THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
MIDTERM EXAM - SPRING SESSION 2006
110.402 - ADVANCED ALGEBRA II.

Examiner: Professor C. Consani
Duration: 50 MINUTES (11am-11:50am), March 29, 2006.

No calculators allowed. **Total Marks = 100**

Student Name: ________________________________

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature & date: ____________________________

TA Name: ____________________________

1.	
2.	
3.	
4.	
Total	

1. [25 marks] Let G/H be a p-group and let K be a Sylow p-subgroup of G.

Show that $G = HK$.

2. [25 marks] Give a proof or disprove the following statement:

$\mathbb{Z}[\sqrt{-3}]$ is an Euclidean integral ring (i.e. an Euclidean domain).

3. [25 marks] Consider the domain $R = \mathbb{Z}[\sqrt{3}] := \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$.

a) Which of the following elements of R are invertible

$$5 + 3\sqrt{3}, \quad 2 - \sqrt{3}, \quad 1 + \sqrt{3}, \quad 7 + 4\sqrt{3} \text{ ?}$$

b) Does the following equality of ideals hold in R

$$(5 + 3\sqrt{3}) = (1 + \sqrt{3}) \text{ ?}$$

Explain why.

c) Is $(3 + \sqrt{3})$ a prime ideal in R? Explain.

d) Determine a maximal ideal $\mathfrak{M} \subset \mathbb{Z}[X]$ such that $X^2 - 3 \in \mathfrak{M}$.

4. [25 marks] Consider the ring $R = \mathbb{Z}[X]/(X^4 + 3X^3 + 1)$.

a) Is $(\bar{2}) \subset R$ a maximal ideal in R? Explain.

b) Is R a domain? Is R a field? Explain.

[Hint. You may want to use the fact that if $q(X)$ is an irreducible polynomial in R, which is the image of a non-constant, monic polynomial $p(X)$ in $\mathbb{Z}[X]$, then $p(X)$ is irreducible in $\mathbb{Z}[X]$]

c) Does R have any further unit besides ± 1? If yes, give an example of such unit.