Pg 1

Abstract Algebra Hw #8 Solns

TA: Joe Cutrone

1. p. 248 #8 | Let $R = \mathbb{Z} \times \mathbb{Z}$

a) No : $(1,0)\cdot(1,1) = (1,0) \notin I$

b) Yes : $(c,d)\cdot(2a,2b) = (2ac, 2bd) \in I$

$(2a,2b) + (2c,2d) = (2(a+c), 2(b+d)) \in I$

c) Yes : $(c,d)(2a,0) = (2ac, 0)$

$(2a,0) + (2b,0) = (2(a+b), 0)$

d) No : $(1,0)\cdot(+1,-1) = (1,0) \notin I$

p. 248 #10 | Let $R = \mathbb{Z}[x]$

a) Yes : if $p(x) \in \mathbb{Z}[x]$, $q(0) = 3k$, then $p(0)\cdot q(0) = 3\cdot k'$ and $p(0)+q(0) = 3k_1 + 3k_2 = 3(k_1+k_2)$

b) No : $f(x) = 1+3x^2$, $g(x) = x^2$. Then $f \in I$ but $fg = x^2 + 3x^4$ is not

c) Yes : clearly closed w.r.t +. if $f(x) = a_3x^3 + \dots + a_nx^n$, $g(x) = b_0 + b_1x + \dots + b_nx^n$, then $f \in I \therefore fg \in I$

d) No : $f(x) = x^2 \in I$, $g(x) = x \in R \therefore fg = x^3 \notin I$

e) Yes : The sum of coeff of poly is $f(1)$. If $f(1), g(1) = 0$, $(f+g)(1) = 0$

if $f(1) = 0$, $g \in \mathbb{Z}[x]$, then $(fg)(1) = f(1)g(1) = 0$

f) No : if $f = \sum^n a_i x^i$, then $f'(0) = 0 \longleftrightarrow a_1 = 0$. if $f(x) = 1$, $g(x) = x$, $f \in I$, but $fg = x \notin I$

2. p. 256 #9 | I is an ideal since it is the kernel of the ring homomorphism $\varphi: R \to R \times R$ def by

$f(x) \mapsto (f(1/2), f(1/3))$

I not prime since $(x - 1/2)(x - 1/3) \in I$, but $(x - 1/2) \notin I$, $(x - 1/3) \notin I$

p. 257 #14 | Let R be comm, $f(x) \in R[x]$ monic of deg $n \geq 1$.

a) if $f(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0$, then $\bar{x}^n = -\overline{(b_{n-1}x^{n-1} + \dots + b_0)}$

$\therefore$ for powers of x greater than or equal to x^n, replace with to get $R[x]/(f(x)) = \{\bar{a}_0 + a_1\bar{x} + \dots + \overline{a_{n-1}x^{n-1}} \mid a_i \in R\}$

b) Let $p(x) \neq q(x) \in R[x]$, both of deg $< n$

$\overline{p(x)} \neq \overline{q(x)}$ since if $p(x) \equiv q(x) \mod f(x)$, $p(x) - q(x) \equiv 0 \mod f(x)$

ie $f(x) \mid p(x) - q(x)$. But $\deg(p(x) - q(x)) < \deg f(x)$ $\rightarrow\leftarrow$

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c) if $f(x)=a(x)b(x)$, where $\deg(a(x))<n$, $\deg(b(x))<n$, then in $R[x]/(f(x))$,

$\overline{a(x)}\cdot\overline{b(x)}=\overline{f(x)}=0$, where $\overline{b(x)}\neq 0$ since else $b(x)=0$. (since $\deg b(x)<n$)

$\therefore \overline{a(x)}$ is a zero divisor.

d) if $f(x)=x^n-a$ for $a\in\mathfrak{N}(R)$, then $\bar{x}\in R[x]/(f(x))$ is nilpotent since $\bar{x}^n=\bar{a}$ and if $a^m=0$,

$\bar{x}^{mn}=\bar{a}^m=0$

e) Let $R=\mathbb{F}_p$, $f(x)=x^p-a$ for $a\in\mathbb{F}_p$. $\therefore (\overline{x-a})^p=\overline{x^p-a^p}=\bar{x}^p-\bar{a}^p=\bar{a}-\bar{a}=0$

3. $\mathbb{R}^{\{0,1\}}$ = set maps from $\{0,1\}$ to $\mathbb{R}$

$=\{ar+br' \mid a,b\in\{0,1\},\ r,r'\in\mathbb{R}\}$

The zero divisors of A are elements of the form $(0,r)$ and $(r',0)$ for $r,r'\in\mathbb{R}$

All prime ideals are of the form $\mathbb{R}\times p$ and $p\times\mathbb{R}$ (for prime ideal $p\subseteq R$), so since $\mathbb{R}$ is a field, this is just $\mathbb{R}\times\{0\}$, $\{0\}\times\mathbb{R}$

These prime ideals are maximal since $\mathbb{R}^{\{0,1\}}/\mathbb{R}\times\{0\}\cong\mathbb{R}^{\{0,1\}}/\{0\}\times\mathbb{R}\cong\mathbb{R}$

Clearly $\mathbb{R}^{\{0,1\}}=\{ar+br'\}=\mathbb{R}\times\{0\}\oplus\{0\}\times\mathbb{R}\cong\mathbb{R}\oplus\mathbb{R}$ as $\mathbb{R}$ vector spaces

The idempotent elements are $(0,0),(0,1),(1,0)$, & $(1,1)$

4. Let A be a commutative ring with a unit, let I,J be comaximal ideals. Then map is surjective by Chinese Remainder Thm, with kernel $IJ=I\cap J$.

5. Let A be commutative ring without proper non-trivial ideals. If $1\in A$, then A is a field by Prop 9, p. 253

Now assume $1\notin A$. If $\exists\, d,a\in R$ s.t $da\neq 0$, then Ra is an ideal, so $Ra=R$

$\therefore \exists\, b\in R$ s.t $ba=a$, where $a,b\neq 0$.

Then $\forall r\in R$, $r=r'a$ for some $r'\in R$

$\therefore br=br'a=r'ba=r'a=r \quad \therefore b=1 \rightarrow\leftarrow$

$\therefore$ if $1\notin A$, the product of any two elements is zero

Now all ideals are just subgroups, so R must contain no proper non-trivial subgroups, ie $R\cong(\mathbb{Z}_p,+,\times)$, where mult is $\underline{0}$.

6. Let (p) be the prime ideal in R. Then $R/(p)$ is a finite integral domain; which by Cor 3 p. 228 is a field, so by Prop 12 p. 254, (p) is maximal

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7. This is prop 45 p. 717

$R = \{ \frac{p(x)}{q(x)} \mid q(x) \neq 0 \}$ is a local ring since this ring has unique max ideal $\{ f(x) \in A[x] \mid f(a) = 0 \}$

8. Let A be local ring. Let $x \in A$. Since $x + (1-x) = 1$ is invertible by 7i, either x or $1-x$ is invertible

Let $e = e^2$ be idempotent

Then $(1-e)^2 = 1-e$ so $(1-e)$ idempotent as well

$\therefore$ either e or $1-e$ is invertible.

if e invertible, $\exists e^{-1} \in A$ $\therefore e^2 = e \overset{(\text{mult by } e^{-1})}{\Rightarrow} e = 1$

if $1-e$ invertible, $(1-e)^2 = (1-e) \overset{\text{mult by inverse}}{\Rightarrow} (1-e) = 1 \Rightarrow e = 0$

$\therefore e = 0$ or 1.