Miscellaneous Exercices

1. Determine the ideals of the ring $\mathbb{R}[X]/(X^3 - X)$. Which among them are primes, which are maximals? How many ideals $\mathbb{Z}[X]/(X^3 - X)$ has?

2. Show that if $f : A \to B$ is a homomorphism of rings and B is an integral domain, then ker(f) is a prime ideal.

3. Show that $\mathbb{Z}[\sqrt{-3}]$ is not Euclidean.

4. Show that $(2X^4 + X^2 - X + 1, 2X + 1) = (1)$ in $\mathbb{Q}[X]$.

5. Is $\mathbb{R}[X]/(X^2 + X + 1)$ a field? Determine, if exists, the inverse of $\overline{X + 2}$.
Is $\mathbb{C}[X]/(X^2 + X + 1)$ a field? Why?

6. Is $(\bar{2})$ a prime ideal in $\mathbb{Z}_6[X]$?

7. Consider the $\mathbb{Z}$-module (=abelian group) $M = \mathbb{Z} \oplus \mathbb{Z}$. Is $\{(1, 0),\ (0, 2),\ (0, 3)\}$ a basis of M? Why?

8. Is $\mathbb{Q}[\sqrt[5]{3}] \simeq \mathbb{Q}[X]/(X^5 - 3)$ as isomorphism of rings? Why?

9. Show that there are no fields K such that $\mathbb{R} \subsetneq K \subsetneq \mathbb{C}$.

10. Is $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$ a homomorphism of $\mathbb{C}$-vector spaces? Is f a homomorphism of $\mathbb{R}$-vector spaces?

11. Determine the minimum polynomial of $\sqrt[3]{3}$ over $\mathbb{Q}[\sqrt{3}]$.

12. Which among the following $\mathbb{Q}$-extensions are Galois: $\mathbb{Q}(i\sqrt{3})$, $\mathbb{Q}(\sqrt{2}+\sqrt{3})$, $\mathbb{Q}(i, \sqrt[3]{2})$?

13. Determine the Galois group of the extension $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$.

14. Determine the Galois correspondence for the extension of $\mathbb{Q}$ associated to the polynomial $f(X) = (X^2 - 2)(X^2 - 3) \in \mathbb{Q}[X]$. Namely, find the Galois group G of the splitting field K of f and determine the correspondence between subgroups of G and subfields of K.

15. Let K be the splitting field of $f(X) = X^3 + 2X + 1 \in \mathbb{Q}[X]$. Determine the Galois group Gal(K/$\mathbb{Q}$) and the complete Galois correspondence.

16. Let ζ be a primitive 6-th root of unit. Determine $\mathbb{Q}(\zeta)$ and the related Galois group.