1. Let $G$ be a group with identity $e$. Let $x, y \in G$ such that
   \[ x^4 = y^2 = e, \quad x^3 y = (yx)^2. \]
   Determine $(xy)^3$.

2. Let $G$ be a group and let $x \in G$. Consider the set $H_x = \{ g \in G \mid gx = xg \}$. Is $H_x$ a subgroup of $G$? If yes, prove it.

3. Which among the following subgroups of $GL(2, \mathbb{R})$ are normal?
   \[ H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid ab \neq 0 \right\}, \quad K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}, \]
   \[ L = \left\{ \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \]

4. Let $G = \mathbb{Z}/12\mathbb{Z}$. Is $H = \{0, 4, 8\}$ a normal subgroup of $G$? If yes, describe $G/H$.

5. Describe the group made by the invertible elements of $\mathbb{Z}/8\mathbb{Z}$. Is this group a cyclic group?

6. Find a subgroup of $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ of order 8. Is there any cyclic subgroup of $G$ of order 8? If yes, find it, if not explain why.

7. Let $G$ be the multiplicative group of the invertible elements in $\mathbb{Z}_{20}$. Show that $H = \{ x \in G \mid \pi(x) \leq 2 \}$ ($\pi(\cdot)$ = period) is a subgroup of $G$. Is $H$ cyclic? Is $G/H$ cyclic?

8. Let $G = GL(n, \mathbb{R})$ and let $H = \{ M \in G \mid \det(M) = 1 \}$. Find a group isomorphic to $G/H$.

9. Let $G = \mathbb{Z}_6$. Describe the group homomorphisms $f : G \to G$. Determine which among them are injective and which are surjective.

10. Does there exist a surjective group homomorphism $f : \mathbb{Z}_4 \times \mathbb{Z}_8 \to \mathbb{Z}_{16}$. Explain.

11. List, up-to-isomorphism, all subgroups of $G = (\mathbb{Z}/18\mathbb{Z}, +)$, i.e. for every different isomorphism class, write a corresponding subgroup of $G$. How many isomorphism classes of subgroups of $G$ do exist for each given order?

12. Let $G$ be a non-commutative group of order 8, show that $G$ contains an element of period 4.

13. Let $N$ be a normal subgroup of a group $G$; let $n$ be the index of $N$ in $G$. Show that if $g \in G$, then $g^n \in N$. Give an example to show that this may be false when $N$ is not normal.

14. State Lagrange’s Theorem and give an example of application of that theorem.

15. State the definition of a group action on a set and give an example of at least two different types of group actions.


17. Show that if $H < G$ is such that $[G : H] = 2$, then $H$ is a normal subgroup in $G$.

18. Show that if $\varphi : G \to G_1$ is a group homomorphism, then $\text{Image}(\varphi)$ is a subgroup of $G_1$. Is this a normal subgroup of $G_1$? If yes, prove it, if not give a counter-example.


20. Determine the cycle decomposition (in disjoint cycles) of the following permutation of $S_{10}$:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 4 & 2 & 5 & 9 & 3 & 8 & 10 & 1 & 6 \end{bmatrix}$$