On the notion of equivalence relation

Let $\sim$ be an equivalence relation on a set S. For any $x \in S$ we group together all the elements equivalent to x into an equivalence class

$$S_x = \{y \in S \mid x \sim y\}.$$

By the reflexivity, $x \in S_x$.

Claim Any two equivalence classes S_x, S_y are either disjoint or coincide.

proof Suppose $S_x \cap S_y \neq \emptyset$, then we must prove that $S_x = S_y$ and we begin by showing that $x \sim y$. Since $S_x \cap S_y \neq \emptyset$, there exists $z \in S_x \cap S_y$; by definition this means that $x \sim z$ and $y \sim z$. By symmetry then, $z \sim y$ and hence by transitivity $x \sim y$. Now let $u \in S_y$; then $y \sim u$, hence $x \sim u$ (by transitivity) and so $u \in S_x$; this proves that $S_y \subseteq S_x$. A similar argument shows that $S_x \subseteq S_y$.

The different S_x provide a partition of S into non-empty subsets, any two of which are disjoint. This is called a partition of S

$$S = \coprod_{x \in S} S_x.$$

Given an equivalence relation $\sim$ on a set S, we can form a new set $S/\sim$, whose elements are the different (sub)sets S_x

$$S/\sim = \{S_x \mid x \in S\}$$

Then we have a natural map

$$\varphi : S \to S/\sim, \quad \varphi(x) = S_x.$$

φ is surjective but not injective unless the equivalence on S was just equality. $S/\sim$ is called a quotient set.

Conversely, every partition on a set S arises in this way from an equivalence. For, suppose that S is partitioned into sets A, B,... Then each $x \in S$ belongs just to one set of the partition, say $x \in A$. We put $x \sim y$ if x and y lie in the same set. This defines an equivalence on S with blocks A, B,...