take a prime $p|n$. Then $p|a^k$ and hence $p|a$. If every prime dividing n also divides a , then we can choose a power of a large enough so that n divides it.

(c) Let $f : X \rightarrow F$ be nilpotent. Then, there exists n such that $f^n(x) = 0$. Since $f(x) \in F$ and F is a field, then $f(x) = 0$. Thus, f is identically zero.