THE GEOGRAPHY OF LOG MODELS AND ITS APPLICATIONS.

by

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ABSTRACT

We use the Log Minimal Model Program (LMMP) to investigate the stratification of the set of $\mathbb{R}$-divisors on an algebraic variety X.

We first refine the classical definition of Iitaka dimension of an $\mathbb{R}$-divisor D on a variety. Using the equivalence relation on $\mathbb{R}$-boundary divisors defined by LMMP, we prove that the set of $\mathbb{R}$-boundary divisors on some fixed support is decomposed into finitely many equivalence classes which are open and rational polyhedral. Each class is called a country and such decomposition is called the geography log models.

As the first application of geography, we prove that the cone of effective divisors on an FT variety up to $\mathbb{R}$-linear equivalence relation $\sim_{\mathbb{R}}$ is rational polyhedral. Secondly, we prove the finiteness theorem for projective wlc models of a given log pair $(X/Z, B)$ with klt singularities such that $K_X + B$ is big over Z.

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Contents

1 Introduction

1.1 Introduction

The Log Minimal Model Program, (LMMP), is one of the most important and the deepest theories in algebraic geometry, especially in higher dimensions. The primary goal of the program is to find a good model in a given (log) birational class of a variety, which in some sense is minimal. Since the pioneering work of Mori in the 1980's, the progress in this area has been remarkable until very recently. See [KMM], [KoMo], [Mat] or [IsSh] for general theories. See [HM] and [BCHM] for the recent developments.

A resulting model of a given pair $(X/Z, B)$, where B is an $\mathbb{R}$-boundary on a variety X/Z, by LMMP is either a log minimal model or a log Mori fibration. For a fixed reduced divisor F on a variety X/Z, it was conjectured [IsSh, Conjecture 2.10] that the set of $\mathbb{R}$-boundary divisors B with support in $SuppF$ is decomposed by LMMP into open rational polyhedra according to the resulting models. In this thesis, we are interested in the case where the resulting models are log minimal models. More precisely, we study the case where the resulting model is a *weakly log canonical model* $(Y/Z, B_Y^{log})$ *of* $(X/Z, B)$ (see Definition 1.2.6). Assuming LMMP, we prove that the set $\mathcal{N}_F$ of $\mathbb{R}$-boundary divisors B with support in $SuppF$ forms a closed rational polyhedron and the decomposition of $\mathcal{N}_F$ is indeed as conjectured (see Main Theorem 1). See Chapter 3 for details. These phenomena were first observed by Shokurov in [Sho4, Section 6] and he called the decomposition *the Geography of Log Models*, (GLM). The paper [BCHM] also studied GLM in the same style at the same time as an application of their astonishing result (see Theorem 3.1.1 or the original paper [BCHM]). While our approach is quite similar to theirs, our treatment

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is done in a slightly more general setting with more elementary methods. In this thesis, the precise equations and inequalities determining the *countries* (which are called 'Shokurov polytopes' in [BCHM]) in the set $\mathcal{N}_F$ are given in Main Theorem 2.

In the rest of Chapter 1, we will give some standard preliminaries, including the definitions, used in the study of birational algebraic geometry and then briefly review the notions of convex polyhedra. We will also fix some notations about the spaces of divisors.

In Chapter 2, we first recall the definition of Iitaka dimension $\iota(D)$ and volume $v(D)$ of a divisor D, which can be found in the literature, for example, [La1, Chapter 2] and [Nak, §3 of Chapter 2]. We point out an obstacle of these definitions when we deal with $\mathbb{R}$-linear equivalence relations among $\mathbb{R}$-divisors. We introduce an improved version of the Iitaka dimension of a divisor D in section 2.2, which is compatible with $\mathbb{R}$-linear equivalence relation. We also introduce the modified definition of log Kodaira dimension accordingly. The new versions of definitions are suitable for our purposes and hopefully for other researchers' purposes as well.

Chapter 3 is the main ingredient of this thesis. We define the log model equivalence relations ($\sim_{wlc}$, $\sim_{lc}$) on $\mathbb{R}$-boundary divisors and D-model equivalence relations ($\sim_{Dm}$, $\sim_{Dmb}$) on $\mathbb{R}$-divisors. Next, we state Main Theorem 1 and 2. Before giving the proof of Main Theorem 2, we study some properties of log model equivalence classes and look at simple examples. The (rather lengthy) proof of Main Theorem 2 is given in the last section.

In Chapter 4, we give a few applications of GLM. As the first application, we give a conceptual proof of the rational polyhedrality of the cone of effective divisors of an FT variety (see Definition 4.1.2). There have been several attempts to prove this by generalizing the method of [Bat], where the theorem was obtained in dimension 3 by using the duality of the cones of curves and divisors. See [Bat], [Ara] and [Bar]. We also prove the finiteness of wlc models. Both applications are not conditional due to [BCHM].

1.2 Definitions and Preliminaries

In this section, we recall some definitions and fix notations, which will be used throughout this thesis. All the standard definitions follow from [Ha].

By a variety X, we always mean a normal variety over a field k, in most cases, of characteristic 0. By a variety X over another variety Z, we mean a normal variety X locally over a scheme point z_0 which is not necessarily closed in Z. The variety Z is usually a normal affine irreducible variety. We abbreviate this situation as X/Z. X_{z_0} denotes the fibre over the point z_0.

By a divisor $D = \sum_{i=1}^{m} r_i D_i$ on a variety, we always mean an $\mathbb{R}$-divisor, that is, $r_i \in \mathbb{R}$. An $\mathbb{R}$-boundary B is a boundary divisor with $\mathbb{R}$-coefficients in $[0, 1]$. For a real number r, $\lfloor r \rfloor$ denotes the round down of r and $\{r\}$ denotes the (Gaussian) fractional part of r, that is, $\{r\} = r - \lfloor r \rfloor$. If $D = \sum r_i D_i$ is a divisor, we let $\{D\} = \sum \{r_i\} D_i$. For any prime divisor D', the equality $\{\text{mult}_{D'} D\} = \text{mult}_{D'}\{D\}$ holds. By (B, B'), we mean the open segment $\{D \,|D = \alpha B + \beta B' \text{ where } 0 < \alpha, \beta \text{ and } \alpha + \beta = 1\}$. The closed segment $[B, B']$ and half closed segment $(B, B']$ are defined similarly.

The canonical divisor $K = K_X$ of X is the Weil divisor of zeros and poles of a rational differential form of the highest degree.

We will need the notion of *b-divisors* [Sho4, Section 1], [Isk], [F34, 2.3.2].

Let X be a normal variety. We call a proper birational map $f : X' \dashrightarrow X$ or just the variety X' a *model* of X.

Definition 1.2.1. A *b-divisor* $\mathbb{D}$ of X is an element of

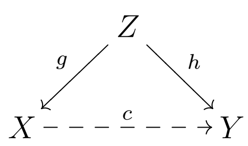
$$\mathbb{D}\mathrm{iv}(X) := \lim_{Y \to X} \mathrm{Div}(Y),$$

where the projective limit is taken over all models $Y \to X$. The divisor $\mathbb{D}_Y$ on a model Y of X which is given by a b-divisor $\mathbb{D}$ of X is called the *trace* of $\mathbb{D}$ on Y.

Definition 1.2.2. Let D be an $\mathbb{R}$-Cartier divisor on a variety X. Then the $\mathbb{R}$-*Cartier closure* of D is a b-divisor $\overline{D}$ with $\overline{D}_Y = f^*D$ for the models $f : Y \to X$ of X.

Prime b-divisors of X are in $1-1$ correspondence with the set of divisorial discrete valuations of the field of rational functions of X. A b-divisor is usually an $\mathbb{R}$-linear combination of infinitely many prime b-divisors, whose any trace is a divisor. It is well-known that the canonical divisor K, the discrepancy divisor of a log pair (X, B) and many other divisors can be realized as b-divisors. For details, see [F34, 2.3.2] or [Isk].

By LMMP, we mean that of [KMM], [KoMo], [Mat], and [IsSh]. An *extremal curve* C is an irreducible, reduced and proper curve which generates an extremal ray R of Kleiman-Mori cone $\overline{NE}(X/Z)$ of a variety X/Z and has the minimal degree for the ray R [Sho6, Definition 1]. A *contraction* of Y is a proper surjective morphism $f : Y \to X$ with $f_*\mathcal{O}_Y = \mathcal{O}_X$. We also say that it is an *extraction* of X. A *rational* 1-*contraction* $c : X \dashrightarrow Y$ is a proper dominant rational map which does not blow up any divisors [Pro, 1.1], [Sho5, Definition 3.1]. For a rational 1-contraction $c : X \dashrightarrow Y$, there exist a birational contration g and a contraction h such that the following diagram commutes:



where a prime divisor on Z contracted by g is also contracted by h. The pull back of an $\mathbb{R}$-Cartier divisor D on Y by a rational 1-contraction is defined as $c^*D := g_*h^*D$.

Definition 1.2.3. We define the $\mathbb{R}$-linear system $|D|_{\mathbb{R}}$ of an $\mathbb{R}$-divisor D on X/Z as a set of $\mathbb{R}$-divisors

$$|D|_{\mathbb{R}} = \{\ D' \in \mathrm{Div}_{\mathbb{R}}(X/Z) \mid D \sim_{\mathbb{R}} D' \geq 0\ \}.$$

Its base locus $\mathrm{Bs}|D|_{\mathbb{R}}$ is defined as

$$\mathrm{Bs}|D|_{\mathbb{R}} = \bigcap_{D' \in |D|_{\mathbb{R}}} \mathrm{Supp} D'.$$

We say that a Weil divisor D on X/Z is $\mathbb{R}$-*mobile* if $\mathrm{codim}_X \mathrm{Bs}|D|_{\mathbb{R}} \geq 2$.

A divisor D on a variety X is said to be *b-semiample* if there exists a rational 1-contraction $c : X \dashrightarrow Y$ and a numerically ample $\mathbb{R}$-divisor H on Y such that $|D|_{\mathbb{R}} = c_*^{-1}|H|_{\mathbb{R}}$. By definition, this implies that for any effective $D' \sim_{\mathbb{R}} D$, there exists an effective $H' \sim_{\mathbb{R}} H$ such that $D = c_*^{-1}H'$. Thus with each b-semiample divisor D on X, we can associate a rational 1-contraction c and a model Y. Moreover, c and Y are unique up to isomorphism [Pro, Section 1].

Let $f : Y \dashrightarrow X$ be a birational modification of normal varieties. If D' is a divisor on Y, then we denote $f_*D' := D'_X$, the proper image of D' on X. Thus if D is a divisor on X, we define $f_*^{-1}D := (f^{-1})_*D$. We define the *log birational transform* D_Y^{log} of D on Y as follows:

$$D_Y^{log} := f_*^{-1}D + \sum E_i$$

where E_i are f-exceptional prime divisors.

Let (X, D) be a log pair with an $\mathbb{R}$-divisor D such that $K + D$ is $\mathbb{R}$-Cartier. Suppose that $f : Y \to X$ is a log resolution of (X, D). Then we have $K_Y + f_*^{-1}D =$

$f^*(K+D)+\sum e_iE_i$ where E_i are f-exceptional prime divisors and we call $e_i := e(E_i, X, D)$ the *discrepancy* of (X, D) at E_i. The value $a(E_i, X, D) = e_i + 1$ is called the *log discrepancy* of (X, D) at E_i. If E is a prime non-exceptional divisor on X, then we define $a(E, X, D) = 1 - mult_E D$. The pair (X, D) has only *log canonical (lc) singularities* if $e_i \geq -1$ for all exceptional E_i. It has only *log terminal (lt) singularities* if $e_i \geq -1$ for all E_i. If, furthermore, X is $\mathbb{Q}$-factorial and projective over Z, then we say that $(X/Z, D)$ is *strictly log terminal (slt).* If D is a strict $\mathbb{R}$-boundary, that is, $\lfloor D \rfloor = 0$ and $e_i > -1$ for all i, then (X, D) has *Kawamata log terminal (klt) singularities.*

We use the standard abbreviations: lc, lt, klt, and slt for log canonical, log terminal, Kawamata log terminal and strictly log terminal, respectively. We often say that a pair $(X/Z, B)$ is lc, lt, etc if it has the corresponding singularities.

Lemma 1.2.4. *Let D and D' be $\mathbb{R}$-divisors with $SuppD = SuppD'$. Suppose that (X, D) is slt and (X, D') is lc. Then for all $D'' \in [D, D')$, (X, D'') is also slt.*

Proof. Let $S = SuppD = SuppD'$. To compute the log discrepancy of the pair (X, D'') with $D'' \in (D, D')$, consider a log resolution $f : Y \to (X, S)$ of (X, S). Note that this is also a log resolution of (X, D'') with $D'' \in (D, D')$. Indeed, $SuppD'' \subseteq S$ for any $D'' \in (D, D')$.

Therefore, $SuppD_Y''^{log} = Supp(\sum E_i \cup f^{-1}(SuppD'')) \subseteq Supp(\sum E_i \cup f^{-1}(S))$ where E_i are f-exceptional prime divisors. Since $Supp(\sum E_i \cup f^{-1}(S))$ is simple normal crossing, so is $SuppD_Y''^{log}$.

We have

$$K_Y + D_Y''^{log} = f^*(K+D'') + \sum_{i \in I} a(E_i, X, D'')E_i,$$

where the divisors E_i are prime f-exceptional. By our assumptions and definition, each $a(E_i, X, D) > 0$ and each $a(E_i, X, D') \geq 0$. Thus, for every $D'' \in (D, D')$, each

$a(E_i, X, D'') > 0$ by the linear property of the function $a(E_i, X, D'')$ with respect to D'', and (X, D'') is slt. Note that X is $\mathbb{Q}$-factorial because (X, D) is slt. □

As the next example shows, if $SuppD \neq SuppD'$, then the lemma does not hold in general. Furthermore, unlike lc singularities, slt property is neither convex nor closed.

Example 1.2.5. [Sho6, page 8] Let L_1, L_2 and L_3 be 3 distinct lines in the plane $\mathbb{P}^2$ passing through a point P. Then for $F = L_1 + L_2 + L_3$, the set of divisors D in $\oplus[0,1]L_i$, for which $(\mathbb{P}^2, D)$ is lc, is a closed rational convex polyhedron. The set of boundaries $D = \sum b_i L_i$ in $\oplus[0,1]L_i$ where $\sum b_i = 2$ gives a non slt, lc pair with 3 exceptions: $F - L_i$, $i = 1, 2, 3$. These exceptions give slt pairs $(\mathbb{P}^2, F - L_i)$. Indeed, for $D = \sum b_i L_i$ with $\sum b_i = 2$ and $0 < b_i \leq 1$, we consider a log resolution $(Bl_P(\mathbb{P}^2), \sum b_i \widetilde{L}_i + E)$ of $(\mathbb{P}^2, D)$ where $Bl_P(\mathbb{P}^2)$ is the blow up of $\mathbb{P}^2$ at P and E is the induced exceptional divisor. Then $a(E, \mathbb{P}^2, D) = codim_{\mathbb{P}^2} P - \sum b_i mult_P L_i = 2 - 2 = 0$ and $(\mathbb{P}^2, D)$ is lc, not slt. The pair $(\mathbb{P}^2, F - L_i)$ is log-nonsingular and it is clear that it is slt.

A pair $(Y/Z, B)$ with an $\mathbb{R}$-boundary B is called a *relative weakly log canonical (wlc) model* if it is lc and the divisor $K_Y + B$ is nef over Z. If $K_Y + B$ is ample over Z, then the pair is called a *relative log canonical model.* If the pair $(Y/Z, B)$ is lt and the divisor $K_Y + B$ is nef over Z, then it is called a *relative log minimal model.* Similarly, we can also define a *terminal model* and a *canonical model* for log pairs.

The following is the most important object in this thesis.

Definition 1.2.6. Let $(X/Z, B)$ be a pair with an $\mathbb{R}$-boundary B on X and let $X \dashrightarrow Y$ be a birational map. Then the pair $(Y/Z, B_Y^{log})$ is called a *relative wlc model* or *relative lc model of* $(X/Z, B)$ if $(Y/Z, B_Y^{log})$ is a relative wlc model or a relative lc model, respectively, and the inequality $a(E, X, B) \leq a(E, Y, B_Y^{log})$ holds for any prime non-exceptional divisor E on X, which is exceptional on Y. If $(Y/Z, B_Y^{log})$ is a relative log minimal model and the above inequality is strict, then $(Y/Z, B_Y^{log})$ is a *relative log minimal model of* $(X/Z, B)$. Furthermore, if Y/Z is $\mathbb{Q}$-factorial and projective over Z, then $(Y/Z, B_Y^{log})$ is a *relative strictly log minimal(slm) model of* $(X/Z, B)$.

When $(Y/Z, B_Y^{log})$ is a wlc model and the divisor $K_Y + B_Y^{log}$ is semiample [IsSh, Conjecture 1.16], then the contraction $I : Y \to Y'$ defined by the linear system $|m(K_Y + B_Y^{log})|$ is called the *Iitaka fibration* and its image *lc Iitaka model* of $(Y/Z, B_Y^{log})$.

The relative situation will be assumed by writing $/Z$. Thus we will often omit 'relative' and just say, for example, a wlc or a lc model $(Y/Z, B_Y^{log})$ of $(X/Z, B)$.

Finally in this section, we define and state some basic notions about convex polyhedra in a finite dimensional $\mathbb{R}$-vector space V.

A *polyhedron* P is defined as a bounded convex set in V which is an intersection of a finite number of open (or closed) half spaces. Let P be a nonempty convex set in V. The *linear span of* P in V is the set $L + p$ where $p \in P$ and L is the sub-vector space of V over $\mathbb{R}$ generated by the vectors in $P - p$. The set P is said to be *open* (closed) if it is open (closed, respectively) in the linear span of P. In particular, any point in V is open. For a fixed set $\mathcal{N} \subseteq V$, a subset $P \subseteq \mathcal{N}$ is said to be *open in* $\mathcal{N}$ if there exists an open set $P' \subseteq V$ such that $P = P' \cap \mathcal{N}$. The *interior* $intP$ of P is defined as the (usual) interior of the convex set $\overline{P}$ in the linear span of P. Note that since $dimP \leq dimV$ in general, the (usual) interior of P may be empty whereas $intP$ is always nonempty.

Let P be a convex polyhedron, not necessarily closed, in a finite dimensional $\mathbb{R}$-space V. Then by a *face* of P, we always mean a proper face of $\overline{P}$, that is, the set

$\mathcal{G} = \overline{P} \cap H$, where H is a supporting hyperplane to $\overline{P}$ and $dim\overline{P} \cap H \leq dimP - 1$. If $dim\overline{P} \cap H = dimP - 1$, then it is a *maximal face* of P. Note that a maximal face can be disjoint from P and that any face $\mathcal{G}$ of P is a finite intersection of maximal faces of P containing $\mathcal{G}$.

1.3 Terminology and Notation.

In this section, we fix and introduce some terminologies and notations about the spaces of divisors.

We will denote the set of Weil divisors on X as $\mathrm{Div}(X)$ and the set of Cartier divisors on X as $\mathrm{CDiv}(X)$. We also let $\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes \mathbb{R}$ and $\mathrm{CDiv}_{\mathbb{R}}(X) := \mathrm{CDiv}(X) \otimes \mathbb{R}$. The $\mathbb{R}$-vector space $\mathrm{Div}_{\mathbb{R}}(X)$ is in general an infinite dimensional space. For example, $\mathrm{Div}_{\mathbb{R}}(\mathbb{P}^1) = \bigoplus_{x \in \mathbb{P}^1} \mathbb{R} \cdot x$.

Notation 1.3.1. Let $F = \sum_{i=1}^{m} D_i$ be a fixed reduced Weil divisor where D_i are distinct prime divisors on X. We define the following sets:

$$\mathfrak{D}_F := \bigoplus_{i}^{m} \mathbb{R} D_i \cong \mathbb{R}^m$$
$$\mathfrak{D}_F^{+} := \bigoplus_{i}^{m} \mathbb{R}_{\geq 0} D_i \cong (\mathbb{R}_{\geq 0})^m$$
$$\mathfrak{B}_F := \bigoplus_{i}^{m} [0,1] D_i \cong [0,1]^m$$

Clearly, $\mathfrak{B}_F \subseteq \mathfrak{D}_F^{+} \subseteq \mathfrak{D}_F$ and $\mathfrak{D}_F^{+}$ is a convex rational polyhedral cone.

Let $\mathcal{E}(= \mathcal{E}_X)$ denote the $\mathbb{R}$-vector space of $\mathbb{R}$-Weil divisors on X that are $\mathbb{R}$-linearly equivalent to an effective divisor. That is,

$$\mathcal{E}(= \mathcal{E}_X) = \{D \in \mathrm{Div}_{\mathbb{R}}(X) |\ D \sim_{\mathbb{R}} D' \text{ for some } D' \geq 0\}.$$

The divisors in the closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ are called $\mathbb{R}$-*linear pseudo-effective* divisors. Similarly, let $\mathcal{E}\!f\!f (= \mathcal{E}\!f\!f_X)$ denote the $\mathbb{R}$-vector space of $\mathbb{R}$-Weil divisors on X that are numerically equivalent to an effective divisor. That is,

$$\mathcal{E}\!f\!f\,(= \mathcal{E}\!f\!f_X) = \{D \in \mathrm{Div}_{\mathbb{R}}(X) |\ D \equiv D' \text{ for some } D' \geq 0\}.$$

The divisors in the closure $\overline{\mathcal{E}\!f\!f}$ of $\mathcal{E}\!f\!f$ are called *pseudo-effective* divisors. Clearly, $\mathbb{R}$-linear pseudo-effective divisors are pseudo-effective: $\mathcal{E} \subseteq \mathcal{E}\!f\!f$ and $\overline{\mathcal{E}} \subseteq \overline{\mathcal{E}\!f\!f}$.

Notation 1.3.2. For a divisor B on X, we define the following subset of $\mathfrak{D}_F$:

$$\mathcal{E}_F(B) := \Big\{ D \in \mathfrak{D}_F \ \Big|\ D \sim_{\mathbb{R}} D' \text{ for some } D' \geq B \Big\}.$$

If we denote $\mathcal{E}_F := \mathcal{E}_F(0)$, then we have

$$\begin{aligned} \mathcal{E}_F &= \Big\{ D \in \mathfrak{D}_F \ \Big|\ D \sim_{\mathbb{R}} D' \text{ for some } D' \geq 0 \Big\} \\ &= \text{the set of effective divisors in } \mathfrak{D}_F \text{ up to } \sim_{\mathbb{R}}. \end{aligned}$$

The set $\mathcal{E}_F$ is a convex cone with vertex at the origin 0 in the space $\mathfrak{D}_F$. The set $\mathcal{E}_F(B)$ can be also interpreted as $\mathcal{E}_F(B) = B + \mathcal{E}_F$. Therefore the cone $\mathcal{E}_F(B)$ is isomorphic to the cone $\mathcal{E}_F$, but their vertices are B and 0, respectively.

If we want to specify on which variety the above sets are defined, we add the variety X in the notation, i.e. for example, $\mathfrak{D}_{X,F}$, $\mathfrak{D}^+_{X,F}$, $\mathfrak{B}_{X,F}$, $\mathcal{E}_{X,F}$, and $\mathcal{E}_{X,F}(B)$.

Notation 1.3.3. In the numerical space, $N^1(X/Z) = \mathrm{Div}_{\mathbb{R}}(X/Z)/\equiv$, we define the following subsets:

$$\begin{aligned}\mathrm{Mb}(X/Z) &= \text{the convex cone in } N^1(X/Z) \text{ generated by the classes} \\ &\quad \text{of } \mathbb{R}\text{-mobile divisors in } \mathrm{Div}_{\mathbb{R}}(X/Z), \\ \mathrm{Eff}(X/Z) &= \text{ the convex cone in } N^1(X/Z) \text{ generated by the classes} \\ &\quad \text{of effective divisors in } \mathrm{Div}_{\mathbb{R}}(X/Z),\end{aligned}$$

We have the following inclusion relations:

$$\mathrm{Mb}(X/Z) \subseteq \mathrm{Eff}(X/Z) \subseteq N^1(X/Z)$$

Notation 1.3.4. In the numerical space, $CN^1(X/Z)$=$\{\mathrm{CDiv}(X/Z)/\equiv\} \otimes \mathbb{R}$, we define the following subsets:

$$\begin{aligned}\mathrm{CMb}(X/Z) &= \text{the convex cone in } CN^1(X/Z) \text{ generated by the classes} \\ &\quad \text{of } \mathbb{R}\text{-mobile divisors in } \mathrm{CDiv}_{\mathbb{R}}(X/Z), \\ \mathrm{CEff}(X/Z) &= \text{ the convex cone in } CN^1(X/Z) \text{ generated by the classes} \\ &\quad \text{of effective divisors in } \mathrm{CDiv}_{\mathbb{R}}(X/Z),\end{aligned}$$

We have the following inclusion relations:

$$\mathrm{CMb}(X/Z) \subseteq \mathrm{CEff}(X/Z) \subseteq CN^1(X/Z)$$

Remark. 1. Since X is a normal variety, there is a natural injective homomorphism

$$\mathrm{CDiv}(X/Z) \hookrightarrow \mathrm{Div}(X/Z).$$

Thus we can identify $\mathrm{CDiv}(X/Z)$ as a subgroup of $\mathrm{Div}(X/Z)$ and by this identification, we also have

$$\begin{array}{ccccc} \mathrm{CMb}(X/Z) & \subseteq & \mathrm{CEff}(X/Z) & \subseteq & CN^1(X/Z) \\ \cap| & & \cap| & & \cap| \\ \mathrm{Mb}(X/Z) & \subseteq & \mathrm{Eff}(X/Z) & \subseteq & N^1(X/Z). \end{array}$$

Since $CN^1(X/Z)$ is a linear section of $N^1(X/Z)$, each $\mathrm{CMb}(X/Z) = \mathrm{Mb}(X/Z) \cap CN^1(X/Z)$ and $\mathrm{CEff}(X/Z) = \mathrm{Eff}(X/Z) \cap CN^1(X/Z)$ is a linear section.

2.Let $\rho_C(X/Z)$ denote the rank of $CN^1(X/Z)$ and let $\rho(X/Z)$ denote the rank of $N^1(X/Z)$. In general, we have $\rho_C(X/Z) \leq \rho(X/Z)$. If X is $\mathbb{Q}$-factorial, then $CN^1(X/Z) = N^1(X/Z)$ and $\rho_C(X/Z) = \rho(X/Z)$.

For an $\mathbb{R}$-divisor D effective up to $\sim_{\mathbb{R}}$, the following decomposition is called an $\mathbb{R}$-*mobile decomposition*:

$$D = \mathrm{Fx}D + \mathrm{Mb}D$$

where $\mathrm{Fx}D = \min\{E | \ D = E + M \text{ with } E \geq 0 \text{ and } M \in \mathrm{Mb}(X/Z)\}$ if such a minimum exists and $\mathrm{Mb}D$ is b-semiample. We can also define the $\mathbb{R}$-mobile decomposition for $D \in \overline{\mathcal{E}}$ by letting $\mathrm{Fx}D := \lim_{\varepsilon \to 0} \mathrm{Fx}(D + \varepsilon H)$ for an ample divisor H. Then the pseudo-mobile part $\mathrm{Mb}D$ of the decomposition of D belongs to $\overline{\mathrm{Mb}(X/Z)}$.

2 Dimensions

2.1 Introduction.

In this section, we recall the definitions of the Iitaka dimension $\iota(D)$ and the volume $vol(D)$ of a divisor D on a variety X/Z, which can be found in the existing literature, for example, [La1, Definition 2.1.3 and 2.2.C], [Nak, Definition 3.2] and [Bo]. In the next section, we introduce slightly improved versions of the Iitaka dimension and of the volume of a divisor.

For a relative variety X/Z, X_{z_0} denotes the fibre in X over a scheme point z_0 of Z, that is, over the generic point of $\overline{z_0}$ (see Section 1.2).

Definition 2.1.1. Let D be an $\mathbb{R}$-divisor on X/Z and let $\mathbb{N}(D) := \{m \in \mathbb{N} \mid |mD| \neq \varnothing\}$.

(1) The *relative Iitaka dimension* $\iota(D) = \iota(X/Z, D)$ of D over Z is defined as follows:

$$\iota(D)(=\iota) = \begin{cases} \max_{m\in\mathbb{N}(D)} \dim\Phi_m(X_{z_0}) & \text{if } \mathbb{N}(D) \neq \varnothing, \\ -\infty & \text{if } \mathbb{N}(D) = \varnothing. \end{cases}$$

where Φ_m is a rational map defined on X/Z by the linear system $|mD|$.

(2) The *pseudo-volume* $\upsilon(D) = \upsilon(X/Z, D)$ of D over Z is defined as follows:

$$\upsilon(D) = \begin{cases} \lim_{m\to\infty} \frac{\iota!}{m^\iota} h^0(X_{z_0}, mD|_{X_{z_0}}) \text{ where } m \in \mathbb{N}(D) & \text{if } \mathbb{N}(D) \neq \varnothing, \\ 0 & \text{if } \mathbb{N}(D) = \varnothing. \end{cases}$$

Remark. The relative Iitaka dimension $\iota = \iota(D)$ and the pseudo-volume $\upsilon = \upsilon(D)$ are related by the following equation:

$$h^0(X_{z_0}, mD|_{X_{z_0}}) = \frac{\upsilon}{\iota!} m^\iota + O(m^{\iota-1}).$$

The above Iitaka dimension $\iota(D)$ has the following property.

Proposition 2.1.2. *If D and D' are $\mathbb{R}$-divisors on X such that $D \sim_{\mathbb{R}} D' \geq 0$, then $\iota(D) \leq \iota(D')$.*

Proof. By our assumption, we have

$$0 \leq D' = D + P \tag{1}$$

for some $\mathbb{R}$-principal divisor $P = \sum_{i=1}^{k} r_i(s_i)$ where each r_i is a real number. We can suppose that the numbers r_i are integrally (or rationally) independent:

$$\sum_{i=1}^{k} n_i r_i = n, \quad n_i, n \in \mathbb{Z} \text{ for all } i \text{ implies } n_i = n = 0 \quad \text{for all } i.$$

Indeed, suppose that $n_1 r_1 + n_2 r_2 + \cdots + n_k r_k = n$ for some integers n_i and n with $n_k \neq 0$. Then

$$\begin{aligned} n_k D' &= n_k D + n_k P \\ &= n_k D + \textstyle\sum_{i=1}^{k-1} n_k r_i(s_i) + n_k r_k(s_k) \\ &= n_k D + \textstyle\sum_{i=1}^{k-1} n_k r_i(s_i) + (n - n_1 r_1 - \cdots - n_{k-1} r_{k-1})(s_k) \\ &= n_k D + n(s_k) + \textstyle\sum_{i=1}^{k-1} r_i(s_i') \end{aligned}$$

where $s_i' = s_i^{n_k} \cdot s_k^{-n_i}$. Therefore, $D' \sim_{\mathbb{R}} D + \frac{n}{n_k}(s_k)$. However, since $D + \frac{n}{n_k}(s_k) \sim_{\mathbb{Q}} D$ and the dimension ι is invariant up to $\sim_{\mathbb{Q}}$ [Nak, Definition 3.2], we obtain $\iota(D) = \iota(D + \frac{n}{n_k}(s_k))$. Thus we can replace D by $D + \frac{n}{n_k}(s_k)$. Now by induction on the set $\{r_1, r_2, \cdots, r_{k-1}\}$, we can find a maximal integrally independent subset. Thus without loss of generality, we may assume that the real numbers r_i are integrally independent.

By definition, there exists a natural number m such that $H^0(X, mD)$ has sufficiently many sections giving rise to a rational map from X with an image of dimension

$\iota(D)$. The same holds for the natural numbers n divisible by m. It is sufficient to find such n with an injection

$$H^0(X, nD) \hookrightarrow H^0(X, (n+1)D')$$
$$f \longmapsto f/g$$

for some appropriately chosen rational function $g \neq 0$.

Now we are to find such g and n. First, set a positive real number

$$\delta = \min\{ \ \mathrm{mult}_{D_i} D' \mid D_i \in \mathrm{Supp} D' \ \text{ and } \ \mathrm{mult}_{D_i} D' > 0\}.$$

Since r_i are integrally independent, there is a simultaneous approximation:

$$r_i = \frac{m_i}{n} + \frac{\delta_i}{n} \tag{2}$$

where $m|n$, $m_i \in \mathbb{Z}$ for all i and $0 < |\delta_i|$. However by choosing a sufficiently large integer n with $m|n$, we can assume that $0 < |\delta_i| \ll 1$. Thus if we let

$$\alpha = \max\left\{ |\mathrm{mult}_{D_j} \sum \delta_i(s_i)| \ \Big| \ D_j \in \mathrm{Supp} \sum \delta_i(s_i) \right\},$$

then we can suppose that $\alpha < \min\{ \ \frac{1}{2}, \delta \ \}$.

We claim that the rational function $g = \prod s_i^{m_i}$ defines the above injection. Indeed, for any $f \neq 0$ in $H^0(X, nD)$,

$$\begin{aligned}
(f/g) + (n+1)D' &= (f) - \sum m_i(s_i) + (n+1)D' \\
&= (f) - \sum m_i(s_i) + nD + nP + D' && \text{by (1)} \\
&= (f) - \sum m_i(s_i) + nD + \sum nr_i(s_i) + D' \\
&= (f) + nD + \sum (nr_i - m_i)(s_i) + D' \\
&= (f) + nD + \sum \delta_i(s_i) + D' && \text{by (2).}
\end{aligned}$$

Now we have to verify that the last divisor is effective, or equivalently

$$\mathrm{mult}_{D_j}\big((f) + nD + \sum \delta_i(s_i) + D'\big) \geq 0$$

for any prime divisor D_j. Note that by our assumption, $(f) + nD \geq 0$, or more precisely, $(f) + \lfloor nD \rfloor \geq 0$ by definition.

Suppose first that $D_j \in \mathrm{Supp} D'$. Then the following inequality holds:

$$\mathrm{mult}_{D_j} D' \geq \delta > \alpha \geq \left| \mathrm{mult}_{D_j} \sum \delta_i(s_i) \right| \tag{3}$$

So we obtain $\mathrm{mult}_{D_j} \left(\sum \delta_i(s_i) + D' \right) > 0$ and thus for $D_j \in \mathrm{Supp} D'$,

$$\mathrm{mult}_{D_j} \left((f) + nD + \sum \delta_i(s_i) + D' \right) > 0.$$

Next suppose $D_j \notin \mathrm{Supp} D'$. Then the following equality holds:

$$\mathrm{mult}_{D_j} D' = \mathrm{mult}_{D_j} nD' = \mathrm{mult}_{D_j} (nD + nP) = 0.$$

We have the following three cases:

case 1: $\mathrm{mult}_{D_j} \sum \delta_i(s_i) > 0$.

Then $\mathrm{mult}_{D_j} \left((f) + nD + \sum \delta_i(s_i) + D' \right) > \mathrm{mult}_{D_j}((f) + nD) \geq 0$ because $(f) + nD$ is effective.

case 2: $\mathrm{mult}_{D_j} \sum \delta_i(s_i) < 0$.

We have

$$\begin{aligned} 0 = \mathrm{mult}_{D_j}(nD + nP) &= \mathrm{mult}_{D_j} \left(nD + \sum nr_i(s_i) \right) \\ &= \mathrm{mult}_{D_j} \left(nD + \sum m_i(s_i) + \sum \delta_i(s_i) \right). \end{aligned}$$

Note that since $\mathrm{mult}_{D_j} \sum m_i(s_i)$ is integral and $-1/2 < \mathrm{mult}_{D_j} \sum \delta_i(s_i) < 0$, the following holds:

$$\{\mathrm{mult}_{D_j} nD\} + \mathrm{mult}_{D_j} \sum \delta_i(s_i) = 0.$$

Since $(f) + \lfloor nD \rfloor \geq 0$, we have

$$\begin{aligned} \mathrm{mult}_{D_j} \left((f) + nD + \sum \delta_i(s_i) + D' \right) &= \mathrm{mult}_{D_j} \left((f) + \lfloor nD \rfloor + \{nD\} + \sum \delta_i(s_i) + D' \right) \\ &\geq \mathrm{mult}_{D_j} \{nD\} + \mathrm{mult}_{D_j} \sum \delta_i(s_i) \\ &= 0. \end{aligned}$$

case 3: $\text{mult}_{D_j} \sum \delta_i(s_i) = 0$.

In this case, $\text{mult}_{D_j} D' = \text{mult}_{D_j} \sum \delta_i(s_i) = 0$. So it follows easily that

$$\text{mult}_{D_j}\big((f) + nD + \sum \delta_i(s_i) + D'\big) = \text{mult}_{D_j}\big((f) + nD\big) \geq 0.$$

Now we obtain

$$\text{mult}_{D_j}\big((f) + nD + \sum \delta_i(s_i) + D'\big) \geq 0$$

for ANY prime divisor D_j and thus $(f) + nD + \sum \delta_i(s_i) + D'$ is effective. □

Definition 2.1.3. Let D be a divisor on X/Z. The *volume* $vol(D)$ of D is defined as follows:

$$vol(D) := \limsup_{m \to \infty} \frac{h^0(X/Z, mD)}{m^d/d!},$$

where $d = dimX$.

By [La1, Remark 2.2.50], '$\limsup$' can be replaced by 'lim'. If D is a nef divisor on X/Z, then it is well known that $vol(D) = (D^\iota \cdot X_{z_0})$. By definition, we see that D is big/Z if and only if $vol(D) > 0$. Both $vol(D)$ and $\upsilon(D)$ measure the growth rate of $h^0(X/Z, mD)$ as m grows. For detailed properties of volumes of divisors, see [La1, Section 2.2] and [Bo].

Corollary 2.1.4. *If $D \sim_{\mathbb{R}} D'$ and both $D \geq 0$ and $D' \geq 0$, then $\iota(D) = \iota(D')$, $\upsilon(D) = \upsilon(D')$, and $vol(D) = vol(D')$.*

Proof. Immediate from the Proposition 2.1.2 and by definition. □

Note, however, that the volume $vol(D)$ and the pseudo-volume $\upsilon(D)$ can be different by the following remark.

Warning-Example: If D is nef and big/Z, then $\iota(D) = \dim X_{z_0}$ and $\upsilon(D) = vol(D) = (D^\iota \cdot X_{z_0}) > 0$. If D is nef but not big/Z, then $\iota(D) < \dim X_{z_0}$, and $vol(D) =$

$(D^\iota \cdot X_{z_0}) = 0$. However, $\upsilon(D) = 0$ if and only if $\iota(D) = -\infty$. Thus $\upsilon(D)$ can be different from $vol(D)$.

2.2 Iitaka dimension and volume.

We note that, in fact, the (classical) Iitaka dimension $\iota(D)$ defined in Definition 2.1.1 is compatible with $\sim_{\mathbb{Q}}$ [Nak, Definition 3.2], but not with $\sim_{\mathbb{R}}$. See the following remark.

Remark. Let $div(s) \neq 0$ be a principal divisor defined by a rational function s on a complete variety X. Then by the definition of $\sim_{\mathbb{R}}$, $div(s) \sim_{\mathbb{R}} r \cdot div(s)$ holds for any positive irrational number r and $\iota(div(s)) = 0$. But $\iota(r \cdot div(s)) = -\infty$ since for any positive integer n, $nr \cdot div(s)$ can never be a $\mathbb{Z}$-divisor, in particular a principal divisor $div(s')$. It turns out that this is essentially the only case where dimension $\iota(D)$ fails to be compatible with $\sim_{\mathbb{R}}$. See proposition 2.2.2 below.

Now we introduce a slightly improved version of Iitaka dimension of a divisor, which is compatible even with the relation $\sim_{\mathbb{R}}$.

Definition 2.2.1. Let D be an $\mathbb{R}$-divisor on X/Z. Then we define the *relative invariant Iitaka dimension* $I(X/Z, D)$ of D over Z as follows:

$$I(X/Z, D) = I(D) = \begin{cases} \iota(D') & \text{if } D \sim_{\mathbb{R}} D' \text{ for some } D' \geq 0; \\ -\infty & \text{otherwise.} \end{cases}$$

The dimension $I(X/Z, D)$ is called invariant because it is independent of the choice of $D' \geq 0$ by the transitivity of $\sim_{\mathbb{R}}$ and by corollary 2.1.4.

Proposition 2.2.2. *Let D and D' be any $\mathbb{R}$-divisors on X/Z. Then the following hold:*

(1) We have $\iota(D) \leq I(D)$ and the strict inequality holds if and only if $\iota(D) = -\infty$ and $I(D) \geq 0$.

(2) If $D \sim_{\mathbb{R}} D'$, then we have $I(D) = I(D')$.

Proof. (1): Suppose that there exists an effective divisor D'' such that $D \sim_{\mathbb{R}} D''$. Then by definition $I(D) = \iota(D'')$ and by proposition 2.1.2 we have $\iota(D) \leq \iota(D'')$. Suppose that there exist no effective divisors that are $\mathbb{R}$-linearly equivalent to D. Then $\iota(D) = I(D) = -\infty$.

Suppose that the strict inequality $\iota(D) < I(D)$ holds and $\iota(D) \geq 0$. Then it implies that $\iota(D) = I(D)$, a contradiction. The converse is clear.

(2): If there exists an effective divisor D'' such that $D \sim_{\mathbb{R}} D''$, then we have $I(D) = \iota(D'') = I(D')$ since the relation $\sim_{\mathbb{R}}$ is transitive. Otherwise, $I(D) = I(D') = -\infty$. □

Let $(X/Z, B)$ be an lc pair with an $\mathbb{R}$-boundary B. Then the usual relative log Kodaira dimension $\kappa(X/Z, B)$ is defined as the relative Iitaka dimension of $K + B$ over Z. That is, $\kappa(X/Z, B) := \iota(X/Z, K + B)$. However, if $K + B$ is not lc or big over Z, it is better to use the following version.

Definition 2.2.3. Let B be an $\mathbb{R}$-boundary on X/Z and $g : Y \to X$ be an extraction of X over Z such that the log pair $(Y/Z, B_Y^{log})$ is lc. Then the *relative invariant log Kodaira dimension* $K(X/Z, B)$ of the pair $(X/Z, B)$ is defined as follows:

$$K(X/Z, B) := I(Y/Z, K_Y + B_Y^{log}).$$

In particular, g can be a log resolution of $(X/Z, B)$ in characteristic 0. Such an extraction g exists by Hironaka and the definition is independent of the choice of such

g by the following lemma [Sho5, Proposition 3.20]. In positive characteristic, we need a log resolution.

Lemma 2.2.4. *Let $f : Y \to X$ proper surjective porphism over Z of normal varieties and let D be an $\mathbb{R}$-divisor on X. Then*

$$\iota(f^*D + E) = \iota(D)$$

where E is an effective f-exceptional divisor.

Proof. See [Sho5, Proposition 3.20]. □

Example 2.2.5. As with ι and I, we also have the inequality $\kappa(X/Z, D) \leq K(X/Z, D)$ by proposition 2.2.2, and the strict inequality $\kappa(X/Z, D) < K(X/Z, D)$ holds if and only if $\kappa(X/Z, D) = -\infty$ and $K(X/Z, D) \geq 0$.

Corollary 2.2.6. *Let B be an $\mathbb{R}$-boundary on X/Z. Then the following hold:*

(1) $K(X/Z, B) \leq I(X/Z, K + B)$.

(2) If (X, B) is an lc pair, then $K(X/Z, B) = I(X/Z, K + B)$.

Proof. (1) By definition, $K(X/Z, B) = I(Y/Z, K_Y + B_Y^{log})$ where $(Y/Z, B_Y^{log})$ is lc for an extraction $g : Y \to X$ as in the definition above. Then for any divisor D on Y/Z, we have $I(Y/Z, D) \leq I(X/Z, g_*D)$, because for any integer m, we have a natural injection

$$H^0(Y/Z, mD) \hookrightarrow H^0(X/Z, g_*mD).$$

Choosing $D = K_Y + B_Y^{log}$, we obtain $K(X/Z, B) \leq I(X/Z, K + B)$ because $g_*(K_Y + B_Y^{log}) = K + B$.

(2) Immediate by definition 2.2.3. □

Using the numerical data, we can also define the following numerical dimension of a pair $(X/Z, B)$.

Definition 2.2.7. [Sho4, 2.4.4] Suppose that the pair $(X/Z, B)$ has a wlc model $(Y/Z, B_Y^{log})$. Then the *relative numerical log Kodaira dimension* $\nu(X/Z, B)$ is defined to be the following integer:

$$\nu(X/Z, B) := \max\{l \in \mathbb{Z}_{\geq 0} \mid (K_Y + B_Y^{log})^l \cdot C > 0 \text{ for a prime } l\text{-cycle } C \text{ on } X_{z_0}\}.$$

The above definition is independent of the choice of a wlc model $(Y/Z, B_Y^{log})$ of $(X/Z, B)$ by [Sho4, Proposition 2.4]. If $(X/Z, B)$ does not have a wlc model, then we define $\nu(X/Z, B) = -\infty$. Thus $(X/Z, B)$ has a wlc model if and only if $\nu(X/Z, B) \geq 0$ by definition.

Remark. [KMM, remark 6-1-2] We have $K(X, B) \leq \nu(X, B)$ in general. However, it is expected that $K(X, B) = \nu(X, B)$ always. This equality holds if (X, B) is of general type, that is, $K(X, B) = dimX$. It is also known that the divisor $K + B$ is pseudo-effective if and only if $\nu(X, B) \geq 0$.

3 Geography

3.1 Introduction.

It is expected that we can run LMMP on a pair $(X/Z, B)$ with an arbitrary $\mathbb{R}$-boundary B on X/Z. Recently, it was proved in [BCHM] that LMMP indeed works in several settings.

Theorem 3.1.1. [BCHM, Theorem 1.2] *Let (X, Δ) be a klt pair, where $K + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : X \to U$ be a proper morphism, where U is an algebraic space. If either Δ is π-big and $K + \Delta$ is π-pseudo-effective or $K + \Delta$ is π-big, then*

(1) $K + \Delta$ has a log terminal model over U,

(2) $K + \Delta$ has a log canonical model over U, and

(3) if $K + \Delta$ is $\mathbb{Q}$-Cartier, then the $\mathcal{O}_U$-algebra

$$\bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K + \Delta) \rfloor),$$

is finitely generated.

Proof. See [BCHM, Theorem 1.2]. □

Even though the proof of the theorem above does not use the standard LMMP procedures, the result is powerful enough to remove the 'LMMP assumptions' from many theorems. See [BCHM] for numerous corollaries. The geography of log models, we will study in this section, is one of them and it explains the stratification of the cube $\mathfrak{B}_F$ by LMMP according to the resulting models of the pairs $(X/Z, B)$ where $B \in \mathfrak{B}_F$. However, since theorem 3.1.1 does not guarantee LMMP for all pairs $(X/Z, B)$, we prefer to assume LMMP and we remark where we can actually remove the assumption.

3.2 Geography of log models.

Let $(X/Z, B)$ and $(X/Z, B')$ be pairs such that the log divisors $K+B$ and $K+B'$ are $\mathbb{R}$-Cartier. Then the two log pairs $(X/Z, B)$ and $(X/Z, B')$ are said to be *numerically equivalent*/Z if the intersection numbers $(C, K+B)$ and $(C, K+B')$ have the same signatures for all the curves C on X/Z, that is, for all the curves C on X that are contracted to a closed point s in Z.

Definition 3.2.1 (log model equivalences; $\sim_{wlc}$, $\sim_{lc}$). Let B and B' be two $\mathbb{R}$-boundaries on X/Z.

1. The boundaries B and B' are said to be *wlc model equivalent*/Z if both pairs $(X/Z, B)$ and $(X/Z, B')$ have wlc models and the following hold:

   (a) for a $\mathbb{Q}$-factorial model Y/Z of X/Z, $(Y/Z, B_Y^{log})$ is a wlc model of $(X/Z, B)$ if and only if $(Y/Z, B_Y'^{log})$ is a wlc model of $(X/Z, B')$;

   (b) for any model Y/Z in (a), wlc models $(Y/Z, B_Y^{log})$ and $(Y/Z, B_Y'^{log})$ are numerically equivalent over Z.

   If B and B' are wlc model equivalent, then we write $B \sim_{wlc} B'$. (cf. [Sho4, Definition 6.1].)

2. The boundaries B and B' are said to be *lc model equivalent*/Z if both pairs $(X/Z, B)$ and $(X/Z, B')$ have lc Iitaka models and the models are isomorphic. If B and B' are lc model equivalent, then we write $B \sim_{lc} B'$. (cf. [IsSh, Proposition 2.5].)

Let D be an $\mathbb{R}$-divisor on X. If there exists a birational 1-contraction $c : X \dashrightarrow Y/Z$ such that $D_Y = c_*D$ is nef on Y over Z, then we say that Y/Z is a *relative nef model of* X/Z *with respect to* D. Such a *projective* model Y/Z is called a D*-minimal* *model* if there are no dominant nef models over X/Z, that is, if Y'/Z is another relative nef model of X/Z and there is a birational morphism $Y' \to Y/Z$, then it is an isomorphism.

We also define D-model equivalent relation on $\mathbb{R}$-divisors D on X/Z.

Definition 3.2.2 (D-model equivalences; $\sim_{Dm}$, $\sim_{Dmb}$). Let D and D' be two $\mathbb{R}$-divisors on X/Z.

1. The divisors D and D' are said to be D-*minimal model equivalent over* Z if the following hold:

   (a) for a $\mathbb{Q}$-factorial model Y/Z, it is a relative D-minimal model of X/Z with respect to D if and only if it is a relative D-minimal model of X/Z with respect to D';

   (b) for all models Y/Z in (a), the intersection numbers (C, D_Y) and (C, D'_Y) have the same signatures for all the curves C on Y/Z.

   If D and D' are D-minimal model equivalent, then we write $D \sim_{Dm} D'$.

2. The divisors D and D' are said to be D-*mobile equivalent over* Z if both D and $D' \in \mathcal{E}$ (i.e., effective up to $\sim_{\mathbb{R}}$) and have $\mathbb{R}$-mobile decompositions$/Z$

   $$D = \mathrm{Mb}D + \mathrm{Fx}D \quad \text{and} \quad D' = \mathrm{Mb}D' + \mathrm{Fx}D'$$

   such that the relative 1-contractions determined by the b-semiample divisors $\mathrm{Mb}D$ and $\mathrm{Mb}D'$ give the same models, that is, the models are isomorphic. If D and D' are D-mobile equivalent, then we write $D \sim_{Dmb} D'$.

In $\mathfrak{B}_F := \oplus[0,1]D_i$ for a fixed reduced divisor $F = \sum D_i$ with distinct primes D_i on X/Z, a wlc model (a lc model) equivalence class is called a *country* and the decomposition in $\mathfrak{B}_F$, where it is defined, is called *geography of log models* or GLM for short (*geography of lc models* or GLCM for short, respectively). Similarly, in $\mathfrak{D}_F := \oplus \mathbb{R} D_i$, each D-minimal model (D-mobile) equivalence class is called a *D-country* and the decomposition of $\overline{\mathcal{E}} \cap \mathfrak{D}_F$ ($\mathcal{E} \cap \mathfrak{D}_F$, respectively), where it is defined, is called *geography of D-minimal models* or GDM for short (*D-mobile geography* or GDMb for short, respectively).

Remark. 1. Note that the model equivalence in the sense of 2.9 of [IsSh] is slightly weaker than our definition because it omits the numerical equivalence.

2. If the semiampleness conjecture [Sho4, Conjecture 2.6] holds, then the equivalences $\sim_{wlc}$ and $\sim_{lc}$ coincide because the Iitaka contraction is unique and, for all wlc models, it contracts exactly the curves C on Y/Z with $(C, K_Y + B_Y^{log}) = 0$. See [IsSh, Proposition 2.5].

3. It is expected by the semiampleness conjecture [Sho4, Conjecture 2.6] that we can consider wlc model equivalence for non $\mathbb{Q}$-factorial Y/Z.

The following is expected to hold for GLM. See [Sho4, Section 6] and [IsSh, Conjecture 2.10].

Conjecture 3.2.3 (Geography of log models : GLM). *Fix a reduced divisor $F = \sum D_i$ on a variety X/Z of dimension d. Let $\mathfrak{B}_F = \bigoplus [0,1] D_i$ and let $\mathcal{N}_F$ be a subset of divisors B in $\mathfrak{B}_F$ such that the numerical log Kodaira dimension $\nu(X/Z, B) \geq 0$. Then the following hold:*

(1) The set $\mathcal{N}_F$ is closed convex rational polyhedral.

(2) The set $\mathcal{N}_F$ is decomposed into a finite number of wlc model equivalence classes, countries, which are convex rational polyhedral. The countries are open in $\mathcal{N}_F$.

GLM in dimension d will mean Conjecture 3.2.3 in dimension d. We can also formulate the similar statements for GDM or GDMb with cones instead of polyhedra. However, GDM and GDMb hold (or are expected to hold) only for certain classes of varieties, such as FT varieties (see Definition 4.1.2). It is actually possible to define the log model equivalence relation and D-model equivalence relation on the whole set of divisor in $\mathfrak{B}_F$ and $\mathfrak{D}_F$, respectively and similar results as in Conjecture 3.2.3 are also expected to hold everywhere in $\mathfrak{B}_F$ and $\mathfrak{D}_F$. See [IsSh, Conjecture 2.10]. This part of geography will be studied elsewhere.

Main Theorem 1. *LMMP in dimension d implies GLM in dimension d. More precisely, we can relax LMMP to the following statement: for any slt pair $(X/Z, B)$, there exists a terminated sequence of extremal log flips, that is, the last pair in the sequence is either a slm model or a Mori log fibration.*

Remark. To run LMMP on an arbitrary pair $(X/Z, B)$, (if necessary) we first take a log birational model $(Y/Z, B_Y^{log})$ of $(X/Z, B)$, which is lc. Thus if there is a divisor on Y which is exceptional on X, that is, $B_Y^{log} \neq B_Y$, then theorem 3.1.1 is not applied in this situation. Therefore we cannot remove the LMMP assumption completely.

Proof of the Main Theorem 1.

Convexity of $\mathcal{N}_F$: Note that for an lc pair (X, B) with an $\mathbb{R}$-boundary B, the pseudo-effectivity of $K + B$ and the condition $\nu(X, B) \geq 0$ are equivalent. Let B and B' be $\mathbb{R}$-divisors in $\mathcal{N}_F$. Then for an extraction $f : Y \to X$ of X such that both $(Y/Z, B_Y^{log})$ and $(Y/Z, B_Y'^{log})$ are lc, the divisors $K_Y + B_Y^{log}$ and $K_Y + B_Y'^{log}$ are pseudo-effective. We can take a log resolution $g : Y \to X$ of $(X, \mathrm{Supp}(B + B'))$ as a model Y/Z. Moreover, we can suppose that Y/Z is projective. Then for any divisor

$B'' \in (B, B')$, the pair $(Y, B_Y''^{log})$ is lc and $K_Y + B_Y''^{log}$ is pseudo-effective since the lc and pseudo-effective properties are convex. On the other hand, $(Y/Z, B_Y''^{log})$ is slt and has a resulting model which is a slm model. Thus B'' is in $\mathcal{N}_F$.

Closedness of $\mathcal{N}_F$: Let $f : Y \to X$ be an extraction of X such that (Y, F_Y^{log}) is lc. Note that by construction, the pair (Y, B_Y^{log}) is lc for any $B \in \mathfrak{B}_F$. Then since the lc and pseudo-effective properties are closed, the set $\mathcal{N}_{F_Y^{log}}$ is closed. There is an injection $\varphi : \mathfrak{B}_F \hookrightarrow \mathfrak{B}_{F_Y^{log}}$ which maps $D = \sum d_i F_i \mapsto D_Y^{log} = \sum d_i (F_i)_Y + \sum 1E_j$ where E_j are f-exceptional prime divisors. The intersection of two closed sets $\mathcal{N}_{F_Y^{log}} \cap (\mathfrak{B}_F \oplus 1E_j)$ is also closed and it is isomorphic to $\mathcal{N}_F$ via φ.

Rationality of $\mathcal{N}_F$ *and finite decomposition of* $\mathcal{N}_F$: Since $\mathcal{N}_F (\subseteq \mathfrak{B}_F)$ is compact, it is enough to prove this locally. This follows from Proposition 3.2.5.

Rationality of countries and local Finiteness: Immediate by proposition 3.2.5 and the compactness of $\mathcal{N}_F$.

Convexity of a country: Immediate by lemma 2 of [Sho6]. □

A subset $\mathcal{C}$ in a finite dimensional $\mathbb{R}$-vector space $V = \mathbb{R}^n$ is said to be *locally conical at* a point $O \in \mathbb{R}^n$ if there exists a real number $\varepsilon > 0$ such that $\mathcal{C}$ is conical with the vertex O in the ε-neighborhood $U_\varepsilon(O) = \{O' \in \mathbb{R}^n \mid \|O' - O\| < \varepsilon\}$. In other words, if $O' \in \mathcal{C}$ and $0 < \|O - O'\| < \varepsilon$, then $(O, O + \varepsilon(O' - O)/\|O' - O\|) \subseteq \mathcal{C}$. A set $\mathcal{C}$ is said to be *locally conical* if it is locally conical at each point in $\mathbb{R}^n$. A decomposition of a set $\mathcal{N}$ in a finite dimensional $\mathbb{R}$-vector space $\mathbb{R}^n$ into disjoint subsets $\{\mathcal{C}_i\}$ is said to be locally conical at a point $O \in \mathbb{R}^n$ if each subset $\mathcal{C}_i$ is locally conical at O. Such decomposition is said to be locally conical if each $\mathcal{C}_i$ is locally conical at each point in $\mathbb{R}^n$. As we will see shortly, GLM is a locally conical decomposition of the set $\mathcal{N}_F$.

Proposition 3.2.4. *Under the assumptions of the Main Theorem 1, GLM is locally conical in dimension d.*

Proof. Immediate by the stability of wlc models, [Sho6, Corollary 11]. Indeed, LMMP allows us to replace the dlt property by lc property [Sho3, Conjecture]. □

Proposition 3.2.5. *Geography is locally finite and rational polyhedral.*

Proof. Note that the locally finite decomposition on each hyperplane section of $\mathcal{N}_F$ implies the locally finite decomposition of $\mathcal{N}_F$. Therefore by induction on the dimension of the hyperplane sections, it is enough to prove that a straight line which intersects with $\mathcal{N}_F$ has a locally finite decomposition. However, if there exists an open segment intersecting infinitely many countries, then the open segment contains an accumulation point of the borders of the countries, contradicting the property of locally conical property of GLM, Proposition 3.2.4.

To verify that the polyhedron $\overline{\mathcal{C}}$ is rational, we use rationality of the lc property at a point and of the nef property for intersections with a curve. □

See section 1.2 for the definitions of extremal curves and log discrepancies $a(E, X, B)$.

Main Theorem 2. *Under the assumptions of the Main Theorem 1, each country in GLM is an open convex rational polyhedron in $\mathfrak{B}_F$.*

More precisely, if $\mathcal{C}$ is a country in $\mathcal{N}_F$ containing a fixed divisor B such that the pair $(X/Z, B)$ has a wlc model $(Y/Z, B_Y^{log})$, then there exist finite sets of

- *prime components F_s, F'_t, F''_u of F,*

- *prime divisors E_k, E'_l, exceptional on Y, and*

- *extremal curves* C_i, C'_j *on* Y/Z

such that the country $\mathcal{C} = \{D \in \mathfrak{B}_F \mid D \sim_{wlc} B\}$ *of* B *is a subset of* $\mathfrak{B}_F$ *determined by the following equations and strict inequalities:*

$$(*) \begin{cases} a(F_s, Y, D_Y^{log}) = 0 \textit{ or } = 1; \\ a(F'_t, Y, D_Y^{log}) > 0 \textit{ or } < 1; \\ a(F''_u, Y, D_Y^{log}) > 1 - d_u & \textit{if } F''_u \textit{ is non-exceptional on } X \textit{ and } F''_u \subseteq SuppF \\ & (d_u := mult_{F''_u} D); \\ a(E_k, Y, D_Y^{log}) > 1 & \textit{if } E_k \textit{ is non-exceptional on } X \textit{ and } E_k \not\subseteq SuppF; \\ a(E'_l, Y, D_Y^{log}) > 0 & \textit{if } E'_l \textit{ is exceptional on } X; \\ (C_i, K_Y + D_Y^{log}) = 0; \textit{ and} \\ (C'_j, K_Y + D_Y^{log}) > 0. \end{cases}$$

In particular, the country $\mathcal{C}$ *is open in* $\mathfrak{B}_F$.

Remark. 1. Using the notion of b-divisor $\mathcal{D}^{log}$ of D where

$$mult_E \mathcal{D}^{log} = \begin{cases} mult_E D & \text{if } E \text{ is non-exceptional on } X \text{ and } E \subseteq SuppF; \\ 0 & \text{if } E \text{ is non-exceptional on } X \text{ and } E \not\subseteq SuppF; \\ 1 & \text{if } E \text{ is exceptional on } X, \end{cases}$$

we can unite three of the inequalities above into one as follows:

$$\begin{cases} a(F''_u, Y, D_Y^{log}) > 1 - d_u & \text{if } F''_u \text{ is non-exceptional on } X \text{ and } F''_u \subseteq SuppF \\ & (d_u := mult_{F''_u} D); \\ a(E_k, Y, D_Y^{log}) > 1 & \text{if } E_k \text{ is non-exceptional on } X \text{ and } E_k \not\subseteq SuppF; \\ a(E'_l, Y, D_Y^{log}) > 0 & \text{if } E'_l \text{ is exceptional on } X; \end{cases}$$

$$\Rightarrow \quad a(E_m, Y, D_Y^{log}) > 1 - mult_{E_m} \mathcal{D}^{log},$$

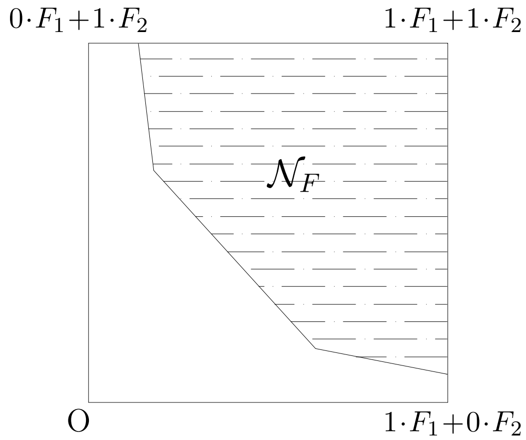
where E_m is exceptional on Y.

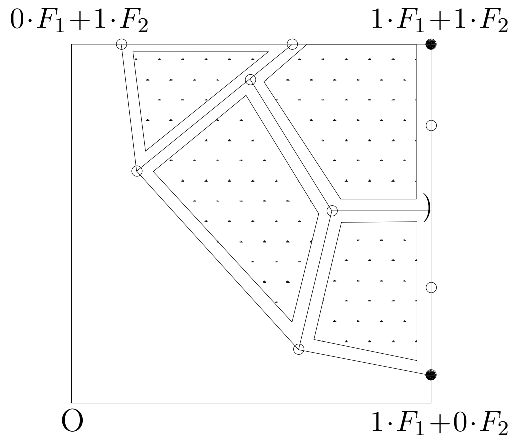
2. Given a wlc model $(Y/Z, D_Y^{log})$, note that the intersection number $(C, K_Y + D_Y^{log})$ for a fixed curve C on Y/Z, the log discrepancy $a(E, Y, D_Y^{log})$ and $mult_{F''_u} D (= d_u)$ for fixed divisors E and F''_u are linear functions with respect to D. Therefore, the equations and inequalities in $(*)$ indeed define a polyhedron in $\mathfrak{B}_F$.

3. It is possible that two equations, for example, $a(F_s, Y, D_Y^{log}) = 0$ (or $= 1$) and $(C_i, K_Y + D_Y^{log}) = 0$ in $(*)$ of the Main Theorem 2 define the same planes. Similarly, the distinct strict inequalities in $(*)$ may define the same open half spaces. However, whenever possible, we describe the planes in terms of prime components F_s rather than curves C_i, and half spaces in terms of prime components F'_t rather than divisors E'_l or curves C'_j. Therefore, the equations $a(F_s, Y, D_Y^{log}) = 0$ (or $= 1$) determine the minimal face $\mathcal{F}$ of $\mathfrak{B}_F$ containing the country $\mathcal{C}$ and the equations $a(F'_t, Y, D_Y^{log}) = 0$ (or $= 1$) determine the faces of $\mathfrak{B}_F$ containing the (proper) maximal faces of $\mathcal{C}$.

The proof of the Main Theorem 2 is given in section 3.5. Before giving the proof, we first study the properties of wlc model equivalent relation $\sim_{wlc}$ and countries, which will be used in the course of the proof. We also give simple examples of geography in section 3.4.

The following figures describe a possible geography of log models when $F = F_1 + F_2$ and $(X/Z, F_1 + F_2)$ is lc.





- Each $\circ$ forms a country.
- Each open segment strictly between two $\circ$'s or a $\circ$ and) forms a country.
- Each half-open segment between $\circ$ and $\bullet$, where $\bullet$ is included and $\circ$ is not, forms a country, unless the segment intersects with some other country.
- Each dotted area forms a country.

The following follows from the easy observation of GLM.

Remark. Let $\mathcal{G}$ be a face of $\mathcal{N}_F$, which does not lie in the boundary of $\mathfrak{B}_F$. Then $\mathcal{G}$ is defined by the planes $(C_i, K_Y + D_Y^{log}) = 0$ for some Y birational to X/Z and for some curve C_i on Y/Z.

Proof. By main theorem 2, the countries in $\mathcal{N}_F$ are open convex polyhedra in $\mathfrak{B}_F$. This implies that if $\mathcal{C} \cap \mathcal{G} \neq \varnothing$ for a country $\mathcal{C}$, then $\mathcal{C} \subseteq \mathcal{G}$. Therefore, there exists a finite set of countries $\{\mathcal{C}_p\}$ such that $\mathcal{G} = \bigcup \mathcal{C}_p$. By the finiteness of countries, there exists a country $\mathcal{C}_p$ having the same dimension as the dimension of $\mathcal{G}$. That is, the linear spans of $\mathcal{G}$ and $\mathcal{C}_p$ are the same. Let $B \in int\mathcal{C}_p$ and let $(Y/Z, B_Y^{log})$ be a wlc model of $(X/Z, B)$. Then the equations $(C_i, K_Y + D_Y^{log}) = 0$ for the country $\mathcal{C}_p$ define the face $\mathcal{G}$. □

3.3 Properties of countries.

In this section, we study some properties of a country $\mathcal{C}$, along with the wlc mode equivalence relation $\sim_{wlc}$.

Lemma 3.3.1. *Let $\mathcal{C}$ be a country in $\mathfrak{B}_F$. Let $B \in \mathcal{C}$ and $D \in \overline{\mathcal{C}}$. If $(Y/Z, B_Y^{log})$ is a wlc model of $(X/Z, B)$, then $(Y/Z, D_Y^{log})$ is a wlc model of $(X/Z, D)$.*

Proof. We first prove that $(Y/Z, D_Y^{log})$ is a wlc model for any $D \in \overline{\mathcal{C}}$. For any divisor $D \in \mathcal{C}$, $(Y/Z, D_Y^{log})$ is a wlc model of $(X/Z, D)$ by definition of $\sim_{wlc}$. Thus, for any curve C on Y/Z, the nef property $(C, K_Y + D_Y^{log}) \geq 0$ holds and, for any b-divisor E of X, the inequality $a(E, Y, D_Y^{log}) \geq 0$ holds. By continuity of the function $(C, K_Y + D_Y^{log})$ with respect to D, we obtain the nef property of $K_Y + D_Y^{log}$ for any $D \in \overline{\mathcal{C}}$. Similarly, by continuity of $a(E, Y, D_Y^{log})$ with respect to D, the log canonicity $a(E, Y, D_Y^{log}) \geq 0$ holds for any $D \in \overline{\mathcal{C}}$.

We now prove that $(Y/Z, D_Y^{log})$ is a wlc model of $(X/Z, D)$ for any $D \in \overline{\mathcal{C}}$. By definition of $\sim_{wlc}$, the inequality $0 \leq a(E, Y, D_Y^{log}) - a(E, X, D)$ holds for any $D \in \mathcal{C}$ and for any prime non-exceptional divisor E on X, which is exceptional on Y. Again, by continuity of the functions $a(E, Y, D_Y^{log})$ and $a(E, X, D)$ with respect to D, the same inequality $0 \leq a(E, Y, D_Y^{log}) - a(E, X, D)$ holds for $D \in \overline{\mathcal{C}}$. □

Lemma 3.3.2. *Let B be an $\mathbb{R}$-boundary in $\mathfrak{B}_F$ and E be a prime divisor on X such that $E \subseteq SuppF$. Suppose that $(X/Z, B)$ has a wlc model $(Y/Z, B_Y^{log})$ and E is exceptional on Y. Then $B \sim_{wlc} D$ where $B = D$ on $SuppF \setminus E$ and $max\{0, 1 - a(E, Y, B_Y^{log})\} \leq mult_E D \leq 1$.*

Proof. Let D be as in the lemma. Since E is exceptional on Y, we have $D_Y^{log} = B_Y^{log}$. Therefore, $(Y/Z, D_Y^{log})$ is a wlc model. Furthermore, two wlc models $(Y/Z, D_Y^{log})$ and $(Y/Z, B_Y^{log})$ are numerically equivalent.

Let E' be a non-exceptional prime divisor on X which is exceptional on Y. Let $M := max\{0, 1 - a(E, Y, B_Y^{log})\}$. If $E' = E$, then

$$a(E', X, D) = 1 - mult_E D \leq 1 - M \leq a(E, Y, B_Y^{log}) = a(E', Y, D_Y^{log}).$$

If $E' \neq E$, then $a(E', X, D) = 1 - mult_{E'} D = 1 - mult_{E'} B = a(E', X, B)$ and $a(E', Y, D_Y^{log}) = a(E', Y, B_Y^{log})$. Since $(Y/Z, B_Y^{log})$ is a wlc model of $(X/Z, B)$, the inequality $a(E', X, D) \leq a(E', Y, D_Y^{log})$ holds.

Therefore, $(Y/Z, D_Y^{log})$ is a wlc model of $(X/Z, D)$ which is numerically equivalent to $(Y/Z, B_Y^{log})$ and $B \sim_{wlc} D$. □

Two log pairs (X, B) and (X', B') are said to be *crepant* to each other if X' is a birational modification of X and $\overline{K_X + B} = \overline{K_{X'} + B'}$. If the context is understood, by writing $(X', B_{X'}^{crep})$, we mean that $(X', B_{X'}^{crep})$ is crepant to some given log pair, usually some wlc model $(Y/Z, B_Y^{log})$.

Proposition 3.3.3. *Let B be an $\mathbb{R}$-boundary divisor on X and $(Y/Z, B_Y^{log})$ a wlc model of $(X/Z, B)$. Suppose that $(Y'/Z, B_{Y'}^{crep})$ is crepant to $(Y/Z, B_Y^{log})$, where $Y \dashrightarrow Y'/Z$ is a birational modification of Y to Y' over Z. Then $B_{Y'}^{crep} = B_{Y'}^{log}$ if and only if the log pair $(Y'/Z, B_{Y'}^{crep})$ is a wlc model of $(X/Z, B)$.*

Proof. Suppose that $B_{Y'}^{crep} = B_{Y'}^{log}$. First, the log pair $(Y', B_{Y'}^{crep}) = (Y', B_{Y'}^{log})$ is lc because $(Y', B_{Y'}^{crep})$ and (Y, B_Y^{log}) have the same log discrepancies, and (Y, B_Y^{log}) is lc. The same argument shows that $K_{Y'} + B_{Y'}^{log}$ is nef/Z. Thus $(Y'/Z, B_{Y'}^{log})$ is a wlc model. Moreover, $(Y'/Z, B_{Y'}^{log})$ is a wlc model of $(X/Z, B)$. According to the above, the required inequality holds for all divisors E which are non-exceptional on X and exceptional on Y and Y'. Suppose now that E is non-exceptional on X and Y, but exceptional on Y'. Then

$$a(E, X, B) = a(E, Y, B_Y^{log}) = a(E, Y', B_{Y'}^{log}).$$

Suppose now that $(Y'/Z, B_{Y'}^{crep})$ is a wlc model of $(X/Z, B)$. Then $B_{Y'}^{log} = B_{Y'}^{crep}$ by definition. □

Remark. By [Sho4, proposition 2.4], it is well known that all the wlc models $(Y/Z, B_Y^{log})$ of $(X/Z, B)$ are crepant to one another.

Proposition 3.3.4. *Let (X, B) be a log pair and $\chi : X \dashrightarrow X'$ be a birational modification of X to a variety X' which is $\mathbb{Q}$-factorial. Suppose that $\overline{K+B} \equiv 0$ over X'. Then there exists a divisor $B_{X'}^{crep}$ on X', that is, the log pair $(X', B_{X'}^{crep})$ is crepant to (X, B).*

Proof. Let B' be a divisor such that $\left(\overline{K+B}\right)_{X'} = K_{X'} + B'$, where $K_{X'} = \mathbb{K}_{X'}$ for the b-divisor $\mathbb{K}$. Since $K_{X'} + B'$ is $\mathbb{R}$-Cartier, by the assumption $\overline{K+B} \equiv 0/X'$, we obtain $\overline{K_{X'} + B'} = \overline{K+B}$. Therefore, $B' = B_{X'}^{crep}$. □

Corollary 3.3.5. *Let (X, B) be a log pair and $\chi : X' \to X$ be a blow up of X. Then there exists a divisor $B_{X'}^{crep}$ on X'.*

Proof. Trivially, the crepant pull back

$$K_{X'} + \chi_*^{-1} B - E = \chi^*(K + B) = \left(\overline{K+B}\right)_{X'}$$

where E is a χ-exceptional divisor, is numerically trivial over X and $\overline{K+B} \equiv 0$ over X'. Therefore, by proposition 3.3.4, there exists $B_{X'}^{crep}$ and, in fact, $B_{X'}^{crep} = \chi_*^{-1}B - E$. □

Proposition 3.3.6. *Let $(Y/Z, B_Y^{log})$ be a wlc model of $(X/Z, B)$ and let $Y \dashrightarrow Y'$ be a birational modification of Y/Z. Suppose that there exists a divisor $B_{Y'}^{crep}$ on Y' such that $(Y', B_{Y'}^{log})$ is crepant to (Y, B_Y^{crep}) and $mult_E B_{Y'}^{log} \neq mult_E B_{Y'}^{crep}$ for some prime divisor E on Y'. Then, for any $B' \sim_{wlc} B$, there exists a divisor $B_{Y'}^{\prime crep}$ on Y' such that $(Y', B_{Y'}^{\prime log})$ is crepant to $(Y, B_Y^{\prime crep})$ and*

$$mult_E B_{Y'}^{\prime log} \neq mult_E B_{Y'}^{\prime crep}.$$

Proof. Since $(Y/Z, B_Y^{\prime log})$ is a wlc model of $(X/Z, B')$ and is numerically equivalent to $(Y/Z, B_Y^{log})$, we see that $\overline{K_Y + B_Y^{\prime log}} \equiv 0$ over Y'. Therefore, by proposition 3.3.4, there exists a crepant divisor $B_{Y'}^{\prime crep}$ on Y'.

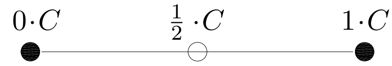
Suppose that $mult_E B_{Y'}^{\prime log} = mult_E B_{Y'}^{\prime crep}$. For a log resolution $f : W \to Y$ of $(Y, B_Y^{\prime log})$ with E non-exceptional on W, let $(Y''/Z, B_{Y''}^{\prime log})$ be a strictly log minimal model of $(W/Z, B_W^{\prime log})$. Then $(Y''/Z, B_{Y''}^{\prime log})$ is a wlc model of $(X/Z, B')$ and E is non-exceptional on Y''.

Now since $B \sim_{wlc} B'$, the log pair $(Y''/Z, B_{Y''}^{log})$ is a wlc model of $(X/Z, B)$. Since wlc models of $(X/Z, B)$ are crepant to each other, we have $B_{Y''}^{log} = B_{Y''}^{crep}$. In particular, $mult_E B_{Y''}^{log} = mult_E B_{Y''}^{crep}$, a contradiction. Indeed, log discrepancies are independent on crepant models, and therefore, $a(E, Y', B_{Y'}^{log}) = a(E, Y', B_{Y'}^{crep})$. □

3.4 Examples.

The following are simple examples.

Example 3.4.1. Let $(X/Z, C)$ be a log minimal surface with a (-2)-curve C on X/Z contracted by the birational $f : X \to Z$. Suppose that the log discrepancy $a(C, Z, 0) = \frac{1}{2}$. Then the geography of X/Z in the closed segment $\mathfrak{B}_C = [0, 1]C$ is as follows:

$\mathfrak{B}_C =$ 

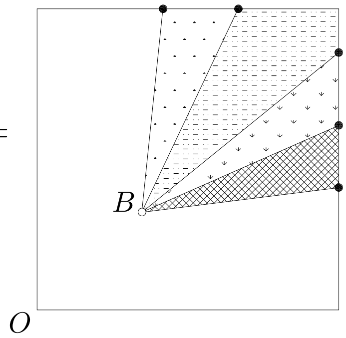
There are three countries in $\mathfrak{B}_C$; $\mathcal{C}_1 = [0, \frac{1}{2})C$, $\mathcal{C}_2 = \frac{1}{2}C$, and $\mathcal{C}_3 = (\frac{1}{2}, 1]C$. Note that

$$\text{if } t \in [0, \tfrac{1}{2}], \quad \text{then } (X/Z, tC) \text{ is a wlc model of } (X/Z, tC), \text{ and}$$
$$\text{if } t \in [\tfrac{1}{2}, 1], \quad \text{then } (Z/Z, 0) \text{ is a wlc model of } (X/Z, tC).$$

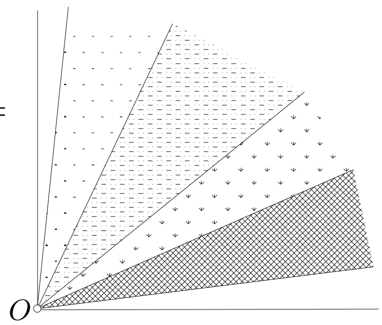
Note that as in lemma 3.3.1, the log pair $(X/Z, \frac{1}{2}C)$ has two wlc models $(X/Z, \frac{1}{2}C)$ and $(Z/Z, 0)$.

A variety X/Z is called an *FT variery* if there exists an $\mathbb{R}$-boundary B such that the log pair $(X/Z, B)$ is a log Fano variety. See Definition 4.1.2.

Example 3.4.2. Let X/Z be an FT variety. Then, by lemma 4.1.3, there exists an $\mathbb{R}$-boundary B such that $(X/Z, B)$ is a 0-log pair. Let $F = SuppB$ and consider the geography of X/Z in $\mathfrak{B}_F$. Note that for a fixed divisor $D \in \mathfrak{B}_F$, LMMP on $(X/Z, B + \varepsilon D)$ for $\varepsilon > 0$ such that $B + \varepsilon D \in \mathfrak{B}_F$ is the same as LMMP on $(X/Z, B + \varepsilon' D)$ for ε' having the same property as ε. Therefore, the geography of log models of X/Z in $\mathfrak{B}_F$ is conical at B as shown below:

$\mathfrak{B}_F =$ 

Since $D \sim_{\mathbb{R}} K + B + \varepsilon D$, D-MMP is the same as LMMP on $(X/Z, B + \varepsilon D)$ for any $\varepsilon > 0$. Therefore, the geography of D-minimal models of X/Z in $\mathfrak{D}_F$ is conical at the origin as shown below;

$\mathfrak{D}_F =$ 

For more examples, see 6.20.2 of [Sho4] and Example 2.11 of [IsSh].

3.5 Proof of Main Theorem 2.

We first state lemmas about convex polyhedrons.

Lemma 3.5.1. *Let $\mathcal{C}$ be a closed convex set in a finite dimensional $\mathbb{R}$-vector space and let dim$\mathcal{C} = n$. Then $\mathcal{C}$ is closed polyhedral if and only if all of its j-dimensional plane sections are closed polyhedral where $2 \le j \le n - 1$.*

Proof. See [Kl]. □

Lemma 3.5.2. *Let $\mathcal{C}$ be a bounded convex subset in a finite dimensional $\mathbb{R}$-space V. Then $\overline{\mathcal{C}}$ is locally conical if and only if $\overline{\mathcal{C}}$ is polyhedral.*

Proof. Suppose that $\overline{\mathcal{C}}$ is locally conical. We use the induction on $dim\overline{\mathcal{C}} = n$. The case where $\dim \overline{\mathcal{C}} = 1$ is trivial. Suppose that $\overline{\mathcal{C}}$ is polyhedral for $dim\overline{\mathcal{C}} < n$. Since the property of being locally conical is preserved under taking sections, all the lower dimensional sections of $\overline{\mathcal{C}}$ are polyhedral by inductive assumption. Therefore, $\overline{\mathcal{C}}$ is polyhedral by lemma 3.5.1.

The converse is clear. □

We now begin the proof of Main Theorem 2.

Proof of Main Theorem 2. Fix a divisor B and let $\mathcal{C}$ be a country in $\mathcal{N}_F$ containing B. By lemma 2 of [Sho6], $\mathcal{C}$ is convex. Furthermore, by proposition 3.2.4, $\mathcal{C}$ is locally conical and by lemma 3.5.2, $\overline{\mathcal{C}}$ is a closed convex polyhedron. Note that the locally conical property and the convex property are preserved under taking the closure. The rationality of $\mathcal{C}$ follows from Proposition 3.2.5.

To verify that $\mathcal{C}$ is an open polyhedron in $\mathfrak{B}_F$, we need to find some prime components F_s, F'_t, F''_u of F, prime exceptional divisors E_k, E'_l on some model Y/Z of X/Z, and curves C_i, C'_j on Y/Z, as in Main Theorem 2, such that the set $\mathcal{C}^*$ in $\mathfrak{B}_F$ defined by the conditions in $(*)$ is equal to the polyhedral $\mathcal{C}$, that is, $\mathcal{C} = \mathcal{C}^*$.

We start the proof with the following verification.

Step 1-1. $int\overline{\mathcal{C}} \subseteq \mathcal{C}$.

Proof. If $dim\overline{\mathcal{C}} = 0$, then the statement is clear. Therefore suppose $d = dim\overline{\mathcal{C}} \geq 1$. Let $B \in int\overline{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is a closed convex polyhedron, for a finite set of points $B_i \in \overline{\mathcal{C}}$,

B is an internal point of the convex hull of points B_i and the hull has dimension d. The same holds for any small perturbation of B_i, that is, for $B'_i \in \overline{\mathcal{C}}$ such that $0 < \|B'_i - B_i\| \ll 1$, B is in the convex hull of the divisors B'_i and the hull has dimension d. Moreover, since $B_i \in \overline{\mathcal{C}}$ we can assume that $B'_i \in \mathcal{C}$. Hence $B \in \mathcal{C}$ by the convexity of $\mathcal{C}$. □

We always have $int\overline{\mathcal{C}} \neq \varnothing$ by definition, even when the set $\mathcal{C}$ is 0-dimensional. Therefore we can always pick a divisor $B \in int\overline{\mathcal{C}}$ and we will always assume this throughout the proof, unless otherwise stated. Then by Step 1-1, $B \in \mathcal{C}$ and the pair $(X/Z, B)$ has a wlc model $(Y/Z, B_Y^{log})$. In particular, Y is $\mathbb{Q}$-factorial. Moreover, by assumptions in the theorem, we may assume that $(Y/Z, B_Y^{log})$ is a slm model of $(X/Z, B)$.

Suppose that we have some prime components F_s, F'_t, F''_u of F, prime divisors E_k, E'_l and curves C_i, C'_j on Y/Z which determine a subset $\mathcal{C}^*$ of $\mathfrak{B}_F$ by the conditions $(*)$ of Main Theorem 2. Then the following inclusion holds.

Step 1-2. $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*}$

Proof. Suppose that we have found some prime components F_s, F'_t, F''_u of F, prime divisors E_k, E'_l exceptional on Y and curves C_i, C'_j on Y/Z, as in Main Theorem 2, such that the conditions in $(*)$ hold for $D = B$. Then the closure $\overline{\mathcal{C}^*}$ is a subset of $\mathfrak{B}_F$ defined by the following conditions:

$$\overline{(*)}\begin{cases} a(F_s, Y, D_Y^{log}) = 0 \text{ or } = 1; \\ a(F'_t, Y, D_Y^{log}) \geq 0 \text{ or } \leq 1; \\ a(F''_u, Y, D_Y^{log}) \geq 1 - d_u & \text{if } F''_u \text{ is non-exceptional on } X \text{ and } F''_u \subseteq SuppF \\ & \quad (d_u := mult_{F''_u} D); \\ a(E_k, Y, D_Y^{log}) \geq 1 & \text{if } E_k \text{ is non-exceptional on } X \text{ and } E_k \nsubseteq SuppF; \\ a(E'_l, Y, D_Y^{log}) \geq 0 & \text{if } E'_l \text{ is exceptional on } X; \\ (C_i, K_Y + D_Y^{log}) = 0; \text{ and} \\ (C'_j, K_Y + D_Y^{log}) \geq 0. \end{cases}$$

It is enough to prove that $int\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*}$. Let B' be a boundary in $\mathfrak{B}_F$ such that $B' \in int\overline{\mathcal{C}}$. Then by Step 1-1, $B' \in \mathcal{C}$, and thus $B' \sim_{wlc} B$. We are to prove that the above equations and inequalities in $\overline{(*)}$ hold for $D = B'$.

Note that any small open neighborhood of $B \in int\overline{\mathcal{C}}$ generate the same linear span as $int\overline{\mathcal{C}}$ does. Therefore for the equations of F_s, it is enough to prove that there exists an open set $U(B) \subseteq int\overline{\mathcal{C}}$ of B such that any divisor $B' \in U(B)$ satisfies $a(F_s, Y, B_Y'^{log}) = 0$ (or $= 1$) since the function $a(F_s, Y, D_Y^{log})$ is linear with respect to D. Suppose that there do not exist such open sets. Then any small open neighborhood $U' \subseteq int\overline{\mathcal{C}}$ of B contains a divisor B'' such that $a(F_s, Y, B_Y''^{log}) \neq 0$ (or $\neq 1$). Since U' is open and $a(F_s, Y, D_Y^{log})$ is a linear function with respect to D, we can assume that $a(F_s, Y, B_Y''^{log}) < 0$ (resp > 1). This implies that $B'' \notin \mathfrak{B}_F$, a contradiction. By construction and definition of $\mathcal{C}$ and $\mathcal{C}^*$, it is immediate for the inequalities $a(F'_t, Y, D_Y^{log}) \geq 0$ or ≤ 1.

Since $B' \sim_{wlc} B$, the log pair $(Y/Z, B_Y'^{log})$ is a wlc model of $(X/Z, B')$. Thus, by definition, if F''_u is non-exceptional on X and $F''_u \subseteq SuppF$, then

$$1 - mult_{F''_u} B' = 1 - d'_u \leq a(F''_u, Y, B_Y'^{log})$$

holds. If E_k is non-exceptional on X with $E'_l \nsubseteq SuppF$ and E_l is exceptional on X, then

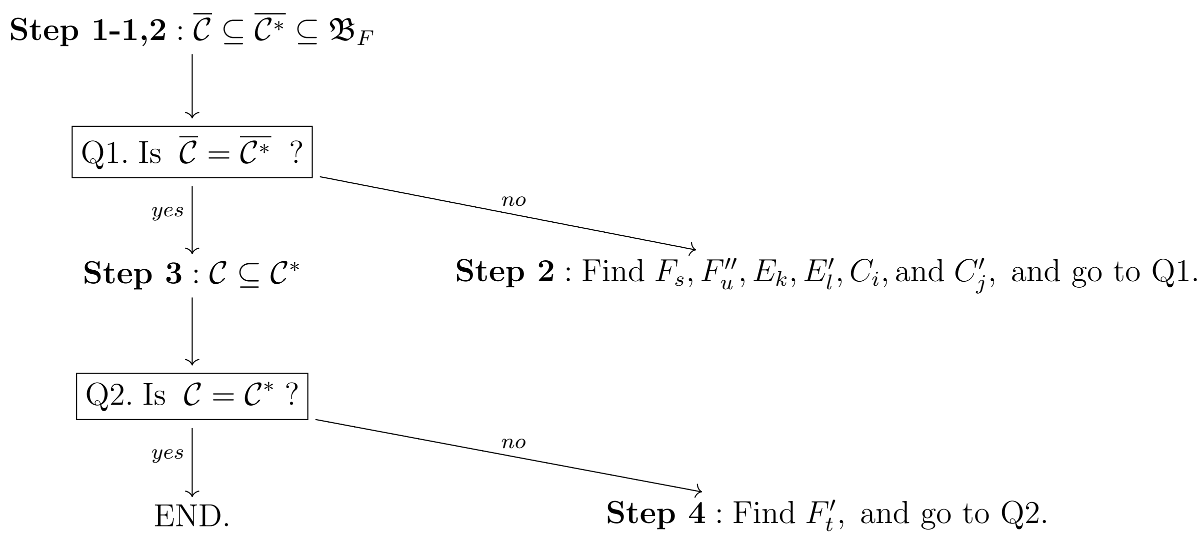
$$a(E_k, Y, B_Y'^{log}) \geq 1 \quad \text{and} \quad a(E'_l, Y, B_Y'^{log}) \geq 0$$

hold, respectively, by the definition of wlc models.

Since the wlc models $(Y/Z, B_Y^{log})$ and $(Y/Z, B_Y'^{log})$ are numerically equivalent, $(C_i, K_Y + B_Y'^{log}) = 0$ holds. Since the divisor $K_Y + B_Y'^{log}$ is nef $/Z$, the last inequality $(C'_j, K_Y + B_Y'^{log}) \geq 0$ also holds. □

To find all the divisors $F_s, F'_t, F''_u, E_k, E'_l$ and the curves C_i, C'_j as in Main Theorem 2, which determine precisely the country $\mathcal{C} = \mathcal{C}^*$ by $(*)$, we proceed as follows:

Plan of Proof

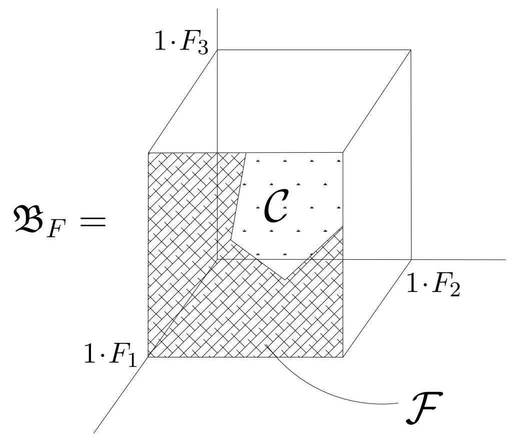


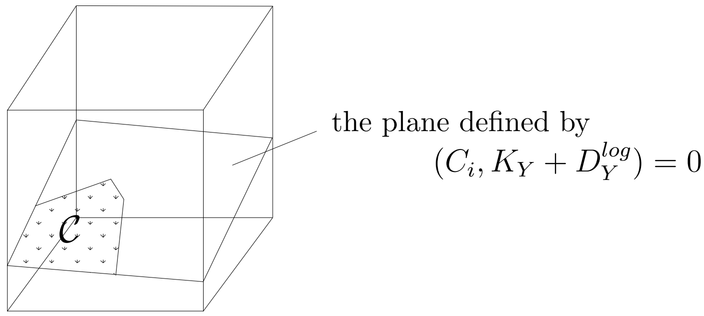
Basic strategy is to find the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ and, then eliminate the faces of $\overline{\mathcal{C}}$ which do not belong to the country $\mathcal{C}$.

By Step 1-2 above, we have the inclusion $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*} \subseteq \mathfrak{B}_F$. First, we deal with the case where the respond to Q1 is *no*.

Warning. Even though we may have some divisors F'_t at hands, we assume that the set of F'_t is empty now. We do not try to find any F'_t until we arrive at Step 4. Indeed, we do not need any F'_t to have $\overline{\mathcal{C}} = \overline{\mathcal{C}^*}$.

Step 2-1. Since Supp$F$ is finite, we may assume that the set of prime components F_s is maximal. Thus, the equations $a(F_s, Y, D_Y^{log}) = 0$ or $= 1$ determine the minimal face $\mathcal{F}$ of $\mathfrak{B}_F$ which contains the country $\mathcal{C}$. Note that $B \in int\overline{\mathcal{C}}$ is also an internal point of the face $\mathcal{F}$ because the set of F_s is maximal. Since the equations $(C_i, K_Y + D_Y^{log}) = 0$ also determine the linear span which contains $\mathcal{C}$, we start working in the intersection of the face $\mathcal{F}$ with the linear span determined by the equations of C_i. See the diagram below.



$\mathcal{F} =$ 

Since the respond to Q1 is *no*, the set $\overline{\mathcal{C}^*} \setminus \overline{\mathcal{C}}$ is nonempty.

Step 2-2. Take a divisor $B' \in \overline{\mathcal{C}^*} \setminus \overline{\mathcal{C}}$.

Case 1. Suppose that B' is not in the linear span of $\overline{\mathcal{C}}$.

Recall that $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*}$ by Step 1-2 and $\overline{\mathcal{C}^*} \subseteq \mathfrak{B}_F$ by construction. Since $B \in int\overline{\mathcal{C}}$ is also an internal point of the face $\mathcal{F}$ of $\mathfrak{B}_F$ defined above, any divisor in $\overline{\mathcal{C}^*}$ sufficiently close to B is a boundary and has support in Supp$B$. Thus by the stability of lt singularities [Sho2, 1.3.4], there exists $\varepsilon' > 0$ such that $\|B - D\| < \varepsilon'$ with $D \in \overline{\mathcal{C}^*} \subseteq \mathcal{F}$ implies that D is a boundary having support in Supp$B$ and (Y, D_Y^{log}) is slt. Without loss of generality, we may assume that $0 < \|B - B'\| < \varepsilon'$.

Since $B \in int\overline{\mathcal{C}}$, for any prime divisor E which is non-exceptional on X and exceptional on Y, the following strict inequalities hold:

$$(\#) \qquad a(E, Y, B_Y^{log}) > \begin{cases} 1 - mult_{E_u} B = 1 - d_E & \text{if } E = F_u'' \subseteq SuppF, \\ 1 & \text{if } E = E_k \nsubseteq SuppF. \end{cases}$$

Since D has support in $SuppB$, by letting $0 < \|D - B\| < \varepsilon'$, we may assume that the same strict inequalities also hold for D.

However, since $B \nsim_{wlc} B'$, by definition $(Y/Z, B_Y^{log})$ and $(Y/Z, B_Y'^{log})$ are not numerically equivalent with respect to some curves C on Y/Z. Let $I = \{C\}$ be the set of all the curves C for which the signatures of $(C, K_Y + B_Y^{log})$ and $(C, K_Y + B_Y'^{log})$ are different. Since $(Y/Z, B_Y^{log})$ is a wlc model, we have one of the following four cases: for $C \in I$,

$$(i) \quad (C, K_Y + B_Y^{log}) > 0, \quad (C, K_Y + B_Y'^{log}) = 0$$

$$(ii) \quad (C, K_Y + B_Y^{log}) > 0, \quad (C, K_Y + B_Y'^{log}) < 0$$

$$(iii) \quad (C, K_Y + B_Y^{log}) = 0, \quad (C, K_Y + B_Y'^{log}) > 0$$

$$(iv) \quad (C, K_Y + B_Y^{log}) = 0, \quad (C, K_Y + B_Y'^{log}) < 0.$$

Suppose that (i) is the case for all $C \in I$. Since the set I contains all the curves C on Y/Z for which the signatures of $(C, K_Y + B_Y^{log})$ and $(C, K_Y + B_Y'^{log})$ are different, the positivity $(C, K_Y + D_Y^{log}) > 0$ holds for any divisor $D \in [B, B')$ and for all $C \in I$. For any curve $C' \notin I$, one of the following holds: 1)$(C', K_Y + B_Y^{log}) > 0$ and $(C', K_Y + B_Y'^{log}) > 0$ or 2)$(C', K_Y + B_Y^{log}) = 0$ and $(C', K_Y + B_Y'^{log}) = 0$. Clearly, the same holds for any divisor $D \in (B, B')$. Thus for all $D \in (B, B')$, the log pair $(Y/Z, D_Y^{log})$ is numerically equivalent to $(Y/Z, B_Y^{log})$. Therefore, by the assumptions above, $D \sim_{wlc} B$ for $D \in (B, B')$. However, since B' is not even in the linear span of $\mathcal{C}$, so is $D \in (B, B')$. Thus $D \nsim_{wlc} B$, a contradiction. So there exists a curve $C \in I$ for which $(ii), (iii)$ or (iv) holds.

Suppose (ii) holds for some curve $C \in I$. Then by cone theorem [KoMo, Theorem 3.7] there exists a rational curve C' such that $R = \mathbb{R}_{\geq 0}[C']$ is an extremal ray with $(C', K_Y + B_Y'^{log}) < 0$. Here, $[C']$ denotes the class of C' in $N_1(Y/Z)$ and we may assume that C' has the minimal degree for this ray. Therefore we can suppose that C' is an extremal curve. Pick a number $\varepsilon > 0$ as in the stability of extremal rays [Sho6, Corollary 9]. Since we may assume that $0 < \|B - B'\| < min\{\varepsilon, \varepsilon'\}$, we have $(C', K_Y + B_Y^{log}) \leq 0$. However, since $(Y/Z, B_Y^{log})$ is a wlc model, the equality $(C', K_Y + B_Y^{log}) = 0$ holds. Now we have found an extremal curve $C' \in I$ for which the case (iv) holds.

If (iii) holds for some curve $C \in I$, then since we may assume that B and B' are arbitrarily close to each other, by replacing B' with $2B - B'$, we can assume that the condition (iv) holds for some $C \in I$. Indeed, $2B - B' \in \mathfrak{B}_F$ and

$$\begin{aligned}(C, K_Y + (2B - B')^{log}_Y) &= (C, K_Y + 2B^{log}_Y - B'^{log}_Y)\\ &= (C, 2K_Y + 2B^{log}_Y - (K_Y + B'^{log}_Y))\\ &= -(C, K_Y + B'^{log}_Y) < 0\end{aligned}$$

Furthermore, the divisor $2B - B'$ satisfies the same properties of B'. That is, $(Y/Z, (2B - B')^{log}_Y)$ is slt and the condition $(\#)$ holds for $2B - B'$. Now by the same argument as above, we can find an extremal curve C' for which the condition (iv) holds.

Now let the extremal curve C' be a new C_i. The intersection of $\overline{\mathcal{C}^*}$ with the plane $(C, K_Y + D^{log}_Y) = 0$ defines a polyhedral $\overline{\mathcal{C}^*}'$ such that $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*}' \subsetneq \overline{\mathcal{C}^*}$ (see Step 1-2). Now go to Q1 replacing $\overline{\mathcal{C}^*}$ with $\overline{\mathcal{C}^*}'$. Note that $dim\overline{\mathcal{C}^*}' = dim\overline{\mathcal{C}^*} - 1$. Therefore Case 1 cannot occur infinitely many times because $\overline{\mathcal{C}}$ is a finite dimensional polyhedron.

Case 2. Suppose now that B' is in the linear span of $\overline{\mathcal{C}}$. Define $B_a := B + a(B' - B)$ and let $r = \max\{a \mid B_a \in \overline{\mathcal{C}}\}$. Perturbing B', if necessary, we may assume that B_r is an internal point of some maximal face of $\overline{\mathcal{C}}$. We may also assume that B' is arbitrarily close to the set $\overline{\mathcal{C}}$, that is, $0 < \|B_r - B'\| \ll 1$. We have the following two subcases.

Subcase 2-1. Suppose that $(Y, (B_r)^{log}_Y)$ is slt.

First note that by lemma 3.3.1, the log pair $(Y/Z, (B_r)^{log}_Y)$ is a wlc model of $(X/Z, B_r)$. Suppose that there exists a prime divisor E which is non-exceptional on X and exceptional on Y such that the following equality holds:

$$a(E, Y, (B_r)^{log}_Y) = \begin{cases} 1 - mult_{F''_u} B_r = 1 - d''_u & \text{if } E = F''_u \subseteq SuppF, \\ 1 & \text{if } E = E_k \nsubseteq SuppF. \end{cases}$$

Then since the functions $a(E, Y, D_Y^{log}) - (1 - mult_E D)$ and $a(E, Y, D_Y^{log}) - 1$ are linear with respect to D and B_r is an interior point of a maximal face of $\overline{\mathcal{C}}$, the equations determine the maximal face of $\overline{\mathcal{C}}$ containing B_r. Let E be a new F''_u if $E = E_k \subseteq SuppF$ and E_k if $E \nsubseteq SuppF$.

Assume that for any prime E which is non-exceptional on X and exceptional on Y, the following strict inequality holds:

$$a(E, Y, (B_r)_Y^{log}) > \begin{cases} 1 - mult_{F''_u} B_r = 1 - d''_u & \text{if } E = F''_u \subseteq SuppF, \\ 1 & \text{if } E = E_k \nsubseteq SuppF. \end{cases}$$

Since B and B' are in the face $\mathcal{F}$ of $\mathfrak{B}_F$ defined in Step 2-1, B_a is a boundary and has support in Supp$B$ for any $0 \leq a \leq 1$. By the stability of lt singularities [Sho2, 1.3.4] and because we may assume $0 < \|B_r - B'\| \ll 1$, the wlc model $(Y, (B_a)_Y^{log})$ is also slt for all $r \leq a \leq 1$. We may also assume that $a(E, X, B_a) < a(E, Y, (B_a)_Y^{log})$ holds for any prime E which is non-exceptional on X and exceptional on Y. Now since $B' \notin \overline{\mathcal{C}}$, B' is not wlc model equivalent to B and thus by the definition of $\sim_{wlc}$, the log pairs $(Y/Z, B_Y^{log})$ and $(Y/Z, B_Y'^{log})$ are not numerically equivalent with respect to some curve C.

Let $I' = \{C\}$ be the set of all the curves for which the signatures of $(C, K_Y + B_Y^{log})$ and $(C, K_Y + B_Y'^{log})$ are different. First, note that $(C, K_Y + (B_r)_Y^{log}) \geq 0$ for any curve C on Y/Z, not necessarily in I', since $(Y/Z, (B_r)_Y^{log})$ is a wlc model.

For $C \in I'$, the following three cases are possible:

$$\begin{aligned} &(i)' \quad (C, K_Y + (B_r)_Y^{log}) > 0, \quad (C, K_Y + B_Y'^{log}) = 0 \\ &(ii)' \quad (C, K_Y + (B_r)_Y^{log}) > 0, \quad (C, K_Y + B_Y'^{log}) < 0 \quad \text{and} \\ &(iii)' \quad (C, K_Y + (B_r)_Y^{log}) = 0, \quad (C, K_Y + B_Y'^{log}) < 0. \end{aligned}$$

Note that $(iv)'$ $(C, K_Y + (B_r)_Y^{log}) = 0$, $(C, K_Y + B_Y'^{log}) > 0$ cannot happen because $(Y/Z, B_Y^{log})$ is a wlc model.

Suppose that $(i)'$ holds for all $C \in I'$. Then the positivity $(C, K_Y + D_Y^{log}) > 0$ holds for all $C \in I'$ and for any divisor $D \in [B, B')$. For any curve $C \notin I'$, one of the following holds: 1)$(C, K_Y + B_Y^{log}) > 0$ and $(C, K_Y + B_Y'^{log}) > 0$ or 2)$(C, K_Y + B_Y^{log}) = 0$ and $(C, K_Y + B_Y'^{log}) = 0$. Clearly, the same holds for any $D \in [B, B']$. Therefore $(Y/Z, B_Y^{log})$ and $(Y/Z, D_Y^{log})$ with any $D \in (B, B')$ are numerically equivalent to each other with respect to any curve on Y/Z. This is a contradiction because $[B, B') \nsubseteq \mathcal{C}$. Therefore $(ii)'$ or $(iii)'$ holds for some $C \in I'$.

Suppose that $(ii)'$ holds for some curve $C \in I'$. Then by cone theorem [KoMo, Theorem 3.7], there exists a rational curve C' such that $R = \mathbb{R}_{\geq 0}[C']$ is an extremal ray with $(C', K_Y + B_Y'^{log}) < 0$. As in Case 1, we can suppose that C' is an extremal curve. Choose $\varepsilon > 0$ as in the stability of extremal rays [Sho6, Corollary 9]. Since we may assume that $0 < \|B_r - B'\| < \varepsilon$, $(C', K_Y + (B_r)_Y^{log}) \leq 0$ holds by the stability of extremal rays. Then since $(Y/Z, (B_r)_Y^{log})$ is a wlc model, we have $(C', K_Y + (B_r)_Y^{log}) = 0$. We have found an extremal curve $C' \in I'$ for which $(iii)'$ holds. Similarly, we may assume that any curve $C \in I'$ satisfying $(iii)'$ is an extremal curve.

Now let the extremal curve C be a new C_j'. Since B_r is an internal point of a maximal face of $\overline{\mathcal{C}}$ and the function $(C_j', K_Y + D_Y^{log})$ is linear with respect to D, the plane defined by $(C_j', K_Y + D_Y^{log}) = 0$ determines a maximal face of $\overline{\mathcal{C}}$ containing B_r.

In Subcase 2-1, we have found an equation $a(F_u'', Y, D_Y^{log}) = 1 - d_u$, $a(E_k, Y, D_Y^{log}) = 1$ or $(C_j', K_Y + D_Y^{log}) = 0$ determined by a divisor F_u'', E_k or an extremal curve C_j', respectively. The intersection of $\overline{\mathcal{C}^*}$ with the corresponding half space $a(F_u'', Y, D_Y^{log}) \geq 1 - d_u$, $a(E_k, Y, D_Y^{log}) \geq 1$ or $(C_j', K_Y + D_Y^{log}) \geq 0$ defines a polyhedron $\overline{\mathcal{C}^*}'$ such that $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*}' \subsetneq \overline{\mathcal{C}^*}$. Now go to Q1 replacing $\overline{\mathcal{C}^*}$ with $\overline{\mathcal{C}^*}'$.

Subcase 2-2. Suppose that $(Y, (B_r)_Y^{log})$ is lc, not slt.

Recall that $B' \in \overline{\mathcal{C}^*} \setminus \overline{\mathcal{C}}$. In particular, $B' \in \mathfrak{B}_F$. Recall also that $B \in int\overline{\mathcal{C}}$ and

$(Y/Z, B_Y^{log})$ is a slm model of $(X/Z, B)$.

Since $B \in int\overline{\mathcal{C}}$ and $B_r \in (B, B')$ are internal points of $\mathcal{F}$ defined in Step 2-1, $SuppB = SuppB_r$ and thus by lemma 1.2.4, (Y, D_Y^{log}) is slt for all $D \in [B, B_r)$. Furthermore, for $D \in (B_r, B']$, $(Y/Z, D_Y^{log})$ is not lc by lemma 1.2.4. Therefore, there exists a divisor E such that $a(E, Y, B_Y^{log}) > 0, a(E, Y, (B_r)_Y^{log}) = 0$, and $a(E, Y, B_Y'^{log}) < 0$. If E is non-exceptional on Y, then $0 > a(E, Y, B_Y'^{log}) = -\text{mult}_E B_Y'^{log} + 1$. Since B' is an internal point of the face $\mathcal{F}$, the coefficients of B' are ≤ 1, a contradiction. Therefore the divisor E is exceptional on Y.

Now we verify that E is also exceptional on X. Suppose that E is non-exceptional on X. Then $E \subseteq SuppF$. Indeed, if $E \nsubseteq SuppF$, then for any $D \in [B, B_r)$, $mult_E D = 0$ and $a(E, Y, D_Y^{log}) \geq a(E, X, D) = 1 - mult_E D = 1$. By the continuity of $a(E, Y, D_Y^{log})$ with respect to D, we have $a(E, Y, (B_r)_Y^{log}) \geq 1$, a contradiction to the fact that $a(E, Y, (B_r)_Y^{log}) = 0$. Thus $E \subseteq SuppF$.

By the same argument and because D is a boundary divisor, $mult_E D > 0$ for some $D \in [B, B_r)$. We have actually $mult_E D = 1$ for any $D \in [B, B_r)$. Indeed, suppose that there exists $D \in [B, B_r)$ such that $mult_E D < 1$. Then since B and B_r are in $int\mathcal{F}$ by construction, $mult_E D < 1 - \varepsilon$ for some $\varepsilon > 0$ for any $D \in [B, B_r]$. By the above argument, we obtain $a(E, Y, (B_r)_Y^{log}) \geq \varepsilon > 0$, a contradiction.

The fact that $mult_E D = 1$ for any $D \in [B, B_r)$ also implies that $mult_E D = 1$ for any $D \sim_{wlc} B$ since $[B, B_r)$ is in $int\overline{\mathcal{C}}$. However, we can find a divisor $D \sim_{wlc} B$ with $mult_E D < 1$. From above, we have $a(E, Y, B_Y^{log}) > 0$ and thus $M = max\{0, 1 - a(E, Y, B_Y^{log})\} < 1$. Therefore, by lemma 3.3.2, $B \sim_{wlc} B - bE$ for any positive $b \leq 1 - M$ and $mult_E(B - bE) = 1 - b < 1$, a contradiction. Therefore E is exceptional on X.

Let E be a new E_l' and consider the equation $a(E_l', Y, D_Y^{log}) = 0$. Since B_r is an internal point of a maximal face of $\overline{\mathcal{C}}$ and $a(E'_l, Y, D_Y^{log})$ is linear with respect to D, the plane $a(E'_l, Y, D_Y^{log}) = 0$ defines a maximal face of $\overline{\mathcal{C}}$ containing B_r. The intersection of $\overline{\mathcal{C}^*}$ with the closed half space $a(E'_l, Y, D_Y^{log}) \geq 0$ defines a polyhedral $\overline{\mathcal{C}^*}'$ such that $\overline{\mathcal{C}} \subseteq \overline{\mathcal{C}^*}' \subsetneq \overline{\mathcal{C}^*}$. Now go to Q1 replacing $\overline{\mathcal{C}^*}$ with $\overline{\mathcal{C}^*}'$.

In Case 1, we have $dim\overline{\mathcal{C}^*}' = dim\overline{\mathcal{C}^*} - 1$. The equations $a(F''_u, Y, D_Y^{log}) = 1 - d_u$, $a(E_k, Y, D_Y^{log}) = 1$ or $(C'_j, K_Y + D_Y^{log}) = 0$ in Subcase 2-1 and $a(E'_l, Y, D_Y^{log}) = 0$ in Subcase 2-2 determine maximal faces of $\overline{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is a finite dimensional convex polyhedron, we cannot have the answer *no* to Q1 infinitely many times.

Step 3. We deal with the case where the respond to Q1 is *yes*. We verify that $\mathcal{C} \subseteq \mathcal{C}^*$ in this case.

Proof. Suppose that $\mathcal{C} \nsubseteq \mathcal{C}^*$ and let $D \in \mathcal{C} \setminus \mathcal{C}^*$. Then since $\mathcal{C} \setminus \mathcal{C}^* \subseteq \overline{\mathcal{C}} \setminus \mathcal{C}^* = \overline{\mathcal{C}^*} \setminus \mathcal{C}^*$ by Step 1-1, D is in a face of the closed polyhedron $\overline{\mathcal{C}} = \overline{\mathcal{C}^*} \subseteq \mathfrak{B}_F$ but $D \notin \mathcal{C}^*$. By construction, each face of $\overline{\mathcal{C}^*}$ is determined in $\mathfrak{B}_F$ by one of the following equations:

- $a(F'_t, Y, D_Y^{log}) = 0$ or $= 1$
- $a(F''_u, Y, D_Y^{log}) = 1 - d_u$
- $a(E_k, Y, D_Y^{log}) = 1$
- $a(E'_l, Y, D_Y^{log}) = 0$
- $(C'_j, K_Y + D_Y^{log}) = 0$

Since we still assume that the set of divisors F'_t is empty [see warning before Step 2-1], D satisfies one of the last three equations. In these three cases, we verify that $D \nsim_{wlc} B$. This contradicts to our choice of $D \in \mathcal{C} \setminus \mathcal{C}^* \subseteq \mathcal{C}$ and thus we conclude that $\mathcal{C} \subseteq \mathcal{C}^*$.

Suppose that $a(F''_u, Y, D_Y^{log}) = 1 - d_u$, $a(E_k, Y, D_Y^{log}) = 1$ or $a(E'_l, Y, D_Y^{log}) = 0$ holds and assume that $D \sim_{wlc} B$. Take a divisorial log resolution $f : W \to Y$ from

$(Y/Z, D_Y^{log})$ where F''_u, E_k or E'_l is non-exceptional on W, and consider the crepant pull back

$$K_W + D_W^{log} = f^*(K_Y + D_Y^{log}).$$

Then $(W/Z, D_W^{log})$ is also a wlc model of $(X/Z, D)$ and moreover, $mult_{F''_u} D_W^{log} = d_u$, $mult_{E_k} D_W^{log} = 0$ or $mult_{E'_l} D_W^{log} = 0$, respectively. Since $B \sim_{wlc} D$, $(W/Z, B_W^{log})$ is also a wlc model of $(X/Z, B)$. But this is possible only if $a(F''_u, W, B_W^{log}) = 1 - b_u$, $a(E_k, W, B_W^{log}) = 1$ or $a(E'_l, W, B_W^{log}) = 0$, respectively. By the invariance of log discrepancies [Sho4, Proposition 2.4], we obtain $a(F''_u, Y, B_Y^{log}) = 1 - b_u$, $a(E_k, Y, B_Y^{log}) = 1$ or $a(E'_l, Y, B_Y^{log}) = 0$, respectively. However, by construction, we have $a(F''_u, Y, B_Y^{log}) > 1 - b_u$, $a(E_k, Y, B_Y^{log}) > 1$ or $a(E'_l, Y, B_Y^{log}) > 0$, respectively, because $B \in int\overline{\mathcal{C}}$. This is a contradiction.

If $(C'_j, K_Y + D_Y^{log}) = 0$, then $(Y/Z, D_Y^{log})$ and $(Y/Z, B_Y^{log})$ are not numerically equivalent because $(C'_j, K_Y + B_Y^{log}) > 0$ by construction. See subcase 2-1 of Step 2. Therefore, $D \nsim_{wlc} B$ and this contradicts to our choice of $D \in \mathcal{C} \setminus \mathcal{C}^* \subseteq \mathcal{C}$. □

In fact, we proved more. Let H be the hyperplane given by one of the last three equations above. Then using the same argument, we obtain that for any $D \in \overline{\mathcal{C}} \cap H$, $D \nsim_{wlc} B$, or equivalently $D \notin \mathcal{C}$. We also note that $\mathcal{C} = \mathcal{C}^*$ in $int\mathcal{F}$.

Now we deal with the two responds *no* and *yes* to Q2.

Step 4. Suppose that the respond to Q2 is *no*. Note that since $\overline{\mathcal{C}} = \overline{\mathcal{C}^*}$, the sets $\mathcal{C}$ and $\mathcal{C}^*$ have the same set of faces. Let $\mathcal{G}$ be a maximal face of $\mathcal{C}$ and suppose that there exists a divisor $D \in int\mathcal{G}$ such that $D \notin \mathcal{C}$. Then we claim that $\mathcal{G} \cap \mathcal{C} = \varnothing$. Note that since $\mathcal{C} = \mathcal{C}^*$ in $int\mathcal{F}$ by Step 3, $D \in \partial\mathcal{F}$ and thus $\mathcal{G} \subseteq \partial\mathcal{F}$.

By lemma 3.3.1, for any $D \in int\mathcal{G}$, the pair $(Y/Z, D_Y^{log})$ is a wlc model of $(X/Z, D)$. Suppose that $(Y/Z, B_Y^{log})$ and $(Y/Z, D_Y^{log})$ are not numerically equivalent. Then there exists a curve C on Y/Z such that $(C, K_Y + B_Y^{log}) > 0$ and $(C, K_Y + D_Y^{log}) = 0$. Moreover, by construction, the strict inequality $(C, K_Y + D_Y^{\prime log}) > 0$ holds for any $D' \in \mathcal{C}$. Thus, since $D \in int\mathcal{G}$ and $dim\mathcal{G} = dim\mathcal{C} - 1$, the equality $(C, K_Y + D_Y^{\prime log}) = 0$ holds for any $D' \in \mathcal{G}$. Therefore, the divisors $D' \in \mathcal{G}$ do not belong to $\mathcal{C}$ because the wlc models $(Y/Z, B_Y^{log})$ and $(Y/Z, D_Y^{\prime log})$ are not numerically equivalent. So, $\mathcal{C} \cap \mathcal{G} = \varnothing$. The inequality $(C, K_Y + D_Y^{\prime log}) > 0$ can also be given by $a(F_t', Y, D_Y^{\prime log}) > 0$ or < 1 for some prime component F_t' of F since the face $\mathcal{G}$ is a subset of some face of $\mathcal{F}$.

Now assume that for all the varieties Y for which $(Y/Z, B_Y^{log})$ is a wlc model of (X, B), the wlc models $(Y/Z, B_Y^{log})$ and $(Y/Z, D_Y^{log})$ are numerically equivalent. Then since $B \not\sim_{wlc} D$, there exists a variety Y' for which $(Y'/Z, D_{Y'}^{log})$ is a wlc model of $(X/Z, D)$, but $(Y'/Z, B_{Y'}^{log})$ is not a wlc model of $(X/Z, B)$. Furthermore, for any $B' \in [B, D)$, the pair $(Y'/Z, B_{Y'}^{\prime log})$ is not a wlc model of $(X/Z, B')$.

Now we prove that there exists a crepant pair $(Y'/Z, B_{Y'}^{crep})$ to $(Y/Z, B_Y^{log})$. Indeed, first of all, note that for the birational modification $\chi : Y \dashrightarrow Y'/Z$, only the curves C on Y/Z with $(C, K_Y + B_Y^{log}) = 0$ are contracted by χ. More precisely, $\overline{K_Y + B_Y^{log}} \equiv 0$ over Y'. Indeed, the wlc models $(Y/Z, D_Y^{log})$ and $(Y'/Z, D_{Y'}^{log})$ of $(X/Z, D)$ are crepant to each other [Sho4, 2.4.2]. Thus $\overline{K_Y + D_Y^{log}} = \overline{K_{Y'} + D_{Y'}^{log}}$. Let $f : W \to Y$ and $g : W \to Y'$ be a model of Y and Y' over Z:

$$\begin{array}{ccc} & W & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y & \overset{\chi}{\dashrightarrow} & Y'. \end{array}$$

Let C be a curve on W/Z which is contracted by g to a point in Y'/Z. Using the projection formula, we see that

$$\left(C, \left(\overline{K_Y + D_Y^{log}}\right)_W\right) = \left(C, \left(\overline{K_{Y'} + D_{Y'}^{log}}\right)_W\right) = (g_* C, K_{Y'} + D_{Y'}^{log}) = 0.$$

However, since $(Y/Z, B_Y^{log})$ and $(Y/Z, D_Y^{log})$ are numerically equivalent by assumption, we also have $\left(C, \left(\overline{K_Y + B_Y^{log}}\right)_W\right) = 0$. Therefore, $\overline{K_Y + B_Y^{log}} \equiv 0$ over Y'/Z. Thus, by proposition 3.3.4, there exists a divisor $B_{Y'}^{crep}$ on Y' such that $(Y'/Z, B_{Y'}^{crep})$ is crepant to $(Y/Z, B_Y^{log})$.

However, since $(Y'/Z, B_{Y'}^{log})$ is not a wlc model of $(X/Z, B)$, we have $B_{Y'}^{crep} \neq B_{Y'}^{log}$ by proposition 3.3.3. In particular, this implies that there exists a prime divisor E on Y' such that $mult_E B_{Y'}^{crep} \neq mult_E B_{Y'}^{log}$. By proposition 3.3.6, the same holds for any $B' \sim_{wlc} B$. Therefore, the inequality $a(E, Y', D_{Y'}^{\prime log}) - a(E, Y', D_{Y'}^{\prime crep}) \neq 0$ holds for any $D' \in \mathcal{C}$.

Since $(Y'/Z, D_{Y'}^{log})$ is a wlc model of $(X/Z, D)$ and is crepant to another wlc model $(Y/Z, D_Y^{log})$ of $(X/Z, D)$, the equality

$$a(E, Y', D_{Y'}^{log}) - a(E, Y', D_{Y'}^{crep}) = a(E, Y, D_Y^{log}) - a(E, Y', D_{Y'}^{crep}) = 0$$

holds. Note that since E is non-exceptional on Y', the log discrepancy $a(E, Y', D_{Y'}^{\prime log})$ is equal to $1 - mult_E D_{Y'}^{\prime log}$, and is thus linear with respect to D'. Recall that $a(E, Y', D_{Y'}^{\prime crep}) = a(E, Y, D_Y^{\prime log})$, which is also linear with respect or D'. Hence, the difference of the two functions

$$a(E, Y', D_{Y'}^{\prime log}) - a(E, Y', D_{Y'}^{\prime crep}) = 1 - mult_E D_{Y'}^{\prime log} - a(E, Y, D_Y^{\prime log})$$

is linear with respect to D' and is actually independent of Y'. Indeed,

$$mult_E D_{Y'}^{\prime log} = \begin{cases} mult_E D' = d'_u & \text{if } E = F''_u \text{ is non-exceptional on } X \text{ and } F''_u \subseteq SuppF, \\ 0 & \text{if } E \text{ is non-exceptional on } X \text{ and } E \nsubseteq SuppF, \\ 1 & \text{if } E \text{ is exceptional on } X. \end{cases}$$

Recall that $D \in int\mathcal{G}$ and $dim\mathcal{G} = dim\mathcal{C} - 1$. Therefore, the equation $a(E, Y', D_{Y'}^{\prime log}) - a(E, Y', D_{Y'}^{\prime crep}) = 0$ holds for all $D' \in \mathcal{G}$ and one of the strict inequalities $a(E, Y', D_{Y'}^{\prime log}) - a(E, Y', D_{Y'}^{\prime crep}) > 0$ or < 0 holds for all $D' \in \mathcal{C}$. This inequality can also be given by the following strict inequality

$$a(F'_t, Y, D_Y^{\prime log}) > 0 \text{ or } < 1$$

for some prime component F'_t of F since the face $\mathcal{G}$ is a subset of some face of $\mathcal{F}$.

Since there are finitely many faces of $\mathcal{C}$, there can be only finitely many such divisors F'_t as above. After finding all such F'_t, we can finally respond to Q2 with a *yes*, and obtain $\mathcal{C} = \mathcal{C}^*$.

Now, the proof of the theorem is complete. □

4 Applications

4.1 Cone of effective divisors.

As a first application, we study the cone of pseudo-effective divisors. As mentioned in chapter 1, in the past the cone of pseudo-effective divisors was studied in several papers, for example, in [Bat], [Ara], and [Bar]. In [Bat], he proved that the cone of effective divisors of a $\mathbb{Q}$-factorial terminal Fano variety of dimension 3 is rational polyhedral using the rationality of the effective thresholds and the duality of the cones. The result was improved by [Ara] and [Bar] reproved the same result on a smooth Fano 3-fold using the duality of the cone of pseudo-effective divisors and the strongly-movable cone of curves.

In this thesis, we give a more conceptual proof the result, using the geography of log models.

Definition 4.1.1. Let X be a normal projective variety over Z.

- A log pair $(X/Z, B)$ with an $\mathbb{R}$-boundary B is called a *weak log Fano*(WLF) variety if the pair has only klt singularities and $-(K+B)$ is nef and big over Z.
- A log pair $(X/Z, B)$ with an $\mathbb{R}$-boundary B is called a *log Fano*(LF) variety if the pair has only klt singularities and $-(K+B)$ is ample over Z.
- A log pair $(X/Z, B)$ with an $\mathbb{R}$-boundary B is called a 0-*log pair* if the pair has only klt singularities and $K+B \sim_{\mathbb{R}} 0/Z$.

Definition 4.1.2. Let X be a normal projective variety over Z. We say that X/Z is a *Fano type*(FT) variety if there exists an $\mathbb{R}$-boundary B such that $(X/Z, B)$ is a WLF variety.

There exist equivalent characterizations of FT varieties [PrSh, Lemma-Definition 2.8]. For example, an FT variety can be defined also as a normal projective variety X over Z for which there exists an $\mathbb{R}$-boundary B such that $(X/Z, B)$ is a LF variety.

Lemma 4.1.3. *Let X/Z be be an FT variety. Then for any fixed reduced divisor F, there exists an $\mathbb{R}$-boundary B such that $(X/Z, B)$ is a 0-log pair and $SuppB \supseteq SuppF$.*

Proof. See Lemma-Definition 2.8 of [PrSh]. □

Remark. For any $\mathbb{R}$-divisor D on a FT variety X/Z, we can run D-MMP under the assumption of LMMP. Indeed, by the above lemma, there exists an $\mathbb{R}$-boundary B on X such that (X, B) is a 0-log pair. Since (X, B) is klt, $B + \varepsilon D$ is an $\mathbb{R}$-boundary for $0 < \varepsilon \ll 1$. Note that $D \sim_{\mathbb{R}} \varepsilon D \sim_{\mathbb{R}} K + B + \varepsilon D$. Thus D-MMP on X/Z is the same as LMMP on $(X/Z, B + \varepsilon D)$.

See Definition 2.2.1 for the relative invariant Iitaka dimension $I(X/Z, D)$ and Definition 2.2.3 for the relative invariant Kodaira dimension $K(X/Z, B)$. We recall the following corollary.

Corollary 4.1.4. *Let B be an $\mathbb{R}$-boundary on X/Z. Then the following hold:*

(1) $K(X/Z, B) \leq I(X/Z, K + B)$.

(2) If (X, B) is an lc pair, then $K(X/Z, B) = I(X/Z, K + B)$.

Proof. See Corollary 2.2.6 for proof. □

Lemma 4.1.5. *Suppose that X is $\mathbb{Q}$-factorial and that $D \in \mathfrak{D}_F$ for some fixed reduced divisor $F = \sum D_i$. Assume also that the log pair (X, D) has only klt singularities. Then there exists a neighborhood U of D in $\mathfrak{D}_F$ such that for any divisor $D' \in U$, the log pair (X, D') also has only klt singularities.*

Proof. By the stability [Sho2, 1.3.4], for a prime divisor D_i of F, there exists $\epsilon_i > 0$ such that $(X, D + r_i D_i)$ has only klt singularities where $-\epsilon_i < r_i < \epsilon_i$. Let U be the interior of the convex hull of the finite set of points$\{D \pm \epsilon_1 D_1, D \pm \epsilon_2 D_2, \cdots, D \pm \epsilon_m D_m\}$. By the convexity [Sho2, 1.3.1], the log pair (X, D') has only klt singularities for any $D' \in U$. □

Notation 4.1.6. For a fixed reduced divisor F on X/Z, let $\mathcal{M}_F$ denote the subset of divisors $B \in \mathfrak{B}_F$ such that the relative invariant log Kodaira dimension $K(X/Z, B) \geq 0$. By the remark at the end of section 2.2, we have $\mathcal{M}_F \subseteq \mathcal{N}_F$ and $\overline{\mathcal{M}_F} = \mathcal{N}_F$.

Proposition 4.1.7. *Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial 0-log pair and let F =SuppB. Then there exists a neighborhood $N(B) \subseteq \mathfrak{B}_F$ of B such that*

$$N(B) \bigcap \mathcal{M}_F = N(B) \bigcap \mathcal{E}_F(B).$$

Proof. Let $B = \sum_{i=1}^m b_i D_i$. Then by definition, $(X/Z, B = \sum_{i=1}^m b_i D_i)$ is klt and $0 < b_i < 1$ for all i. By lemma 4.1.5, there exists an open neighborhood $N(B) \subseteq \mathfrak{B}_F$ of B such that (X, B') has only klt singularities (in particular, lc singularities) for any $B' \in U$.

Let $B' \in N(B) \cap \mathcal{M}_F$. Then $B' \in \mathfrak{B}_F$ and $I(X/Z, K + B') = K(X/Z, B') \geq 0$ by corollary 2.2.6. Thus by definition 2.2.1, $K + B' \sim_{\mathbb{R}} D$ for some $D \geq 0$. On the other hand, $K + B \sim_{\mathbb{R}} 0/Z$. Hence

$$B' - B \sim_{\mathbb{R}} B' - B + K + B = K + B' \sim_{\mathbb{R}} D \geq 0.$$

Therefore we have $B' \sim_{\mathbb{R}} D + B \geq B$ and $B' \in N(B) \bigcap \mathcal{E}_F(B)$.

Conversely, let $B' \in N(B) \bigcap \mathcal{E}_F(B)$. Then $B' \sim_{\mathbb{R}} D + B$ for some $D \geq 0$. As above,

$$K + B' = K + B + B' - B \sim_{\mathbb{R}} B' - B \sim_{\mathbb{R}} D \geq 0.$$

Hence by corollary 2.2.6, $K(X, B') = I(X, K + B') \geq 0$, and $B' \in N(B) \bigcap \mathcal{M}_F$. $\square$

Corollary 4.1.8. *Assume GLM in dimension d. Let $(X/Z, B)$ be a $\mathbb{Q}$-factorial 0-log pair of dimension d and let $F = SuppB$. Then the cone $\overline{\mathcal{E}_F(B)}$ is rational polyhedral.*

Addendum. *The cone $\overline{\mathcal{E}_F}$ is also rational polyhedral.*

Proof. By proposition 4.1.7, there exists a neighborhood $N(B) \subseteq \mathfrak{B}_F$ of B such that

$$N(B) \bigcap \mathcal{M}_F = N(B) \bigcap \mathcal{E}_F(B).$$

Since $\overline{\mathcal{M}_F} = \mathcal{N}_F$, we have the equality:

$$N(B) \bigcap \mathcal{N}_F = N(B) \bigcap \overline{\mathcal{M}_F} = N(B) \bigcap \overline{\mathcal{E}_F(B)}.$$

By GLM in dimension $d = \dim X$, the set $\overline{\mathcal{M}_F} = \mathcal{N}_F$ is rational polyhedral. This implies that the cone $\overline{\mathcal{E}_F(B)}$ is also rational and polyhedral near B. Therefore $\overline{\mathcal{E}_F(B)}$ is a rational polyhedral cone.

The cone $\overline{\mathcal{E}_F}$ is the image of $\overline{\mathcal{E}_F(B)}$ by the translation map $D \mapsto D - B$ which preserves the structure of the cones. $\square$

Corollary 4.1.9. *Assume GLM and LMMP in dimension d. Let $(X/Z, B)$ be a 0-log pair of dimension d and let $F = SuppB$. Then the cone $\overline{\mathcal{E}_{X,F}(B)}$ is rational polyhedral.*

Addendum. *The cone $\overline{\mathcal{E}_{X,F}}$ is also rational polyhedral.*

Proof. There exists a $\mathbb{Q}$-factorialization Y of X. That is, under the assumption of LMMP in dimension $d = \dim X$, there exists a small blow up

$$g : Y \longrightarrow X$$

such that Y is $\mathbb{Q}$-factorial [IsSh, Corollary 6.7]. Since g is a small blow up, we have $\mathfrak{D}_{Y,F} \cong \mathfrak{D}_{X,F}$ and since $\mathbb{R}$-linear equivalence is compatible with a small blow up, we also have $\mathcal{E}_{Y,F}(B) \cong \mathcal{E}_{X,F}(B)$, where we identify the divisors $g_*^{-1}F$ and $g_*^{-1}B$ on Y with the divisors F and B on X, respectively. By corollary 4.1.8, $\overline{\mathcal{E}_{Y,F}(B)}$ is rational polyhedral. Therefore so are $\overline{\mathcal{E}_{X,F}(B)}$ and $\overline{\mathcal{E}_{X,F}}$. □

Corollary 4.1.10. *Assume GLM in dimension $d = \dim X$. Let X/Z be a $\mathbb{Q}$-factorial FT variety. Then for any fixed reduced divisor F on X/Z, the cone $\overline{\mathcal{E}_F}$ is rational polyhedral. Furthermore, $\mathcal{E}_F = \overline{\mathcal{E}_F}$, that is, the cone $\mathcal{E}_F$ is closed.*

Proof. Fix a reduced divisor $F = \sum_{i=1}^{m} D_i$. By lemma 4.1.3, there exists an $\mathbb{R}$-boundary B such that $(X/Z, B)$ is a 0-log pair and Supp$B \supseteq$ Supp$F$. So we have $\mathfrak{D}_{\lceil B \rceil} \cong \mathfrak{D}_F \oplus \mathbb{R}^{m'} \cong \mathbb{R}^m \oplus \mathbb{R}^{m'}$ for some integer $m' \geq 0$. By corollary 4.1.8 (or by its Addendum), $\overline{\mathcal{E}_{\lceil B \rceil}}$ is a rational polyhedral cone in $\mathfrak{D}_{\lceil B \rceil}$. Consider the natural injection $i : \mathfrak{D}_F \hookrightarrow \mathfrak{D}_{\lceil B \rceil}$ such that $i(D) = D$. We identify $\mathfrak{D}_F$ as a subspace of $\mathfrak{D}_{\lceil B \rceil}$ via this injection.

We claim that $\overline{\mathcal{E}_F} = \overline{\mathcal{E}_{\lceil B \rceil}} \cap \mathfrak{D}_F$. Indeed, first of all, it is clear that $\overline{\mathcal{E}_F} \subseteq \overline{\mathcal{E}_{\lceil B \rceil}} \cap \mathfrak{D}_F$ since Supp$F \subseteq$ Supp$B$. Conversely, let $D \in \overline{\mathcal{E}_{\lceil B \rceil}} \cap \mathfrak{D}_F$. Then $SuppD \subseteq SuppF$ and D is a limit of divisors effective up to $\sim_{\mathbb{R}}$. So by definition, $D \in \overline{\mathcal{E}_F}$.

Now since $\overline{\mathcal{E}_{\lceil B \rceil}}$ is rational polyhedral and the $\mathbb{R}$-linear space $\mathfrak{D}_F$ is defined over $\mathbb{Q}$, $\overline{\mathcal{E}_F} = \overline{\mathcal{E}_{\lceil B \rceil}} \cap \mathfrak{D}_F$ is also rational polyhedral. Furthermore, by the semiampleness on FT varieties, we have $\overline{\mathcal{E}_F} = \mathcal{E}_F$. □

Remark. For any FT varieties, the cone of nef divisors is rational polyhedral. Thus the semiampleness of nef divisors on FT varieties follows from that of $\mathbb{Q}$-divisors [Sho1], [KMM, Theorem 3-1-1].

Corollary 4.1.11. *Assume GLM and LMMP in dimension d and let X/Z be a relative FT variety of dimension d. Then for any fixed reduced divisor F on X/Z, the cone $\overline{\mathcal{E}_F}$ is rational polyhedral.*

Addendum. *By* [BCHM]*, we can remove the assumptions GLM and LMMP.*

Proof. According to [IsSh, Corollary 6.7], there exists a $\mathbb{Q}$-factorialization $g : Y \to X$. The $\mathbb{Q}$-factorialization Y/Z is also a relative FT variety [PrSh, Lemma 2.9 (ii)] and $\mathcal{E}_F \cong \mathcal{E}_{Y,F}$ where we identify F on X with its image $g_*^{-1}F$ on Y (cf. the proof of Corollary 4.1.9). The cone $\overline{\mathcal{E}_{Y,F}}$ is rational polyhedral by corollary 4.1.10. So is $\overline{\mathcal{E}_F}$. □

We will need the following lemma.

Lemma 4.1.12. *Let X/Z be a relative FT variety over the base field of characteristic 0. Then the following hold:*

(1) *The two equivalence relations '$\equiv$' and '$\sim_{\mathbb{R}}$' coincide on X/Z.*

(2) *There exist a finite number of prime divisors $D_1, D_2, \ldots, D_m$ with $m \geq \rho(X/Z)$ such that if we let $F = \sum_{i=1}^{m} D_i$, then we have*

$$N^1(X/Z) = \mathfrak{D}_F/\sim_{\mathbb{R}} .$$

Proof. (1): It is enough to prove that the two notions '$\mathbb{R}$-linearly trivial ($\sim_{\mathbb{R}} 0/Z$)' and 'numerically trivial ($\equiv 0/Z$)' are equivalent. We first define the following two $\mathbb{R}$-linear spaces:

$$\mathcal{P}_{\mathbb{R}}(X/Z) = \{D \in \mathrm{Div}(X/Z) \mid D \sim_{\mathbb{R}} 0/Z\}$$

$$\mathrm{Num}_0(X/Z) = \{D \in \mathrm{Div}(X/Z) \mid D \text{ is } \mathbb{R}\text{-Cartier and } D \equiv 0/Z\}.$$

We need to verify that $\mathcal{P}_{\mathbb{R}}(X/Z) = \mathrm{Num}_0(X/Z)$. First of all, we note that both spaces are defined over $\mathbb{Q}$. More precisely, for any fixed reduced divisor F, both $\mathcal{P}_{\mathbb{R}}(X/Z) \cap \mathfrak{D}_F$ and $\mathrm{Num}_0(X/Z) \cap \mathfrak{D}_F$ are $\mathbb{R}$-linear subspaces of $\mathfrak{D}_F$ defined over $\mathbb{Q}$. Thus it is enough to verify that these intersections coincide or they have the same $\mathbb{Q}$-vectors (-points). Here $\mathbb{Q}$-vectors (-points) are identified with $\mathbb{Q}$-divisors. Therefore, our claim in fact follows from the following special case: for $\mathbb{Q}$-divisor D, $D \sim_{\mathbb{R}} 0/Z$ if and only if D is $\mathbb{Q}$-Cartier and $D \equiv 0/Z$. Note that a $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier if and only if it is $\mathbb{R}$-Cartier, and it is $\sim_{\mathbb{R}} 0/Z$ if and only if it is $\sim_{\mathbb{Q}} 0/Z$.

Let D be a $\mathbb{Q}$-divisor. If $D \sim_{\mathbb{Q}} 0/Z$, then clearly $D \equiv 0/Z$ since each principal divisor is numerically trivial. Conversely, assume that D is $\mathbb{Q}$-Cartier and $D \equiv 0/Z$. Since X/Z is an FT variety, there exists a $\mathbb{Q}$-boundary divisor B such that $(X/Z, B)$ is a log Fano (LF) variety. Note that $D - (K + B) \equiv -(K + B)/Z$ and is ample over Z. Thus by the Base Point Free theorem, $|nD|$ is a base point free linear system for some integer $n \gg 0$.

It defines a contraction $f : X \to X'$ over Z and the divisor nD is a pull back of some Cartier divisor M on X'/Z. Since $D \equiv 0/Z$, X'/Z is finite and any Cartier divisor on X' is $\sim 0/Z$. Thus $M \sim 0/Z$. Therefore $nD = f^*M \sim 0/Z$ and $D \sim_{\mathbb{Q}} 0/Z$.

(2): We first claim that the rank $\rho(X/Z)$ of $N^1(X/Z)$ is finite. Consider a resolution of singularities $f : X' \to X/Z$. Define a subset $\mathrm{Num}_0'(X'/Z)$ of $\mathrm{Num}_0(X'/Z)$ as follows:

$$\mathrm{Num}_0'(X'/Z) = \{D \in \mathrm{CDiv}(X'/Z) \mid D \equiv 0/Z \text{ and } D \sim_{\mathbb{R}} 0/X\}.$$

We note that $\mathrm{Num}_0(X'/Z)\big/\mathrm{Num}_0'(X'/Z)$ is finitely generated since an FT variety X/Z has only rational singularities [KMM, Theorem 1-3-6] and $\mathrm{Pic}(X'/X)$ is finitely generated [Ka2, Lemma 1.1]. Furthermore, we have the following well-defined surjection:

$$\mathrm{Div}(X'/Z)\big/\mathrm{Num}_0'(X'/Z) \longrightarrow \mathrm{Div}(X/Z)\big/\mathrm{Num}_0(X/Z) = N^1(X/Z).$$

Therefore to prove our claim, it is enough to show that $\mathrm{Div}(X'/Z)\big/\mathrm{Num}_0'(X'/Z)$ is finitely generated.

Consider the following exact sequence:

$$0 \to \mathrm{Num}_0(X'/Z)\Big/\mathrm{Num}'_0(X'/Z) \to \mathrm{Div}(X'/Z)\Big/\mathrm{Num}'_0(X'/Z) \to N^1(X'/Z) \to 0,$$

where $N^1(X'/Z) = \mathrm{Div}(X'/Z)\big/\mathrm{Num}_0(X'/Z)$. Since $\mathrm{Num}_0(X'/Z)\big/\mathrm{Num}'_0(X'/Z)$ and $N^1(X'/Z)$ are finitely generated, $\mathrm{Div}(X'/Z)\big/\mathrm{Num}'_0(X'/Z)$ is also finitely generated. So it follows that $\rho(X/Z)$ is finite.

There exist a finite number of prime divisors $D_1,\ D_2,\ \ldots\ D_m$ with $m \geq \rho(X/Z)$ such that for $F = \sum_{i=1}^m D_i$,

$$N^1(X/Z) = \mathfrak{D}_F / \equiv .$$

By (1), we can replace $\equiv$ by $\sim_{\mathbb{R}}$ to have the required result. □

Proposition 4.1.13. *Assume that the variety X/Z is defined over a field of characteristic 0.*

(1) *For a fixed reduced divisor F on X/Z, there exists a natural $\mathbb{R}$-linear map*

$$\varphi : \mathfrak{D}_F \longrightarrow N^1(X/Z)$$
$$D \longmapsto \varphi(D) = [D],$$

where $[D]$ denotes the numerical class of D in $N^1(X/Z)$. For this map, we have

$$\varphi(\mathcal{E}_F) \subseteq \mathrm{Eff}(X/Z) \cap \varphi(\mathfrak{D}_F). \qquad (*)$$

(2) *Suppose that X/Z is an FT variety. Then for any fixed reduced divisor F on X/Z, the inclusion $\subseteq$ in $(*)$ of (1) is an equality $=$.*

Proof. (1): Fix a reduced divisor F on X/Z. By definition, we obtain the following inclusion:

$$\begin{aligned}\varphi(\mathcal{E}_F) &= \{[D] \mid D \in \mathcal{E}_F\} \\ &= \{[D] \mid D \sim_{\mathbb{R}} D' \geq 0 \text{ and } D \in \mathfrak{D}_F\} \\ &\subseteq \{[D] \mid D \equiv D' \geq 0 \text{ and } D \in \mathfrak{D}_F\} \\ &= \mathrm{Eff}(X/Z) \cap \{[D] \mid D \in \mathfrak{D}_F\} \\ &= \mathrm{Eff}(X/Z) \cap \varphi(\mathfrak{D}_F).\end{aligned}$$

(2): If X/Z is a relative FT variety, we can change $\subseteq$ above to $=$ by (1) of Lemma 4.1.12. □

Corollary 4.1.14. *Let X/Z be a relative FT variety over a field of characteristic 0. Then there exists a reduced divisor F on X/Z such that*

$$\mathrm{Eff}(X/Z) = \varphi(\mathcal{E}_F).$$

In particular, GLM and LMMP in dimension $d = \dim X$ imply that $\mathrm{Eff}(X/Z)$ is rational polyhedral.

Proof. Suppose that X/Z is an FT variety. By lemma 4.1.12 (2), there exist prime divisors $D_1, D_2, \ldots D_m$ which generate $N^1(X/Z)$. Let $F = \sum D_i$. Since $\varphi(\mathfrak{D}_F) = N^1(X/Z) \supseteq \mathrm{Eff}(X/Z)$, we have

$$\mathrm{Eff}(X/Z) = \mathrm{Eff}(X/Z) \cap \varphi(\mathfrak{D}_F).$$

Therefore,

$$\begin{aligned}\mathrm{Eff}(X/Z) &= \mathrm{Eff}(X/Z) \cap \varphi(\mathfrak{D}_F) \\ &= \varphi(\mathcal{E}_F) \qquad \text{by proposition 4.1.13 (2),}\end{aligned}$$

which is rational polyhedral by lemma 4.1.11. □

4.2 Finiteness Theorems for wlc models.

As the second application of GLM, we prove the finiteness theorem for wlc models [IsSh, Conditional Theorem 3.12], [Sho4, 6.22].

Theorem 4.2.1. *Let $(X/Z, B)$ be a klt log pair such that $K+B$ is big over Z. Then there are only finitely many projective wlc models $(Y_i/Z, B_{Y_i})$ of the pair $(X/Z, B)$.*

Proof. We follow the proof of [IsSh, 3.12]. By theorem 3.1.1, there exists a log canonical model $(Y'/Z, B_{Y'})$ of $(X/Z, B)$. There exists a finite set of prime mobile b-divisors $\{D_i\}$ of X such that on any wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$, the divisors in $Supp(B_Y + \sum D_i)$ along with the divisors E_j exceptional on Y'/Z generate the numerical space $N^1(Y/Z)$. Let $F = Supp(B + \sum D_i)$ be a reduced divisor on X/Z and consider GLM [Main Theorem 1] in $\mathfrak{B}_F$.

By construction, we can find an ample divisor D on any wlc model Y/Z of $(X/Z, B)$ such that $SuppD = Supp(B_Y + \sum D_i + \sum E_j)$ where E_j are exceptional on Y'/Z. We can also find an effective ample divisor H on the log canonical model Y'/Z such that $SuppH = SuppF_{Y'}$, where $F_{Y'}$ is the birational transform of F on Y'. After adding some multiple of f^*H to D, we may assume that D is an effective ample divisor.

Let $(Y/Z, B_Y)$ be a wlc model of $(X/Z, B)$. Then the pair $(Y/Z, B_Y + \varepsilon D)$ is a lc model of $(X/Z, B+\varepsilon D_X)$ for $0 < \varepsilon \ll 1$, and it is also a wlc model. In particular, it is the unique model with this property. Therefore, there is an injection from the set of (projective) wlc models of $(X/Z, B)$ into the set of countries in $\mathfrak{B}_F$ near B. However, by the finiteness of countries in $\mathfrak{B}_F$ [Main Theorem 2], a small open neighborhood of B in $\mathfrak{B}_F$ intersects with only finitely many countries. Therefore, there are only finitely many such varieties Y/Z. □

Corollary 4.2.2. *Let X/Z be a relative variety of general type, having only cn singularities. Then there are only finitely many projective minimal models of X/Z.*

Proof. In theorem 4.2.1, choose $B = 0$. □

References

[Ara] Carolina Araujo; *The cone of effective divisors of log varieties after Batyrev.* arXiv:math.AG/0502174, 2005. Preprint.

[Bat] V. V. Batyrev; *The cone of effective divisors of threefolds*, Proceedings of the Inter- national Conference on Algebra, Part 3 (Novosibirsk, 1989), Contemp. Math., vol. 131, AMS, 1992, pp. 337-352.

[Bar] Sammy Barkowski; *The cone of moving curves of a smooth Fano-threefold.* arXiv:math/0703025, 2007. Preprint.

[BCHM] C. Birkar, P. Cascini, C. Hacon and J. McKernan; *Existence of minimal models for varieties of log general type*, arXiv:math/0610203, 2006. Preprint.

[Bo] S. Boucksom; *On the volume of a line bundle*, Internat. J. Math. 13 (2002), no. 10, 1043-1063.

[F34] Corti et al.; *Flips for 3-folds and 4-folds*, Oxford Lecture Series in Mathematics and Its Applications: Oxford University Press, 2007.

[Ha] Hartshorne, R.; *Algebraic Geometry*, Graduate Texts inMathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[HM] C. Hacon, J. McKernan; *On the existence of flips.* arXiv:math/0507597, 2005. Preprint.

[Isk] V. A. Iskovskikh; *b-divisors and functional algebras according to Shokurov*, Tr. Mat. Inst. Steklov. 240 (2003), 8-20; English transl., Proc. Steklov Inst. Math. 240 (2003), 4-15.

[IsSh] V.A. Iskovskikh, and V.V. Shokurov; *Birational models and flips*, Russian Mathematical Surveys, 60 (2005), 27-94

[Ka] Y. Kawamata; *Flops connect minimal models*, arXiv:0704.1013, 2007. Preprint.

[Ka2] Y. Kawamata; *Crepant blowing-up of* 3 *dimensional canonical singularities and its application to degenerations of surfaces*, Ann. Math. 127 (1988), 93-163.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki; *Introduction to the minimal model program*, Algebraic Geometry, Sendai (T. Oda, ed.), Kinokuniya-North- Holland, 1987, Adv. Stud. Pure Math, vol 10, pp. 283-360.

[KoMo] J. Kollar and S. Mori; *Birational geometry of algebraic varieties*, Cambridge University Press 134, 1998.

[Kl] V. Klee; *Some characterizations of convex polyhedra.* Acta Math. 102(1959), 79-107.

[La1] Lazarsfeld, R.; *Positivity in Algebraic Geometry*, I. Ergeb. Math. Grenzgeb., vols. 48. Berlin: Springer, 2004.

[La2] Lazarsfeld, R.; *Positivity in Algebraic Geometry*, II. Ergeb. Math. Grenzgeb., vols. 49. Berlin: Springer, 2004.

[Mat] Kenji Matsuki; *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.

[Nak] N. Nakayama; *Zariski-decomposition and abundance,* MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[Nik] V. V. Nikulin; *The diagram method for 3-folds and its application to the K ähler cone and Picard number of Calabi-Yau 3-folds, I, with an appendix by V. V. Shokurov.* Higher-Dimensional Complex Varieties. (Trento, Italy, 1994), de Gruyter, Berlin, 1996, 261-328.

[Pro] Yu. G. Prokhorov; *On Zariski decomposition problem.* Proc. Steklov Inst. Math., 240:37-65, 2003.

[PrSh] Yu. G. Prokhorov, V. V. Shokurov; *Toward the second main theorem on complements: from local to global.* arXiv:math/0606242v4, Preprint, to appear in Journal of Algebraic Geometry.

[Sho1] V. V. Shokurov; *The nonvanishing theorem*, Izv. Akad. Nauk SSSR Ser. Mat. 49(3), 635-651 (1985); English transl., Math. USSR Izv. 26 (1986), 591-604.

[Sho2] V. V. Shokurov; *Three-dimensional log perestroikas*, Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 1, 105-203.

[Sho3] V. V. Shokurov; *Anticanonical boundedness for curves*, Appendix to [Nik].

[Sho4] V. V. Shokurov; *3-fold log models.* Algebraic Geometry, 4,J.Math. Sci. 81, Consultants Bureau, New York, 1996, 2667-2699.

[Sho5] V. V. Shokurov; *Prelimiting flips*, Proc. Steklov Inst. of Math. 240 (2003), 82-219.

[Sho6] V. V. Shokurov; *Letters of a bi-rationalist VII: Ordered termination.* arXiv:math/0607822, Preprint. 2006.

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