No calculators allowed. Total Marks = 100

Student Name: __________________________

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature & date: __________________________

TA Name: __________________________

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1. [25 marks] Let $G/H$ be a $p$-group and let $K$ be a Sylow $p$-subgroup of $G$.

Show that $G = HK$. 

2. [25 marks] Give a proof or disprove the following statement:

\[ \mathbb{Z}[\sqrt{-3}] \text{ is an Euclidean integral ring (i.e. an Euclidean domain).} \]
3. [25 marks] Consider the domain $R = \mathbb{Z}[\sqrt{3}] := \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$.

a) Which of the following elements of $R$ are invertible

$5 + 3\sqrt{3}$, $2 - \sqrt{3}$, $1 + \sqrt{3}$, $7 + 4\sqrt{3}$?

b) Does the following equality of ideals hold in $R$

$(5 + 3\sqrt{3}) = (1 + \sqrt{3})$?

Explain why.

c) Is $(3 + \sqrt{3})$ a prime ideal in $R$? Explain.

d) Determine a maximal ideal $\mathfrak{m} \subset \mathbb{Z}[X]$ such that $X^2 - 3 \in \mathfrak{m}$. 
4. [25 marks] Consider the ring \( R = \mathbb{Z}[X]/(X^4 + 3X^3 + 1) \).

a) Is \((\bar{2}) \subset R\) a maximal ideal in \( R \)? Explain.

b) Is \( R \) a domain? Is \( R \) a field? Explain.

[Hint. You may want to use the fact that if \( q(X) \) is an irreducible polynomial in \( R \), which is the image of a non-constant, monic polynomial \( p(X) \) in \( \mathbb{Z}[X] \), then \( p(X) \) is irreducible in \( \mathbb{Z}[X] \)]

c) Does \( R \) have any further unit besides \( \pm 1 \)? If yes, give an example of such unit.