Abstract Algebra - Hw #5 Solutions Joe Cutrone

1\. p. 85 #3

Let A be abelian, $B \leq A$. Then for $aB, bB \in A/B$, $aB \cdot bB = abB = baB = bB \cdot aB$ $\therefore$ A/B is abelian

For example of non-abelian group G containing proper normal subgroup N s.t G/N is abelian, let $G = Q_8$, $N = \langle -1 \rangle$

Then $Q_8/\langle -1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is abelian.

p. 85 #6

Let $\varphi: \mathbb{R}^\times \to \{\pm 1\}$ def by $x \mapsto \frac{x}{|x|}$.

φ is a homomorphism since $\varphi(xy) = \frac{xy}{|xy|} = \frac{x}{|x|} \cdot \frac{y}{|y|} = \varphi(x) \cdot \varphi(y)$

Fibers of φ are $\varphi^{-1}(+1) = \{x \in \mathbb{R} \mid x > 0\}$ and $\varphi^{-1}(-1) = \{x \in \mathbb{R} \mid x < 0\}$

p. 87 #20

Let $G = \mathbb{Z}/24\mathbb{Z}$, $\tilde{G} = G/\langle 12 \rangle$

By 3rd Iso Thm, $\tilde{G} \cong (\mathbb{Z}/12\mathbb{Z})/(12\mathbb{Z}/24\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, which gives part c)

a) trivially follows and use the formula that $|a| = \frac{|G|}{(|G|, a)} = \frac{12}{(12, a)}$

2\. p. 101 #1

I've done this in section! Let $\varphi: GL_n(\mathbb{F}_q) \twoheadrightarrow \mathbb{F}_q^\times$ be homomorphism defined by $A \mapsto \det A$

Then $\ker \varphi = SL_n(\mathbb{F}_q)$, so by 1st Iso Thm, $GL_n(\mathbb{F}_q)/SL_n(\mathbb{F}_q) \cong \mathbb{F}_q^\times$ since φ is surjective.

$\therefore |GL_n(\mathbb{F}_q) : SL_n(\mathbb{F}_q)| = |\mathbb{F}_q^\times| = q - 1$

3\. See Cor 9 p. 125

4\. Let G group, $H, K \leq G$ s.t $|G:H| = |G:K| = 2$, and $H \cap K = \{e\}$

The lattice of G is then

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ H \arrow[dr, dash, "2"'] & & K \arrow[dl, dash, "2"] \\ & 1 & \end{tikzcd}$$

and by the 2nd Iso Thm, $|H| = |K| = 2$.

By example 2 pg 91, both $H \trianglelefteq G$ and $K \trianglelefteq G$, so by Thm 9 p. 171, $G \cong H \times K$ so $|G| = |H| \cdot |K| = 2$, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

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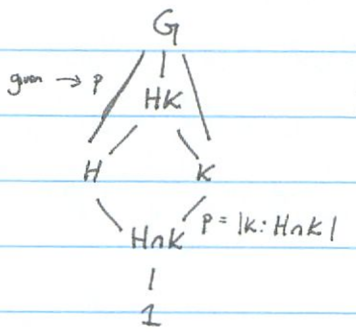
5. Define a map $\varphi: Q_8 \longrightarrow \langle \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} \rangle$ defined by $i \longmapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $j \longmapsto \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$.
Recalling that a presentation for $Q_8 = \langle i, j \mid i^4 = j^4 = 1, i^2 = j^2, i^{-1} j i = j^{-1} \rangle$ it is an easy verification that φ is an isomorphism. Then the image of $\langle -1 \rangle$ is the only element of order 2, which is also the center, and since every subgroup of Q_8 has ~~order~~ index 2, they are normal (with the exception of the center, which is clearly normal)

6. Draw Lattice:



Assume $K \not\leq H$. Then $H \subset HK$ (proper containment)

By 2nd Iso Thm, $|K : H \cap K| = p$

$\therefore |G : H| = |G : HK| \cdot |HK : H| = p$, so since $H \subset HK$,

$|HK : H| \neq 1$ $\therefore |HK : H| = p$ and $|G : HK| = 1$

$\therefore G = HK$

7. Look at map $\pi: H \longrightarrow G/N$ defined by $h \mapsto hN$

Then $|\pi(H)| \mid |H|$ since $\pi(H)$ is a quotient of H

and $|\pi(H)| \mid |G : N|$ by Lagrange

$\therefore$ since $(|H|, |G:N|) = 1$, $|\pi(H)| = 1$ $\therefore$ H gets mapped to the identity coset, so $H \leq N$

8. $(aH)^n = a^n H = H$ $\therefore |aH| \mid n$

9. Let $\varphi: A \times B \longrightarrow A/C \times B/D$ be natural quotient map. Then φ is surjective and $\ker \varphi = (C \times D)$
$\therefore$ by 1st Iso Thm, $A \times B / C \times D \cong A/C \times B/D$

10. 1) if G is finite, $[G:K] = |G|/|K|$, so since $|G|/|K| = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = [G:H][H:K]$

2) if ~~the~~ $[G:H][H:K]$ is finite, say $[G:H] = p$, $[H:K] = q$, then have cosets $g_1H, g_2H, \ldots, g_pH$ which partition G and $h_1K, h_2K, \ldots, h_qK$ (cosets of K in H) partition H. Then let $g \in g_1 H = \bigcup_{j=1}^{q} g_1 h_j K$, so $g \in g_i h_j K$

$\therefore |G:K|$ is finite

If $|G:K| = n < \infty$, $g_1K, g_2K, \ldots, g_nK$ partition G. Pick cosets in H, say $g_1K, \ldots, g_qK \subseteq H$
Then there are finitely many $(n-q)$ cosets $|G:H|$. $\therefore |G:H||H:K|$ will be finite as well.

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$\therefore$ equality proved if $[G:K]=\infty$ or $[G:H][H:K]=\infty$

Now, if all index's are finite, say $n=[G:K]$ and $[G:H][H:K]=p\cdot q$

considering everything mod K, $G/K = |G:K| = n$ is finite, so by part a)

$$|G/K : K/K| = |G/K : H/K||H/K : K/K|$$

so by the 3rd Iso Thm, $|G:K| = |G:H||H:K|$ as desired.

11. Let $G=\{z\in\mathbb{C} \mid \exists n \text{ s.t. } z^n=1\}$. Let p prime.

1. $\varphi: G\to G$ def by $z\mapsto z^p$ is surjective since given any $z=e^{i\theta}\in G$, consider $e^{\frac{i\theta}{p}}\in G$.

   Then $\varphi(e^{i\theta/p}) = e^{i\theta} = z$ as desired

2. Then by first iso thm, $G/\ker\varphi \cong G$