Miscellaneous Exercices

1. Determine the ideals of the ring \( \mathbb{R}[X]/(X^3 - X) \). Which among them are primes, which are maximals? How many ideals \( \mathbb{Z}[X]/(X^3 - X) \) has?

2. Show that if \( f : A \rightarrow B \) is a homomorphism of rings and \( B \) is an integral domain, then \( \ker(f) \) is a prime ideal.

3. Show that \( \mathbb{Z}[\sqrt{-3}] \) is not Euclidean.

4. Show that \((2X^4 + X^2 - X + 1, 2X + 1) = (1)\) in \( \mathbb{Q}[X] \).

5. Is \( \mathbb{R}[X]/(X^2 + X + 1) \) a field? Determine, if exists, the inverse of \( X + 2 \).

6. Is \((2)\) a prime ideal in \( \mathbb{Z}_6[X] \)?

7. Consider the \( \mathbb{Z}\)-module (=abelian group) \( M = \mathbb{Z} \oplus \mathbb{Z} \). Is \( \{(1, 0), (0, 2), (0, 3)\} \) a basis of \( M \)? Why?

8. Is \( \mathbb{Q}[\sqrt{3}] \simeq \mathbb{Q}[X]/(X^5 - 3) \) as isomorphism of rings? Why?

9. Show that there are no fields \( K \) such that \( \mathbb{R} \subseteq K \subseteq \mathbb{C} \).

10. Is \( f : \mathbb{C} \rightarrow \mathbb{C}, f(z) = \bar{z} \) a homomorphism of \( \mathbb{C}\)-vector spaces? Is \( f \) a homomorphism of \( \mathbb{R}\)-vector spaces?

11. Determine the minimum polynomial of \( \sqrt{3} \) over \( \mathbb{Q}[\sqrt{3}] \).

12. Which among the following \( \mathbb{Q}\)-extensions are Galois: \( \mathbb{Q}(i\sqrt{3}), \mathbb{Q}(\sqrt{2} + \sqrt{3}), \mathbb{Q}(i, \sqrt{2}) \)?

13. Determine the Galois group of the extension \( \mathbb{Q}(\sqrt{5})/\mathbb{Q} \).

14. Determine the Galois correspondence for the extension of \( \mathbb{Q} \) associated to the polynomial \( f(X) = (X^2 - 2)(X^2 - 3) \in \mathbb{Q}[X] \). Namely, find the Galois group \( G \) of the splitting field \( K \) of \( f \) and determine the correspondence between subgroups of \( G \) and subfields of \( K \).

15. Let \( K \) be the splitting field of \( f(X) = X^3 + 2X + 1 \in \mathbb{Q}[X] \). Determine the Galois group \( \text{Gal}(K/\mathbb{Q}) \) and the complete Galois correspondence.

16. Let \( \zeta \) be a primitive 6-th root of unit. Determine \( \mathbb{Q}(\zeta) \) and the related Galois group.