On the notion of signature of a permutation

Let S_n be the symmetric group associated to the bijections of the set $M = \{1, 2, \ldots, n\}$.

A *transposition* is a 2-cycle $c \in S_n$. It is known that transpositions generate S_n. Equivalently stated, every permutation can be written as a product of transpositions. Notice though, that unlike the decomposition of σ into disjoint cycles, the decomposition of a permutation as a product of transpositions is *not* unique! However, the *parity* of the number of transpositions which appear in any such a decomposition is *independent* of the chosen decomposition.

Let s be the number of transpositions which appear in a decomposition of $\sigma \in S_n$.

Definition The signature of σ is

$$sgn(\sigma) = (-1)^s.$$

There is an equivalent way to state the definition of the signature of a permutation σ by considering the *canonical* decomposition of σ into *disjoint* cycles. Let assume that the decomposition of σ into disjoint cycles is given by $\sigma = c_1 \cdots c_r$.

Definition 1 The signature of a r-cycle $c \in S_n$ is

$$sgn(c) = \begin{cases} -1 & \text{if } r \text{ is odd} \\ 1 & \text{if } r \text{ is even} \end{cases}$$

Consequently we set

Definition 2 The signature of the permutation $\sigma = c_1 \cdots c_r$ is

$$sgn(\sigma) = \prod_{i=1}^{r} sgn(c_i).$$

This is the definition of signature of a permutation I gave in class (09/24/2008).

Example Let $\sigma = (1\ 3\ 5)(5\ 4\ 3\ 2)(5\ 6\ 7\ 8) \in S_8$. Then, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 5 & 6 & 7 & 8 & 4 \end{pmatrix}$. The above decomposition of σ is not the canonical one, as the cycles appearing in the decomposition are not disjoint. We then proceed by decomposing first σ into a product of disjoint cycles, following the well-known algorithm

$$\sigma = (1\ 3\ 2)(4\ 5\ 6\ 7\ 8).$$

Then by applying Definition 2 we get $sgn(\sigma) = (-1)(-1) = 1$.

Alternatively, we can find a decomposition of σ into 2-cycles and apply Definition 1 for the computation of the signature. For example, σ can be decomposed as a product of (not disjoint) 2-cycles as follows

$$\sigma = (3\ 5)(5\ 1)(2\ 3)(4\ 2)(2\ 5)(7\ 8)(6\ 8)(5\ 8).$$

By applying Definition 1 we find again that $sgn(\sigma) = (-1)^8 = 1$.