Test-type exercices

1. Let G be a group with identity e. Let $x, y \in G$ such that
$$x^4 = y^2 = e, \quad x^3 y = (yx)^2.$$
Determine $(xy)^3$.

2. Let G be a group and let $x \in G$. Consider the set $H_x = \{g \in G \mid gx = xg\}$. Is H_x a subgroup of G? If yes, prove it.

3. Which among the following subgroups of $GL(2, \mathbb{R})$ are normal?
$$H = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid ab \neq 0\}, \quad K = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0\},$$
$$L = \{\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \mid a \in \mathbb{R}\}.$$

4. Let $G = \mathbb{Z}/12\mathbb{Z}$. Is $H = \{\bar{0}, \bar{4}, \bar{8}\}$ a normal subgroup of G? If yes, describe G/H.

5. Describe the group made by the invertible elements of $\mathbb{Z}/8\mathbb{Z}$. Is this group a cyclic group?

6. Find a subgroup of $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ of order 8. Is there any cyclic subgroup of G of order 8? If yes, find it, if not explain why.

7. Let G be the multiplicative group of the invertible elements in $\mathbb{Z}_{20}$. Show that $H = \{x \in G \mid \pi(x) \leq 2\}$ ($\pi(.) =$ period) is a subgroup of G. Is H cyclic? Is G/H cyclic?

8. Let $G = GL(n, \mathbb{R})$ and let $H = \{M \in G \mid \det(M) = 1\}$. Find a group isomorphic to G/H.

9. Let $G = \mathbb{Z}_6$. Describe the group homomorphisms $f : G \to G$. Determine which among them are injective and which are surjective.

10. Does there exist a surjective group homomorphism $f : \mathbb{Z}_4 \times \mathbb{Z}_8 \to \mathbb{Z}_{16}$. Explain.

11. 3. List, up-to-isomorphism, all subgroups of $G = (\mathbb{Z}/18\mathbb{Z}, +)$, i.e. for every different isomorphism class, write a corresponding subgroup of G. How many isomorphism classes of subgroups of G do exist for each given order?

12. Let G be a non-commutative group of order 8, show that G contains an element of period 4.

13. Let N be a normal subgroup of a group G; let n be the index of N in G. Show that if $g \in G$, then $g^n \in N$. Give an example to show that this may be false when N is not normal.

14. State Lagrange's Theorem and give an example of application of that theorem.

15. State the definition of a group action on a set and give an example of at least two different types of group actions.

16. Prove that if $K < H < G$, then $[G : K] = [G : H][H : K]$.

17. Show that if $H < G$ is such that $[G : H] = 2$, then H is a normal subgroup in G.

18. Show that if $\varphi : G \to G_1$ is a group homomorphism, then $Image(\varphi)$ is a subgroup of G_1. Is this a normal subgroup of G_1? If yes, prove it, if not give a counter-example.

19. Let $A_4 < S_4$ be the alternating subgroup of the symmetric group S_4. Is A_4 normal in S_4? Determine $[S_4 : A_4]$. Is A_4 abelian? Determine a *cycle* not in S_4.

20. Determine the cycle decomposition (in disjoint cycles) of the following permutation of S_{10}:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 4 & 2 & 5 & 9 & 3 & 8 & 10 & 1 & 6 \end{bmatrix}$$