Some Results On Stable Compact Embedded Minimal Surfaces in 3-Manifolds

by

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ABSTRACT

In this thesis, we discuss several results concerning stable compact embedded minimal surfaces in 3-manifolds. We show that for any 3-manifold N and any positive integer γ, there exists an open nonempty set of metrics on N (in the C^2-topology on the space of metrics on N) for each of which there are stable compact embedded minimal surfaces of genus γ with arbitrarily large area. This extends the result of Colding and Minicozzi for $\gamma = 1$. We then focus on 3-manifolds with positive Ricci curvature. We state a result of Ross which gives a uniform upper area bound for stable compact embedded minimal surfaces in such a 3-manifold depending only on the lower bound for the Ricci curvature of the 3-manifold. We then discuss some of the difficulties involved in trying to extend this result to higher dimensional stable minimal hypersurfaces in manifolds of positive Ricci curvature. Finally, we examine genus bounds for stable compact embedded minimal surfaces in a compact 3-manifold of positive Ricci curvature. For a given 3-manifold, such a bound exists; however, it is not a uniform bound. We place additional restrictions on the 3-manifold, obtaining a class of compact 3-manifolds for which there are uniform upper bounds for both the diameter and the genus of stable compact embedded minimal surfaces.

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1 Introduction

The mathematical concept of minimal surfaces is well-represented in nature. Perhaps the best-known examples from nature are soap films and soap bubbles (see [22] and [11]). The minimal surface known as the helicoid is approximated in architecture by the double spiral staircase, and in the human body by the part of the inner ear called the cochlea and, most notably, by the structure of DNA, the double helix of Watson and Crick (see [11]). Minimal surfaces conform with the notion that Nature does things as efficiently as possible.

This paper examines some aspects of stable compact embedded minimal hypersurfaces. In almost all cases, we will be dealing with minimal surfaces in 3-manifolds; higher-dimensional minimal hypersurfaces will be considered briefly in Section 4. The paper is written at such a level that the reader is expected to have some background in topology and differential geometry. A familiarity with the fundamental concepts of minimal surface theory is also expected; however, we will discuss a few of the main concepts in Section 2 for completeness.

The rest of Section 2 will be devoted to a few results which will be used throughout the remainder of the paper. We will state a curvature estimate, first proved by Schoen in [26], for stable minimal surfaces. We will use a version of Schoen's result given by Colding and Minicozzi in [10]. We will also discuss two results of Choi and Wang (see [5]). The first result is a first eigenvalue estimate for compact orientable embedded minimal hypersurfaces (of any dimension) in a compact manifold of positive Ricci curvature. This is then used to find an upper bound for the area of a compact orientable embedded minimal surface M in a compact 3-manifold N of positive Ricci curvature, depending on the genus of M and the lower bound for Ric_N. We will also discuss some results on the existence and embeddeness of area-minimizing surfaces. For spheres, these results are due to Sacks and Uhlenbeck in [25] and Meeks and Yau in [21]. The corresponding results for higher genus surfaces are due to Schoen and Yau in [27] and Freedman, Hass, and Scott in [16]. These higher genus results, which we state as Theorems 2.10 and 2.11, will be of particular importance to us in Section 3.

In Section 3, we examine stable compact embedded minimal surfaces in an arbitrary 3-manifold. In [7], Colding and Minicozzi showed that for any 3-manifold N there exists an open nonempty set of metrics on N for each of which there are stable compact embedded minimal tori of arbitrarily large area. Our first main result extends this to higher genus surfaces.

Theorem 1.1. *Let N be a 3-manifold (possibly with boundary), and let γ be a positive integer. There exists an open nonempty set of metrics on N (in the C^2-topology on the space of metrics on N) for each of which there are stable compact embedded minimal surfaces of genus γ with arbitrarily large area.*

It remains an open question whether the theorem holds for $\gamma = 0$. The argument used for $\gamma > 0$ breaks down when $\gamma = 0$, most notably since the aforementioned Theorems 2.10 and 2.11 hold only for $\gamma > 0$. The main problem with proving Theorem 1.1 for the genus 0 case seems to be that $\pi_1(S^2)$ is trivial. The non-trivial nature of π_1 when $\gamma > 0$ appears to be a crucial part of being able to prove Theorem 1.1.

Section 4 is a discussion of the possibility of finding a uniform area bound for stable compact embedded minimal hypersurfaces in a manifold of positive Ricci curvature, depending only on the lower bound for the Ricci curvature of the ambient manifold. It is easy to see that any such hypersurface must be nonorientable. Ross showed in [24] that an area bound exists when the ambient manifold is three-dimensional (and so the minimal hypersurface is two-dimensional). It is an open question whether a uniform area bound exists in higher dimensions. However, we will examine ways in which Ross' argument cannot be extended to higher dimensions. For one thing, Ross' argument involves several concepts and results which are unique to the case where the minimal hypersurface is two-dimensional. Also, when the minimal hypersurface M is two-dimensional, there is an upper bound for $\lambda_1(M)\text{Area}(M)$ which holds uniformly for all metrics on M (see [29]). This is the basis for an important lemma in [24], which we state as Lemma 4.1. It would therefore be desirable to find an upper bound for $\lambda_1(M)\text{Area}(M)$ when $\dim(M)$ is arbitrary, but a scaling argument shows that no such bound exists when the dimension of M is not two. Instead, as Berger correctly noted in [1], the appropriate thing to try to bound above is $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$, where

$n = \dim(M)$, which is the familiar $\lambda_1(M)\text{Area}(M)$ when $n = 2$. However, Bleecker conjectured in [2] that, for every manifold M of dimension greater than or equal to three, there are metrics with fixed area and arbitrarily large λ_1. This conjecture was proved independently by Colbois and Dodziuk in [6] and by Xu in [28]. Thus, there is no upper bound for $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$ which holds uniformly for all metrics on M.

Finally, in Section 5, we look at genus bounds for stable compact embedded minimal surfaces in a compact 3-manifold of positive Ricci curvature. As mentioned above, any such surface must be nonorientable. We show that, for a given compact 3-manifold of positive Ricci curvature, there is an upper bound for the genus of a nonorientable stable compact embedded minimal surface, but the bound is not uniform–it depends on the ambient 3-manifold. In fact, Ross (see [24]) cites an argument of Rubinstein in which it is shown that there is a sequence of compact 3-manifolds, all with the same lower bound for Ricci curvature, for which a nonorientable stable compact minimal surface of arbitrarily large genus can be embedded in some element of the sequence. However, we show that by placing additional restrictions on the compact 3-manifolds, we obtain a class of compact 3-manifolds in which there is a uniform upper bound for the diameter of a nonorientable stable compact embedded minimal surface. We use this to show, in our second main result, that in the same class of compact 3-manifolds, there is a uniform upper bound for the genus of a nonorientable stable compact embedded minimal surface.

Theorem 1.2. *Let N be an element of the class of compact 3-manifolds with Ric $\geq \kappa > 0$, $|sec| \leq k$, and inj $\geq i_0$, where Ric is Ricci curvature, sec is sectional curvature, and inj is injectivity radius. Let M be a nonorientable stable compact embedded minimal surface in N. Then, there exists a constant*

$$G = G(\kappa, k, i_0)$$

such that genus$(M) \leq G$.

2 Background on Stable Minimal Surfaces

2.1 Minimal Graphs; The Minimal Surface Equation

There are multiple points of view from which we can approach minimal surfaces. One is as a critical point of the area functional. Another, when the minimal surface is a graph, is as the solution of a certain uniformly elliptic nonlinear PDE. Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Then, the area of the graph of u is (see [8, p. 1])

$$\text{Area}(\text{Graph}_u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\,.$$

To perturb the graph of u, let φ be a C^1 test function on Ω (i.e., $\varphi|_{\partial\Omega} \equiv 0$), and consider the graphs of $u + t\varphi$ for small t. We obtain

$$\text{Area}(\text{Graph}_{u+t\varphi}) = \int_\Omega \sqrt{1 + |\nabla u + t\nabla\varphi|^2}\,.$$

Then, (see [8, p. 2])

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \text{Area}(\text{Graph}_{u+t\varphi}) &= \int_\Omega \frac{< \nabla u, \nabla\varphi >}{\sqrt{1 + |\nabla u|^2}} \\ &= -\int_\Omega \varphi \, \text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)\,. \end{aligned}$$

If the graph of u is a minimal surface, then it is a critical point for the area functional. Since φ was arbitrary, this means u must satisfy

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0\,.$$

This is the *minimal surface equation*, a uniformly elliptic nonlinear PDE. In non-divergence form, the minimal surface equation in two dimensions is

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0\,.$$

Similar calculations show that, if $u : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is a C^2 function, then Graph_u is a critical point of the area functional if and only if u satisfies

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0\,.$$

Thus, the divergence form of the minimal surface equation is the same in all dimensions.

A few comments are in order. First, a minimal graph is not only a critical point for area, it actually minimizes area amongst all surfaces in the cylinder $\Omega\times\mathbb{R}$ which agree with the graph on the boundary (see [8, p. 2-4]). Second, any minimal hypersurface is locally a minimal graph. Thus, we can say a lot about minimal hypersurfaces from looking at their intersections with small balls. There are many important properties of minimal graphs which are just special cases of results common to all solutions of uniformly elliptic PDEs. One such property which we will use throughout this paper is the maximum principle. In general, the strong maximum principle for solutions to uniformly elliptic PDEs on a connected domain Ω says that the solution attains its maximum on $\partial\Omega$; if the solution has an interior maximum, then it must be constant (see, e.g., [14, p. 332-333]). In this paper, when we cite the maximum principle, we will usually be referring to the following, which is Lemma 1.17 in [8].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^{n-1}$ be an open connected neighborhood of the origin. If $u_1, u_2 : \Omega \to \mathbb{R}$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

Proof [8, p.15]. Since

$$\begin{aligned}
&\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}} \\
&\qquad = \frac{\sqrt{1+|\nabla u_2|^2}\,(\nabla u_1 - \nabla u_2) + (\sqrt{1+|\nabla u_2|^2} - \sqrt{1+|\nabla u_1|^2})\,\nabla u_2}{\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}} \\
&\qquad = \frac{\nabla u_1 - \nabla u_2}{\sqrt{1+|\nabla u_1|^2}} + \frac{(|\nabla u_2|^2 - |\nabla u_1|^2)\,\nabla u_2}{(\sqrt{1+|\nabla u_1|^2} + \sqrt{1+|\nabla u_2|^2})\,\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}}
\end{aligned}$$

and both u_1 and u_2 satisfy the minimal surface equation, we get

$$\begin{aligned}
0 &= \operatorname{div}\left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\right) \\
&= \operatorname{div}\left(\frac{\nabla(u_1-u_2)}{\sqrt{1+|\nabla u_1|^2}}\right) \\
&\quad - \operatorname{div}\left(\frac{<\nabla(u_1-u_2), \nabla(u_1+u_2)>}{(\sqrt{1+|\nabla u_1|^2} + \sqrt{1+|\nabla u_2|^2})\,\sqrt{1+|\nabla u_1|^2}\,\sqrt{1+|\nabla u_2|^2}}\,\nabla u_2\right).
\end{aligned}$$

From this, we conclude that $v = u_1 - u_2$ satisfies an equation of the form

$$0 = \operatorname{div}(a_{i,j} \nabla v) + b_i \nabla v \,.$$

Moreover, if $|\nabla u_1|, |\nabla u_2|$ are sufficiently small, then $\lambda |x|^2 \leq a_{i,j} x_i x_j$ for some $\lambda > 0$. Therefore, the usual strong maximum principle gives the claim. □

As already mentioned, any minimal hypersurface is locally a minimal graph. Therefore, the following is a corollary to the above lemma.

Corollary 2.2 (Strong Maximum Principle). *If $\Sigma_1, \Sigma_2 \subset \mathbb{R}^n$ are complete connected minimal hypersurfaces (without boundaries), $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, and Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.*

The way we will usually apply the maximum principle in this paper is as follows. Given two distinct minimal graphs in a ball, one sitting above the other, the graphs must be disjoint. For, if they touched, the maximum principle would imply that the graphs are identically equal, a contradiction.

2.2 The First and Second Variation Formulas; Stability

So far, we have mentioned two ways to approach minimal surfaces: as a critical point for the area functional, and as locally the graph of a solution of the minimal surface equation. This second approach deals only with hypersurfaces. We will now discuss an approach which defines the concept of minimal submanifolds of arbitrary codimension.

Let M^k be an immersed submanifold of a Riemannian manifold N^n. We assume that M is a critical point for the area functional. Let $F : M \times (-\epsilon, \epsilon) \to N$ be a variation of M with compact support and fixed boundary. In particular, $F(x, 0) = x$ for all $x \in M$, $F(x,t) = x$ for all $t \in (-\epsilon, \epsilon)$ and all x outside a compact set, and $F(x,t) = x$ for all $t \in (-\epsilon, \epsilon)$ and all $x \in \partial M$. Writing $\text{Area}(t)$ as shorthand for $\text{Area}(F(M,t))$, the condition that M is a critical point for the area functional is equivalent to

$$\text{Area}'(0) = 0 \,.$$

The vector field F_t restricted to M is called the *variational vector field*. For convenience, we will assume that F_t is normal to M.

Our goal is to compute the first and second variations of area for this one-parameter family of surfaces. Let x_i be local coordinates on M. Let $g_{i,j}(t) = g(F_{x_i}(t), F_{x_j}(t))$ and $\nu_t = \sqrt{\det(g_{i,j}(t))}\sqrt{\det(g^{i,j}(0))}$, where g is the metric on N and, in general, $a^{i,j}$ is the inverse of the matrix $a_{i,j}$. Note that ν_t is defined so as to be independent of the choice of local coordinates on M. So, we have (see [8, p. 5])

$$\text{Area}(t) = \int_M \nu_t \sqrt{\det(g_{i,j}(0))}\,,$$

and the *first variation formula* for area is (see [8, p. 6])

$$\text{Area}'(0) = -\int_M g(F_t, H)\,,$$

where H is the mean curvature vector of M. Since $\text{Area}'(0) = 0$ for any such variation, we must have $H \equiv 0$. This is often taken as the definition of minimal submanifold; namely, $M^k \subset N^n$ is minimal if its mean curvature H is identically 0 (see [8, p. 6]).

We know that minimal surfaces are critical points for the area functional. We would like to study minimal surfaces which are in fact local minima for the area functional; we have already seen that this is true for minimal graphs. To determine if a given minimal submanifold $M^k \subset N^n$ is a local minimum for area, we must look at the second variation of area. Thus, we want to examine $\text{Area}''(0)$. If $\text{Area}''(0) \geq 0$ for all variations F with fixed boundary, then M is a local minimum for area, and we say that M is a *stable* minimal submanifold.

The *second variation formula* for area is (see [8, p. 16-17])

$$\text{Area}''(0) = -\int_M g(F_t, LF_t)\,.$$

Here, L is a self-adjoint operator called the *stability operator* (or *Jacobi operator*), defined on a vector field X which is normal to M by (see [8, p. 17])

$$LX = \triangle_M^{\text{norm}} X + \sum_{i=1}^{k} R_N(X, E_i)E_i + \widetilde{A}(X)\,,$$

where E_i is an orthonormal basis for T_xM, $\widetilde{A}$ is *Simons' operator*, defined by

$$\widetilde{A}(X) = \sum_{i,j=1}^{k} g(A(E_i, E_j), X)A(E_i, E_j)\,,$$

where A is the second fundamental form of M, and $\triangle_M^{\text{norm}}$ is the Laplacian on the normal bundle (see [8, p. 18]). Thus, M is stable if $-\int_M g(F_t, LF_t) \geq 0$ for all variations F with fixed boundary.

If M is a minimal hypersurface with trivial normal bundle, then the stability operator can be simplified. Let ν be a smooth unit normal on M. Then, normal variations are of the form $\varphi\nu$, where φ is a smooth test function. Writing $L\varphi$ as shorthand for $L(\varphi\nu)$, we get (see [8, p. 18])

$$L\varphi = \triangle_M \varphi + |A|^2\varphi + \text{Ric}(\nu)\varphi\,,$$

where Ric is the Ricci curvature of N. Then, for M to be stable, we must have

$$-\int_M \varphi L\varphi \geq 0$$

for all φ. Writing Area(φ) instead of Area(t) for Area$(F(M,t))$ in the special case $F_t = \varphi\nu$, we have

$$\begin{aligned}
\text{Area}''(\varphi) &= -\int_M \varphi L\varphi \\
&= -\int_M \varphi\triangle_M\varphi + (|A|^2 + \text{Ric}(\nu))\varphi^2 \\
&= \int_M |\nabla^M\varphi|^2 - (|A|^2 + \text{Ric}(\nu))\varphi^2\,,
\end{aligned}$$

where ∇^M is the connection on M. Thus, if M is stable, we must have

$$\int_M \left(\inf_N \text{Ric}_N + |A|^2\right)\varphi^2 \leq \int_M |\nabla^M\varphi|^2$$

for all smooth test functions φ; this is the *stability inequality* (see [8, Lemma 1.20]).

2.3 Some Relevant Results

We will now discuss some results for minimal surfaces in 3-manifolds which, although not fundamental results in the field of minimal surfaces, will nonetheless play important roles in this paper.

There are a number of curvature estimates for minimal surfaces, which give a uniform upper bound for the norm squared of the second fundamental form of the minimal surface within sufficiently small balls. For example, Choi and Schoen proved a curvature estimate in [4] in the case where the minimal surface has small total curvature, and Heinz proved one in [17] for minimal graphs. We are most interested in this paper in a curvature estimate, originally due to Schoen in [26, Thm. 3], for orientable stable minimal surfaces in 3-manifolds of bounded sectional curvature. The version we will use is due to Colding and Minicozzi in [10].

Theorem 2.3 (Schoen Curvature Estimate). *If $M^2 \subset B_{r_0} = B_{r_0}(y) \subset N^3$ is an immersed stable minimal surface with trivial normal bundle, where $|sec_N| \leq k$, $\partial M \subset \partial B_{r_0}$, and r_0 is sufficiently small (depending only on k), then there exists a constant K, depending only on k, such that for all $0 < \sigma \leq r_0$,*

$$\sup_{B_{r_0-\sigma}} |A|^2 \leq \frac{K}{\sigma^2},$$

where A is the second fundamental form of M.

Note that, if the Schoen Curvature Estimate applies to M in a ball of radius r, then M is graphical in smaller balls, i.e., $M \cap B_{r/2}$ is a collection of stable minimal graphs.

Another result which will be used extensively in this paper is an area bound of Choi and Wang for compact orientable embedded minimal surfaces in compact orientable 3-manifolds of positive Ricci curvature. This is actually a consequence of Choi and Wang's main result, a uniform first eigenvalue estimate. This eigenvalue estimate, Theorem 2 in [5], says that if M is a compact orientable embedded minimal hypersurface of a compact orientable Riemannian manifold N with $\mathrm{Ric}_N \geq \kappa > 0$, then $\lambda_1(M) \geq \kappa/2$.

Although this first eigenvalue estimate holds in any dimension, we are interested in the case where N is a 3-manifold. Then, Choi and Wang applied a result of Yang and Yau (see [29]), which states that if M is an orientable Riemann surface of genus γ, then $\lambda_1(M)\mathrm{Area}(M) \leq 8\pi(\gamma + 1)$. Thus, they obtained the following area bound, Proposition 4 in [5].

Proposition 2.4 (Choi/Wang Area Bound). *Let M be a compact orientable embedded minimal surface of genus γ in a compact orientable 3-manifold N with*

$Ric_N \geq \kappa > 0$. *Then,*

$$Area(M) \leq \frac{16\pi}{\kappa}(\gamma + 1)\,.$$

This area bound depends on the topology of M. In Section 4, we will discuss an area bound of Ross (see [24]) in which he shows that, given the additional assumption that M is stable, there is an area bound depending only on κ, i.e., independent of the topology of M.

In [4], Choi and Schoen proved a compactness result for compact embedded minimal surfaces of fixed genus in a compact 3-manifold with positive Ricci curvature. The proof of this result depends on a curvature estimate, which was briefly mentioned earlier and is given here, for compact immersed minimal surfaces with small total curvature.

Proposition 2.5 ([4], p. 389). *Let N be a three-dimensional Riemannian manifold. Let $Q \in N$ and $r > 0$ such that $B_r(Q)$ has compact closure in N. Suppose M is a compact immersed minimal surface (with possibly empty boundary) in N such that $Q \in M$ and $\partial M \cap B_r(Q) = \emptyset$. There exists $\epsilon > 0$ depending only on the geometry of $B_r(Q)$ such that if*

$$\int_{M \cap B_r(Q)} |A|^2 \leq \epsilon,$$

and $r \leq \epsilon$, then

$$\max_{0 \leq \sigma \leq r} \sigma^2 \sup_{B_{r-\sigma}(Q)} |A|^2 \leq C,$$

where C is a constant depending only on the geometry of $B_r(Q)$, i.e., C is independent of M.

The proof of this curvature estimate involves a rescaling argument and the mean value inequality for the Laplacian.

We now state Choi and Schoen's compactness result.

Theorem 2.6 ([4], p. 388). *Let N be a compact 3-manifold with positive Ricci curvature. Then, the space of compact embedded minimal surfaces of fixed topological type in N is compact in the C^k topology for any $k \geq 2$.*

Thus, given a compact 3-manifold N with positive Ricci curvature, any sequence of compact embedded minimal surfaces in N of fixed genus must have a convergent subsequence.

As part of the proof of this compactness result, Choi and Schoen use a universal covering argument. They then use a similar argument to prove the following generalizations of the Choi and Wang eigenvalue estimate and area bound (Proposition 2.4), in which the assumptions of orientability can be dropped. As before, the eigenvalue estimate holds for minimal hypersurfaces in any dimension, while the area bound is specifically for minimal surfaces embedded in 3-manifolds.

Theorem 2.7 ([4], p. 393). *Let M be a compact embedded minimal hypersurface of a compact n-dimensional manifold N with Ricci curvature bounded below by a positive constant κ. Then, $\lambda_1(M) \geq \kappa/2$, where $\lambda_1(M)$ is the first Neumann eigenvalue of the Laplacian of M.*

Corollary 2.8 ([4], p. 393). *Let M and N be as in Theorem 2.7. Assume also that $\dim N$=3. Then,*

$$Area(M) \leq \frac{16\pi}{\kappa}\left(\frac{2}{d} - \frac{1}{2}\chi(M)\right),$$

where $\chi(M)$ is the Euler characteristic of M, and d is the order of $\pi_1(N)$.

Note that, as with the Choi/Wang Area Bound (Proposition 2.4), this area bound is not uniform, as it depends on the topology of M.

2.4 Existence and Embeddedness of Area-Minimizers

Given a compact surface embedded in a 3-manifold, it would be helpful to know whether we can find a surface which minimizes area among all surfaces that share a particular characteristic with the given compact surface. For our purposes in Section 3, that characteristic will be homotopy class, i.e., we will seek a minimal surface which minimizes area, and hence is stable, among all surfaces that induce the same action on the fundamental group of a given surface. In this section, we state some results which give sufficient conditions to ensure existence of area-minimizers, often with the added benefit that the resulting surface is embedded.

For 2-spheres, we have the following. The existence result is due to Sacks and Uhlenbeck in [25], while the embeddedness result was given by Meeks and Yau in [21].

Theorem 2.9 ([25], [21]). *Given any 3-manifold* Ω*, there exist conformal (stable) minimal immersions* $u_1, \ldots, u_m : S^2 \to \Omega$ *which generate* $\pi_2(\Omega)$ *as a* $\mathbb{Z}[\pi_1(\Omega)]$ *module, and such that:*

(i) If $u : S^2 \to \Omega$ *and* $[u]_{\pi_2} \neq 0$*, then* $Area(u) \geq \min_i Area(u_i)$*.*

(ii) Each u_i *is either an embedding or a 2-1 map onto an embedded 2-sided* $\mathbb{R}P^2$*.*

Note that, for spheres, we need to look at π_2, since $\pi_1(S^2)$ is trivial. For higher genus surfaces, we can look at π_1. The following results of Schoen and Yau (for existence) and Freedman, Hass, and Scott (for embeddedness) will be of great use to us in Section 3. They are crucial components in the proof of Theorem 1.1.

Theorem 2.10 ([27], p. 127). *Let* Ω *be a compact 3-manifold and* Σ_γ *a compact Riemann surface of genus* γ*. let* $f : \Sigma_\gamma \to \Omega$ *be a continuous map such that the induced map* $f_\# : \pi_1(\Sigma_\gamma) \to \pi_1(\Omega)$ *is injective. Then, there exists a minimal immersion* $h : \Sigma_\gamma \to \Omega$ *such that* $h_\# = f_\#$ *on* $\pi_1(\Sigma_\gamma)$ *and the induced area of* h *is least among all maps with the same action on* $\pi_1(\Sigma_\gamma)$*.*

The conditions of Theorem 2.10 require that $\gamma > 0$; the theorem does not hold for 2-spheres. The same is true of the corresponding embeddedness result.

Theorem 2.11 ([16]). *Let* Ω*,* Σ_γ*, and* f *be as in Theorem 2.10. If* f*, in addition to inducing an injective map* $f_\#$*, is also an embedding of* Σ_γ *into* Ω*, then there exists a minimal embedding* $h : \Sigma_\gamma \to \Omega$ *such that* $h_\# = f_\#$ *on* $\pi_1(\Sigma_\gamma)$ *and the induced area of* h *is least among all maps with the same action on* $\pi_1(\Sigma_\gamma)$*.*

Our use of these results in Section 3 will be as follows. We will construct embeddings of Σ_γ (for fixed $\gamma \geq 1$) into a particular 3-manifold, such that the induced maps on fundamental groups will be injective. Theorems 2.10 and 2.11 will then give us the existence of stable embedded minimal surfaces with the same actions on π_1 as the surfaces we constructed. The main reason that the result of Theorem 1.1 is an open question for the genus zero case is that Theorems 2.10 and 2.11 don't hold when $\gamma = 0$. The result of Theorem 2.9, which is for spheres, is a weaker result than Theorems 2.10 and 2.11. We will discuss this further in Section 3.5.

3 Compact Embedded Minimal Surfaces of Arbitrary Area

Throughout this section, we use the C^2-topology on the space of metrics on a manifold. Our goal is to prove our first main result, Theorem 1.1, which says that given any 3-manifold N (possibly with boundary) and any positive integer γ, there exists an open nonempty set of metrics on N for each of which there are stable compact embedded minimal surfaces of genus γ with arbitrarily large area.

Although the theorem ensures that there are "many" metrics for which we can embed stable compact genus γ minimal surfaces of arbitrarily large area, the result is false for a large class of metrics. Namely, the Choi/Wang Area Bound and its generalization, Corollary 2.8, show that for any metric in which N has Ricci curvature bounded below by a positive constant, there is an upper bound on the area of compact embedded minimal surfaces of genus γ, depending on γ and the lower bound for Ric_N. In fact, as we will see, the Ross Area Bound (Theorem 4.2) shows that for stable minimal surfaces, the area is bounded above uniformly by a constant depending only on the lower bound for Ric_N.

Colding and Minicozzi (see [7]) have already proved Theorem 1.1 for $\gamma = 1$. We will give their argument in Section 3.2. In Section 3.3, we will prove the theorem for $\gamma = 2$, with an argument borrowing heavily from the genus one case. The theorem will then be extended easily to genus greater than two in Section 3.4. Much of the material in the remainder of Section 3, with the exception of Sections 3.2 and 3.5, also appears in [12], and appears here with kind permission of Kluwer Academic Publishers.

It remains an open question as to whether or not the theorem remains valid for genus zero, i.e., embedded minimal 2-spheres. That is, given any 3-manifold N, do there exist stable embedded minimal spheres in N of arbitrarily large area? If not, does the result at least hold for some 3-manifolds N? We will briefly discuss the genus zero case in Section 3.5.

3.1 Strictly Mean Convex Metrics

Theorem 1.1 says that our result holds for an open nonempty set of metrics on our arbitrary 3-manifold N. Specifically, we will be embedding a 3-manifold Ω with boundary in N, and will be looking at the set of metrics on N in which Ω is strictly mean convex. In general, if Ω is a compact Riemannian manifold with boundary and dimension $n \geq 3$, then Ω is *strictly mean convex* in a given metric if the mean curvature with respect to that metric is strictly positive on $\partial\Omega$. This is the higher-dimensional analogue of a 2-manifold having a convex boundary.

To show that this set of metrics is open and nonempty, we prove the following Proposition, whose proof is inspired by a calculation in [13, p. 4-5].

Proposition 3.1. *Let Ω^n be a compact Riemannian manifold with boundary and dimension $n \geq 3$. Then, the set of metrics on Ω in which Ω is strictly mean convex is open and nonempty.*

Proof. The set of such metrics is clearly open, by the definition of strictly mean convex. To show it is nonempty, let g be any metric on Ω, and let $\widetilde{g} = e^{2f}g$ be a metric conformally related to g. Let $\{e_1, ..., e_n\}$ be a framing for Ω so that $g_{ij} = \delta_{ij}$ and e_n is the unit normal to $\partial\Omega$ in g (and therefore, $e^{-f}e_n$ is the unit normal to $\partial\Omega$ in $\widetilde{g}$). Fix a point $p \in \partial\Omega$, and choose coordinates $\{x_1, ..., x_n\}$ at p so that, at p, $e_i = \frac{\partial}{\partial x_i}$ for all i. Then, the second fundamental form of $\partial\Omega$ in g at p is given by, for $i, j = 1, ..., n-1$,

$$h_{ij} = g(\nabla_{e_i} e_j, e_n) = \sum_{k=1}^{n} g(\Gamma_{ij}^k e_k, e_n) = \Gamma_{ij}^n ,$$

and the mean curvature of $\partial\Omega$ in g at p is given by

$$h = \frac{1}{n-1} \sum_{i,j=1}^{n-1} g^{ij} h_{ij} = \frac{1}{n-1} \sum_{i,j=1}^{n-1} \delta_{ij} \Gamma_{ij}^n = \frac{1}{n-1} \sum_{i=1}^{n-1} \Gamma_{ii}^n .$$

The second fundamental form of $\partial\Omega$ in $\widetilde{g}$ at p is given by, for $i, j = 1, ..., n-1$,

$$\widetilde{h}_{ij} = \widetilde{g}(\widetilde{\nabla}_{e_i} e_j, e^{-f} e_n) = e^{2f} g(\widetilde{\nabla}_{e_i} e_j, e^{-f} e_n) = \sum_{k=1}^{n} e^f g(\widetilde{\Gamma}_{ij}^k e_k, e_n) = e^f \widetilde{\Gamma}_{ij}^n .$$

Now,

$$\widetilde{\Gamma}_{ij}^{n} = \frac{1}{2}\sum_{l=1}^{n}(\widetilde{g}_{jl,i} + \widetilde{g}_{il,j} - \widetilde{g}_{ij,l})\,\widetilde{g}^{ln}\,,$$

where, for example,

$$\widetilde{g}_{jl,i} = \frac{\partial}{\partial x_i}\,\widetilde{g}_{jl}\,.$$

So, at p,

$$\begin{aligned}
\widetilde{\Gamma}_{ij}^{n} &= \frac{1}{2}e^{-2f}\sum_{l=1}^{n}(\widetilde{g}_{jl,i} + \widetilde{g}_{il,j} - \widetilde{g}_{ij,l})\,g^{ln} \\
&= \frac{1}{2}e^{-2f}\sum_{l=1}^{n}(\widetilde{g}_{jl,i} + \widetilde{g}_{il,j} - \widetilde{g}_{ij,l})\,\delta_{ln} \\
&= \frac{1}{2}e^{-2f}(\widetilde{g}_{jn,i} + \widetilde{g}_{in,j} - \widetilde{g}_{ij,n}) \\
&= \frac{1}{2}e^{-2f}\left(2\frac{\partial f}{\partial x_i}e^{2f}g_{jn} + e^{2f}g_{jn,i} + 2\frac{\partial f}{\partial x_j}e^{2f}g_{in} + e^{2f}g_{in,j}\right. \\
&\qquad \left. - 2\frac{\partial f}{\partial x_n}e^{2f}g_{ij} - e^{2f}g_{ij,n}\right) \\
&= \frac{1}{2}(g_{jn,i} + g_{in,j} - g_{ij,n}) + \frac{\partial f}{\partial x_i}\delta_{jn} + \frac{\partial f}{\partial x_j}\delta_{in} - \frac{\partial f}{\partial n}\delta_{ij} \\
&= \Gamma_{ij}^{n} - \frac{\partial f}{\partial n}\delta_{ij}
\end{aligned}$$

since $i, j < n$, where $\frac{\partial f}{\partial n}$ is the normal derivative of f with respect to the unit normal e_n. Therefore, we have

$$\widetilde{h}_{ij} = e^f\widetilde{\Gamma}_{ij}^{n} = e^f\Gamma_{ij}^{n} - e^f\frac{\partial f}{\partial n}\delta_{ij} = e^f\Gamma_{ij}^{n} - \frac{\partial}{\partial n}(e^f)\delta_{ij}\,.$$

The mean curvature of $\partial\Omega$ in $\widetilde{g}$ at p is then given by

$$\begin{aligned}
\widetilde{h} &= \frac{1}{n-1}\sum_{i,j=1}^{n-1}\widetilde{g}^{ij}\widetilde{h}_{ij} \\
&= \frac{1}{n-1}\sum_{i,j=1}^{n-1}e^{-2f}\delta_{ij}\left(e^f\Gamma_{ij}^{n} - \frac{\partial}{\partial n}(e^f)\delta_{ij}\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n-1}\left(e^{-f}\Gamma_{ii}^{n} - e^{-2f}\frac{\partial}{\partial n}(e^f)\right) \\
&= e^{-f}\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\Gamma_{ii}^{n}\right) - e^{-2f}e^f\frac{\partial f}{\partial n} \\
&= e^{-f}\left(h - \frac{\partial f}{\partial n}\right)\,.
\end{aligned}$$

We have shown that this relation holds at an arbitrarily chosen point of $\partial\Omega$, and so it holds everywhere on $\partial\Omega$ since all quantities involved are tensorial. Let m be the minimum of h on $\partial\Omega$, which exists since $\partial\Omega$ is compact. Choose f so that $\frac{\partial f}{\partial n} < m$ everywhere on $\partial\Omega$ and $f \equiv 0$ outside a small tubular neighborhood around $\partial\Omega$. Then, $\widetilde{g} = g$ except for a small tubular neighborhood around $\partial\Omega$, and $\widetilde{h} > 0$ everywhere on $\partial\Omega$, so $\widetilde{g}$ is a metric in which Ω is strictly mean convex. □

3.2 Colding and Minicozzi's Proof For Genus 1

In this section, we give Colding and Minicozzi's proof of Theorem 1.1 in the case $\gamma = 1$. Almost all of the material in this section is taken from Colding and Minicozzi's paper [7].

The main way in which Colding and Minicozzi's argument does not directly extend to the higher genus case is as follows. Instead of working with embedded minimal tori in 3-manifolds, they begin by looking at embedded geodesics of arbitrary length in planar domains (see Proposition 3.3). They then take the cross product of this with S^1, obtaining minimal tori of arbitrarily large area in 3-manifolds. The areas are obviously arbitrarily large, since the lengths of the cross-sections of the minimal tori approach infinity. This idea can not be extended to higher genus: a genus γ surface for $\gamma \geq 2$ can not be written as a cross product with S^1. Thus, it will be harder to show that the stable minimal genus γ surfaces we construct in Sections 3.3 and 3.4 have arbitrarily large area. To prove that in fact they do, we will need a lemma (see Lemma 3.4).

As just mentioned, Colding and Minicozzi begin by looking at embedded geodesics in planar domains. A *planar domain* S is a disc with finitely many subdiscs removed. The fundamental group $\pi_1(S)$ is a free group with one generator for each interior boundary component. For $\alpha \in \pi_1(S)$, we can define the *word metric* $|\alpha|$ by simply counting the number of letters in the word represented by α. For example, if the generators of $\pi_1(S)$ are a, b, and c, then $|a^2ba^{-1}cb^{-1}| = 6$, $|b^3c^2(ab)^{-2}| = 9$, and so on. Let $\pi_1^0(S)$ be the set of conjugacy classes of elements in $\pi_1(S)$ which can be represented by a closed embedded S^1. For $\alpha \in \pi_1^0(S)$, we set

$$|\alpha| \equiv \inf_{\widehat{\alpha} \in \alpha} |\widehat{\alpha}| \, .$$

Colding and Minicozzi begin by proving the following lemma, Lemma 1.3 in [7].

Lemma 3.2. *If S is a connected planar domain with at least three interior boundary components, then*

$$\sup_{\alpha \in \pi_1^0(S)} |\alpha| = \infty \, .$$

In the proof of this lemma, Colding and Minicozzi explicitly construct the desired conjugacy classes. We will use the same construction for one of the generators of the fundamental groups of our higher genus surfaces in Sections 3.3 and 3.4.

Proof [7, p. 1098-1099]. Let $a, b, c \in \pi_1(S)$ be the generators corresponding to one clockwise rotation around each of the punctures. The conjugacy classes of the following elements have the desired properties:

$$(ba)^n c (ba)^{-n} a^{-1} ba \, .$$

Roughly speaking, the loops enclose the second puncture, then circle the first two punctures together n times, hook around the third puncture, and finally reverse their path around the first two punctures n times in the opposite direction. Since $\pi_1(S)$ is a free group, it is easy to see that the above representatives actually minimize the word metric (with respect to these generators) for their conjugacy classes. Since

$$|(ba)^n c (ba)^{-n} a^{-1} ba| = 4n + 4 \, ,$$

we see, by letting $n \to \infty$,

$$\sup_{\alpha \in \pi_1^0(S)} |\alpha| = \infty \, ,$$

as desired. □

We need this lemma to help prove the following Proposition (see [7, Prop. 1.1]), which is Theorem 1.1 in one dimension less, i.e., for closed embedded geodesics in 2-manifolds.

Proposition 3.3. *Given any surface M^2, there exists an open nonempty set of metrics on M for each of which there exist closed embedded geodesics with arbitrarily large length.*

Proof [7, p. 1099]. Let $S \subset M$ be a connected planar domain with three interior boundary components. Let g be a metric on M in which S is strictly convex. The set of such metrics is clearly open and nonempty in the C^2-topology. By Lemma 3.2 and [15], there are closed embedded geodesics γ_i in (S, g) representing classes $\alpha_i \in \pi_1^0(S)$ with $|\alpha_i| \to \infty$ Consequently, we have that the lengths of the γ_i's must also be unbounded. □

Colding and Minicozzi then extend everything to one higher dimension by crossing everything with S^1. Let S be a connected planar domain with three interior boundary components. Then $\Omega_1 = S \times S^1$ is a solid torus with three solid tori removed. So, Ω_1 can be embedded in any N^3. Given $\alpha \in \pi_1^0(S)$, let γ_α be a simple closed curve in S representing α, and set $\Sigma_\alpha = \gamma_\alpha \times S^1$. By construction, Σ_α is an incompressible torus in Ω_1 (see [16]).

Proof of Theorem 1.1 for $\gamma = 1$ [7, p. 1099]. Given any 3-manifold N, we can embed Ω_1 in N. Let g be a metric on N so that Ω_1 is strictly mean convex (by Proposition 3.1, the set of such metrics is open and nonempty). By Lemma 3.2, we may choose classes $\alpha_i \in \pi_1^0(S)$ with $|\alpha_i| \to \infty$ and corresponding embedded incompressible tori Σ_{α_i} in Ω_1. By Theorem 2.10, there are immersed least-area minimal tori $\Gamma_i \subset \Omega_1$ with $\Gamma_i \cap \partial\Omega_1 = \emptyset$ so that Γ_i and Σ_{α_i} induce the same mapping from $\mathbb{Z}^2$ to $\pi_1(\Omega_1)$ for each i. Since the Σ_{α_i} are embedded, Theorem 2.11 implies that the Γ_i are embedded. The action on $\pi_1(\Omega_1)$ implies that the areas of the Γ_i are unbounded. □

3.3 The Genus 2 Case

Let Σ_2 denote the standard genus two surface. This has fundamental group

$$\pi_1(\Sigma_2) = < x_1, y_1, x_2, y_2 \,|\, x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} > ,$$

where x_1 and x_2 are freely homotopic to meridians of the two handles, where the meridians have the same orientation, and y_1 and y_2 are freely homotopic to lines of latitude of the two handles, where the lines of latitude have the same orientation. Let Ω_2 be a solid genus two surface with a solid genus two surface and two solid tori removed, where the solid tori lie in the same handle of the ambient genus two surface. This can be pictured as in Figure 1, where the top and bottom of the picture are identified.

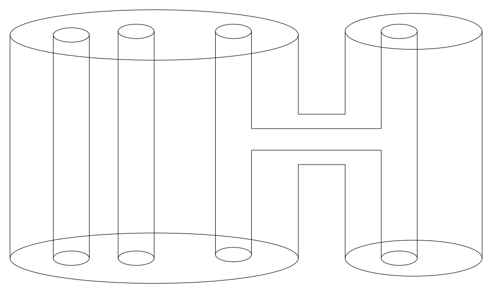

Figure 1: Ω_2. The top and bottom of the picture are identified.

The fundamental group of Ω_2 is

$$\pi_1(\Omega_2) = < a, b, c_1, d_1, c_2, d_2 \,|\, ad_1a^{-1}d_1^{-1},\ bd_1b^{-1}d_1^{-1},\ c_1d_1c_1^{-1}d_1^{-1}c_2d_2c_2^{-1}d_2^{-1} >,$$

where the generators are as follows:

(i) a and b are freely homotopic to meridians of the two removed solid tori (clockwise rotation around the two removed solid tori in Figure 1).

(ii) c_1 is freely homotopic to a meridian of the handle of the removed solid genus two in the same handle of the ambient solid genus two as the removed solid tori (clockwise rotation around the left handle of the removed solid genus two in Figure 1).

(iii) d_1 is freely homotopic to a line of latitude of the left handle of the ambient solid genus two in Figure 1.

(iv) c_2 is freely homotopic to a meridian of the removed solid genus two in the other handle from the meridian which is freely homotopic to c_1 (clockwise rotation around the right handle of the removed solid genus two in Figure 1).

(v) d_2 is freely homotopic to a line of latitude of the right handle of the ambient solid genus two in Figure 1, with the same orientation as the line of latitude which is freely homotopic to d_1.

To prove Theorem 1.1, we will need the following lemma. We will also use this lemma in Section 4.

Lemma 3.4. *Let Ω^3 be a compact Riemannian manifold, and let $\{M_n\} \subset \Omega$ be a sequence of stable, compact, connected, embedded minimal surfaces without boundary such that the following conditions hold:*

(i) *there exists a constant $C_1 > 0$ such that $Area(M_n) \leq C_1$ for all n.*

(ii) *there exists a constant $C_2 > 0$ such that*

$$\sup_{M_n} |A_n|^2 \leq C_2$$

for all n, where A_n is the second fundamental form of M_n.

Then, a subsequence of $\{M_n\}$ converges to a compact, connected, embedded minimal surface without boundary $M \subset \Omega$ of finite multiplicity.

Proof. Take a finite covering $\{B_r(y_j)\}$ of Ω so that $\{B_{r/2}(y_j)\}$ is still a covering of Ω. Then, by [9], a subsequence of $\{M_n\}$ converges in each $B_{r/2}(y_j)$ to a lamination M with minimal leaves. By taking a diagonal subsequence, we have a subsequence of $\{M_n\}$, which we still call $\{M_n\}$, converging to M everywhere. M is clearly minimal, and it is embedded by the maximum principle.

We claim that the number of leaves of M_n in each $B_r(y_j)$ which intersect $B_{r/2}(y_j)$ has an upper bound which is uniform in n and j. Let $\Gamma_{n,j}$ be any such leaf. Then, there exists $x_j \in \Gamma_{n,j} \cap \partial B_{r/2}(y_j)$. So, $B_{r/2}(x_j) \subset B_r(y_j)$, and by monotonicity of area, there exists a constant $C > 0$ so that

$$\text{Area}(\Gamma_{n,j} \cap B_{r/2}(x_j)) \geq C \left(\frac{r}{2}\right)^2 .$$

So, each $\Gamma_{n,j}$ is of at least some fixed positive area, and so the area bound (i) gives an upper bound for the number of such leaves which is uniform in n and j. We can take a subsequence so that the number of leaves of M_n is the same in each $B_{r/2}(y_j)$ for all n, j. Then, the limit M must have finite multiplicity, although the multiplicity may be different in each connected component of M. We have shown that each connected component of M is a closed surface. The diameter of M is bounded, since M is covered by finitely many balls $B_{r/2}(y_j) \cap M$. So, M is compact. M is without boundary since each M_n is without boundary.

It remains to show that M is connected, which would imply that $M_n \to M$ with fixed finite multiplicity. Suppose M is not connected, and let A and B be distinct connected components of M. Then, $\epsilon = \text{dist}(A, B) > 0$. Let $R = \left\{x \in \Omega | \frac{\epsilon}{3} < \text{dist}(x, A) < \frac{2\epsilon}{3}\right\}$. So, R is disjoint from both A and B. Since $M_n \to M$, for large enough n we have $M_n \cap A \neq \emptyset$ and $M_n \cap B \neq \emptyset$, but $M_n \cap R = \emptyset$, contradicting the connectedness of M_n. So, M is connected.

Therefore, M_n converges to a compact, connected, embedded minimal surface without boundary $M \subset \Omega$ of finite multiplicity. □

Proof of Theorem 1.1 for $\gamma = 2$.

Given any three-manifold N, we can embed Ω_2 in N. Choose a metric g on N so that Ω_2 is strictly mean convex (by Proposition 3.1, the set of such g's is open and nonempty). Let $f_n : \Sigma_2 \to \Omega_2$ be a map such that the induced map $f_{n\#} : \pi_1(\Sigma_2) \to \pi_1(\Omega_2)$ is the following:

$$\begin{aligned} f_{n\#}(x_1) &= (ba)^n c_1 (ba)^{-n} a^{-1} ba \\ f_{n\#}(y_1) &= d_1 \\ f_{n\#}(x_2) &= c_2 \\ f_{n\#}(y_2) &= d_2 \,. \end{aligned}$$

It is easy to see that there exist such maps f_n which are embeddings.

One can check that $f_{n\#}(x_1)$ minimizes the word metric for its conjugacy class: by conjugating $f_{n\#}(x_1)$ by any element of $\pi_1(\Omega_2)$ and using the relations of $\pi_1(\Omega_2)$, one can not decrease the length of $f_{n\#}(x_1)$ in the word metric.

So, for each n, we have an embedded incompressible genus two surface $\Sigma_{2,n} = f_n(\Sigma_2)$. By Theorem 2.10, there are immersed least-area minimal genus two surfaces

$\Gamma_{2,n} \subset \Omega_2$ with $\Gamma_{2,n} \cap \partial\Omega_2 = \emptyset$ so that $\Gamma_{2,n}$ and $\Sigma_{2,n}$ induce the same mapping from $\pi_1(\Sigma_2)$ to $\pi_1(\Omega_2)$ for each n. Since the $\Sigma_{2,n}$ are embedded, Theorem 2.11 implies that the $\Gamma_{2,n}$ are embedded.

We claim that the areas of the $\Gamma_{2,n}$ are unbounded. Assume not. Then, there exists a constant $C_1 > 0$ such that $\text{Area}(\Gamma_{2,n}) \leq C_1$ for all n. The $\Gamma_{2,n}$ are stable since they are area-minimizing. So, by the Schoen Curvature Estimate, there exists a constant $C_2 > 0$ such that, for small enough r and all $\sigma \in (0, r]$,

$$\sup_{B_{r-\sigma}} |A_n|^2 \leq \frac{C_2}{\sigma^2}$$

for all n and all balls $B_{r-\sigma} \subset \Omega_2$, where A_n is the second fundamental form of $\Gamma_{2,n}$. Since the $\Gamma_{2,n}$ are all without boundary, we get a uniform curvature estimate on all of $\Gamma_{2,n}$, instead of just on balls. Therefore, by Lemma 3.4, a subsequence of $\{\Gamma_{2,n}\}$ converges to a compact, connected, embedded minimal surface without boundary $\Gamma_2 \subset \Omega_2$ of finite multiplicity.

For large n, the $\Gamma_{2,n}$ are coverings of Γ_2 by the maximum principle, and the degree of the covering is proportional to n. Let $n \to \infty$. Then, Γ_2 has infinite multiplicity, a contradiction. Therefore, the areas of the $\Gamma_{2,n}$ are unbounded. □

3.4 The General Case: $\gamma \geq 2$

We now move to the general case. The arguments for fixed genus $\gamma \geq 2$ are essentially the same as in the genus 2 case.

Let Σ_γ denote the standard genus γ surface, $\gamma \geq 2$. This has fundamental group

$$\pi_1(\Sigma_\gamma) = < x_1, y_1, \ldots, x_\gamma, y_\gamma \,|\, [x_1 y_1] \cdots [x_\gamma y_\gamma] > ,$$

where the x_i are freely homotopic to meridians of the handles, all with the same orientation, the y_i are freely homotopic to lines of latitude of the handles, all with the same orientation, and $[x_i y_i] = x_i y_i x_i^{-1} y_i^{-1}$ for $i = 1, \ldots, \gamma$. Let Ω_γ be a solid genus γ surface with a solid genus γ surface and two solid tori removed, where the solid tori both lie in one of the end handles of the ambient genus γ surface (the case $\gamma = 3$ is shown in Figure 2, where the top and bottom of the picture are identified).

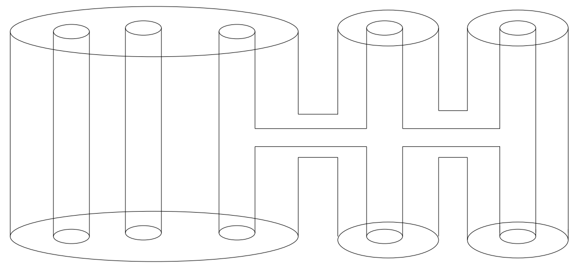

Figure 2: Ω_3. The top and bottom of the picture are identified.

The fundamental group of Ω_γ is

$$\pi_1(\Omega_\gamma) = < a, b, c_1, d_1, \ldots, c_\gamma, d_\gamma \,|\, [ad_1],\, [bd_1],\, [c_1 d_1] \cdots [c_\gamma d_\gamma] > ,$$

where the generators are defined as in the case $\gamma = 2$ (so, a, b, and all c_i are freely homotopic to meridians with the same orientation, and all d_i are freely homotopic to lines of latitude with the same orientation).

Proof of Theorem 1.1.

Given any three-manifold N, we can embed Ω_γ in N. Choose a metric g on N so that Ω_γ is strictly mean convex (by Proposition 3.1, the set of such g's is open and nonempty). Let $f_n : \Sigma_\gamma \to \Omega_\gamma$ be a map such that the induced map $f_{n\#} : \pi_1(\Sigma_\gamma) \to \pi_1(\Omega_\gamma)$ is the following:

$$\begin{aligned} f_{n\#}(x_1) &= (ba)^n c_1 (ba)^{-n} a^{-1} ba \\ f_{n\#}(x_i) &= c_i \text{ for } i = 2, \ldots, \gamma \\ f_{n\#}(y_i) &= d_i \text{ for } i = 1, \ldots, \gamma\,. \end{aligned}$$

It is easy to see that there exist such maps f_n which are embeddings.

The proof then proceeds exactly as in the genus 2 case. Theorems 2.10 and 2.11, and the Schoen Curvature Estimate, again apply. □

3.5 Open Question: The Genus Zero Case

It is unclear whether the result of Theorem 1.1 holds when $\gamma = 0$. Specifically, we pose the following open question.

Question 3.5. *Given a 3-manifold N, does there exist an open nonempty set of metrics on N for each of which there are stable embedded minimal spheres with arbitrarily large area? If not for all N, is it true for at least some N?*

The argument used to prove Theorem 1.1 does not apply to the genus zero case. As mentioned before, Theorems 2.10 and 2.11, which give us existence and embeddedness of area-minimizers, hold only when $\gamma > 0$. The corresponding result for existence and embeddedness of area-minimizing spheres, which we stated as Theorem 2.9, is weaker.

One problem which arises with the genus zero case is the following. Consider the analogous situation in one dimension less, that is, geodesic 1-spheres in 2-manifolds. If we look at 1-spheres going around a cylinder, the shortest S^1 is a horizontal slice of the cylinder, i.e, the generator of the fundamental group of the cylinder. Taking this to have length one, the shortest S^1 going around the cylinder of length n will just be the horizontal slice with multiplicity n; that is, the lengths of the geodesic 1-spheres approach infinity (as n approaches infinity) as immersed curves, but the lengths stay fixed as embedded curves. This multiplicity problem can also arise for minimal 2-spheres in 3-manifolds. If we look at $S^2 \times \mathbb{R}$, the cylinder over S^2, then if the generator of $\pi_2(S^2 \times \mathbb{R})$ has area one, the least-area 2-sphere of area n will just be the generator with multiplicity n. The incompressibility assumption in Theorems 2.10 and 2.11 ensures that this is not a problem in the positive genus case, but there are no such assurances when looking at π_2 in the genus zero case.

4 The Ross Area Bound For Stable Minimal Surfaces in 3-manifolds of Positive Ricci Curvature

Let N be an orientable manifold of positive Ricci curvature, $\mathrm{Ric}_N \geq \kappa > 0$, and let $M \subset N$ be a stable compact embedded minimal hypersurface. We would like to show that $\mathrm{Area}(M)$ is bounded above by a constant depending only on κ. In Section 4.1, we give an argument of Ross (see [24]) which proves that such an upper bound exists when M is a 2-dimensional hypersurface of a 3-dimensional manifold N. However, the argument does not easily extend to higher dimensions–it remains an open question whether a uniform area bound exists for higher dimensional stable compact embedded minimal hypersurfaces. In Section 4.2, we examine reasons why Ross' argument cannot be extended to the higher-dimensional case.

It is easy to see that M must be nonorientable. We saw in Section 2.2 that, for an orientable minimal hypersurface, if we let ν be a smooth unit normal, then normal variations are of the form $\varphi\nu$ for smooth test functions φ, and the second variation formula for area becomes

$$\mathrm{Area}''(\varphi) = \int_M |\nabla^M \varphi|^2 - (|A|^2 + \mathrm{Ric}(\nu))\varphi^2,$$

where ∇^M is the connection on M. For a compact orientable embedded minimal hypersurface, we can take $\varphi \equiv 1$, obtaining

$$\mathrm{Area}''(\varphi) = -\int_M (|A|^2 + \mathrm{Ric}(\nu)).$$

Thus, $\mathrm{Area}''(\varphi) < 0$ since N has positive Ricci curvature, and so a compact orientable embedded minimal hypersurface cannot be stable.

4.1 The Case $M^2 \subset N^3$: Ross' Area Bound

Let N be an orientable 3-manifold with $\mathrm{Ric}_N \geq \kappa > 0$, and let $M \subset N$ be a nonorientable compact embedded minimal hypersurface. We wish to show that any such M which is stable has its area bounded above uniformly by a constant depending only on κ. Let $\widetilde{M}$ be the orientable double cover of M. Note that this gives rise to the projection map $\pi : \widetilde{M} \to M$ and the orientation-reversing antipodal map

$I : \widetilde{M} \to \widetilde{M}$, satisfying $\pi \circ I = \pi$. Let ν be a smooth unit normal on $\widetilde{M}$. Then, ν is antisymmetric, i.e., $\nu \circ I = -\nu$. Let $\varphi : M \to \mathbb{R}$ be a smooth test function on M and let $\widetilde{\varphi} = \varphi \circ \pi : \widetilde{M} \to \mathbb{R}$ be the lift of φ. Then, $(\widetilde{\varphi}\nu) \circ I = -\widetilde{\varphi}\nu$. If we choose any map

$$F = (f_1, f_2, f_3) : \widetilde{M} \to S^2 \subset \mathbb{R}^3$$

for which each f_α is antisymmetric, i.e., $f_\alpha \circ I = -f_\alpha$ for $\alpha = 1, 2, 3$ (we say such an F is antisymmetric), then the vector fields $f_\alpha \widetilde{\varphi}\nu$ on $\widetilde{M}$ satisfy

$$(f_\alpha \widetilde{\varphi}\nu) \circ I = f_\alpha \widetilde{\varphi}\nu,$$

and hence project to vector fields on M. So, we will take $f_\alpha \widetilde{\varphi}\nu$, $\alpha = 1, 2, 3$, as normal variations of $\widetilde{M}$. Applying the second variation formula for area and summing over $\alpha = 1, 2, 3$, we obtain (see [24, p. 3096])

$$\begin{aligned} \sum_{\alpha=1}^{3} \text{Area}''(f_\alpha \widetilde{\varphi}) &= \int_{\widetilde{M}} |\widetilde{\nabla}\widetilde{\varphi}|^2 - (|A|^2 + \text{Ric}(\nu))(\widetilde{\varphi})^2 + |\widetilde{\nabla} F|^2 (\widetilde{\varphi})^2 \\ &= 2\int_M |\nabla^M \varphi|^2 - (|A|^2 + \text{Ric}(\nu))\varphi^2 + |\widetilde{\nabla} F|^2 \varphi^2. \end{aligned}$$

(Note that $|\widetilde{\nabla} F|^2$ is well-defined on M.)

We will need the following Lemma, which is Lemma 2 in [24], but originally appears in [19, p. 481].

Lemma 4.1. *Suppose $\widetilde{M}$ is a compact Riemann surface of genus γ with antipodal map $I : \widetilde{M} \to \widetilde{M}$. Then there is an antisymmetric and conformal $F : \widetilde{M} \to S^2$ with*

$$deg\, F \leq \gamma + 1.$$

Consequently, for any metric compatible with the conformal structure of $\widetilde{M}$,

$$\int_{\widetilde{M}} |\widetilde{\nabla} F|^2 = 8\pi (deg\, F) \leq 8\pi(\gamma + 1).$$

If we let $\varphi \equiv 1$ and apply Lemma 4.1, we obtain

$$\begin{aligned} \sum_{\alpha=1}^{3} \text{Area}''(f_\alpha) &\leq 8\pi(\gamma+1) - \int_{\widetilde{M}} (|A|^2 + \text{Ric}(\nu)) \\ &= 8\pi(\gamma+1) + \int_{\widetilde{M}} (2K - \text{Ric}(\tau_1) - \text{Ric}(\tau_2)) \end{aligned}$$

(see [27, p. 139]), where K is the Gaussian curvature of $\widetilde{M}$ and τ_1, τ_2 is an orthonormal basis for T_pM, for any given $p \in M$. By the Gauss-Bonnet Theorem,

$$\begin{aligned}\sum_{\alpha=1}^{3} \text{Area}''(f_\alpha) &\leq 8\pi(\gamma+1) + 8\pi(1-\gamma) - \int_{\widetilde{M}} (\text{Ric}(\tau_1) + \text{Ric}(\tau_2)) \\ &= 16\pi - \int_{\widetilde{M}} (\text{Ric}(\tau_1) + \text{Ric}(\tau_2)).\end{aligned}$$

Since $\text{Ric}_N \geq \kappa > 0$, we get

$$\begin{aligned}\sum_{\alpha=1}^{3} \text{Area}''(f_\alpha) &\leq 16\pi - 2\kappa \, \text{Area}(\widetilde{M}) \\ &= 16\pi - 4\kappa \, \text{Area}(M),\end{aligned}$$

and so if $\text{Area}(M) > \frac{4\pi}{\kappa}$, then M is unstable. We have therefore proved the following, the Ross Area Bound (see [24, p. 3098]).

Theorem 4.2 (Ross Area Bound). *Let N be an orientable 3-manifold with $Ric_N \geq \kappa > 0$, and let $M \subset N$ be a stable compact embedded minimal hypersurface (which is necessarily nonorientable). Then,*

$$Area(M) \leq \frac{4\pi}{\kappa}.$$

4.2 Problems Extending Area Bound to $M^n \subset N^{n+1}$, $n \geq 3$

We now consider the higher-dimensional case. Let N be an orientable $(n+1)$-manifold, $n \geq 3$, with $\text{Ric}_N \geq \kappa > 0$, and let $M \subset N$ be a nonorientable stable compact embedded minimal hypersurface. It remains an open question whether there exists an upper bound for the area of M depending only on κ. We will show that Ross' argument for the case $n = 2$ does not seem to extend to the higher-dimensional case. If a uniform area bound does exist when $n \geq 3$, it will require a different line of argument to prove it.

Initially, we can proceed as Ross does in [24]. Let $\widetilde{M}$ be the orientable double cover of M, and recall that this gives rise to the projection map $\pi : \widetilde{M} \to M$ and the antipodal map $I : \widetilde{M} \to \widetilde{M}$, and that $\pi \circ I = \pi$. Let ν be a smooth unit normal on $\widetilde{M}$. As before, ν is antisymmetric, i.e., $\nu \circ I = -\nu$. Let $\varphi : M \to \mathbb{R}$ be a smooth test function on M and let $\widetilde{\varphi} = \varphi \circ \pi : \widetilde{M} \to \mathbb{R}$ be the lift of φ. Then, $\widetilde{\varphi}\nu$ is antisymmetric. If we choose any map

$$F = (f_1, \ldots, f_n) : \widetilde{M} \to S^{n-1} \subset \mathbb{R}^n$$

for which each f_α is antisymmetric, $\alpha = 1, \ldots, n$, then the vector fields $f_\alpha \widetilde{\varphi}\nu$ on $\widetilde{M}$ satisfy

$$(f_\alpha \widetilde{\varphi}\nu) \circ I = f_\alpha \widetilde{\varphi}\nu,$$

and hence project to vector fields on M. So, we will take $f_\alpha \widetilde{\varphi}\nu$, $\alpha = 1, \ldots, n$, as normal variations of $\widetilde{M}$. Applying the second variation formula for area and summing over $\alpha = 1, \ldots, n$, we obtain (see [24, p. 3096])

$$\begin{aligned} \sum_{\alpha=1}^{n} \text{Area}''(f_\alpha \widetilde{\varphi}) &= \int_{\widetilde{M}} |\widetilde{\nabla}\widetilde{\varphi}|^2 - (|A|^2 + \text{Ric}(\nu))(\widetilde{\varphi})^2 + |\widetilde{\nabla}F|^2(\widetilde{\varphi})^2 \\ &= 2\int_M |\nabla^M \varphi|^2 - (|A|^2 + \text{Ric}(\nu))\varphi^2 + |\widetilde{\nabla}F|^2\varphi^2. \end{aligned}$$

Again, note that $|\widetilde{\nabla}F|^2$ is well-defined on M.

Up until this point, the argument has proceeded exactly as in Ross' argument for $n = 2$. However, it is at this point that Lemma 4.1 appears for the $n = 2$ case, and that is where our argument falls apart. Lemma 4.1 clearly does not extend to the case $n \geq 3$.

For one thing, Lemma 4.1, and the calculations that follow it in Ross' argument, depend on concepts and results which are unique to the case where M is 2-dimensional. For example, in Lemma 4.1, $\widetilde{M}$ must be a compact Riemann surface of genus γ. Also, F must be a conformal map, and the result holds for any metric compatible with the conformal structure of $\widetilde{M}$. Unfortunately, the concepts of Riemann surface, genus, conformal map, and conformal structure do not exist for $n \neq 2$.

When we look at the calculations in the proof of the Ross Area Bound that follow Lemma 4.1, we find more problems. At one point, Ross uses the identity (see [27, p. 139])

$$|A|^2 + \text{Ric}(\nu) = -2K + \text{Ric}(\tau_1) + \text{Ric}(\tau_2).$$

Neither this identity, nor seemingly any other identity in as nice a form, hold for $n \geq 3$. Even if such an identity did hold, the K in the $n = 2$ case is taken care of by using the Gauss-Bonnet Theorem, which itself only makes sense for $n = 2$. Also, when $n = 2$, the term obtained from the Gauss-Bonnet Theorem which depends on the genus γ cancels exactly with the γ term obtained from Lemma 4.1, and thus Ross eliminates the dependence of his area bound on the topology of M. It is not at all clear how one could eliminate the dependence on the topology of M when $n \geq 3$.

Another problem extending the area bound to higher dimensions is the following. Lemma 4.1 is motivated by the fact that, when $n = 2$, there exists an upper bound for $\lambda_1(M)\text{Area}(M)$ which holds uniformly for all metrics on M. Namely, a result of Yang and Yau (see [29]) says that if M is a Riemann surface and $F : M \to S^2 \subset \mathbb{R}^3$ is a conformal map with $\deg F > 0$, then

$$\lambda_1(M)\text{Area}(M) \leq 8\pi(\deg F).$$

Note that the term $8\pi(\deg F)$ appears in Lemma 4.1. Certainly, the Yang/Yau result cannot be extended directly to higher dimensions; as previously mentioned, the concepts of Riemann surface and conformal map do not exist for $n \geq 3$. However, one would hope to be able to find an upper bound for $\lambda_1(M)\text{Area}(M)$ in a different way if possible.

Unfortunately, it can be shown that, for $n \geq 3$, there is not an upper bound for $\lambda_1(M)\text{Area}(M)$ that holds uniformly for all metrics on M. This is easily seen using a scaling argument. If we rescale M by a factor of h, then

$$\lambda_1(M) = \inf_{\substack{\int_M f = 0 \\ \int_M f^2 \neq 0}} \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \right\}$$

changes by a factor of

$$\frac{h^{n-2}}{h^n} = h^{-2},$$

and $\text{Area}(M)$ changes by a factor of h^n, so $\lambda_1(M)\text{Area}(M)$ changes by a factor of h^{n-2}. Hence, if $n \geq 3$, we can make $\lambda_1(M)\text{Area}(M)$ as large as we want by letting h become sufficiently large.

However, consider $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$. Under rescaling, this changes by a factor of

$$h^{-2}[h^n]^{\frac{2}{n}} = 1,$$

i.e., $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$ is scale-invariant. Thus, there is hope that we might be able to find an upper bound for $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$ which holds uniformly for all metrics on M. Note that, when $n = 2$, we get back $\lambda_1(M)\text{Area}(M)$, for which we know the desired upper bound exists. The idea to try to bound $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$ is due to Berger in [1].

Our attempt to find an upper bound for $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$ also leads us nowhere, however. Bleecker conjectured in [2] that, for any M with $\dim(M) \geq 3$, there exist metrics with fixed area and arbitrarily large λ_1. This conjecture was proved independently by Colbois and Dodziuk in [6] and by Xu in [28]. Thus, when $n \geq 3$, there is no upper bound for $\lambda_1(M)[\text{Area}(M)]^{\frac{2}{n}}$ which holds uniformly for all metrics on M, and our quest to extend the Ross Area Bound to higher dimensions reaches another dead end.

5 Genus Bounds in Compact 3-manifolds of Positive Ricci Curvature

Let N be a compact 3-manifold with positive Ricci curvature. In this section, we will establish restrictions on which compact embedded minimal submanifolds $M \subset N$ can be stable. As we saw in Section 4, M must be nonorientable. We will examine whether there are restrictions on the genus of a nonorientable stable compact embedded minimal surface in a compact 3-manifold with positive Ricci curvature. The genus of a nonorientable surface is defined as follows: for a positive integer γ, M has genus γ if M can be written as the connected sum of γ projective planes.

In Section 5.1, we show that given N, the set of positive integers γ such that there exists a nonorientable stable compact embedded minimal surface $M \subset N$ of genus γ is bounded above. However, there is no uniform upper bound which works for all N. Our goal is to place restrictions on N, creating a class of compact 3-manifolds for which there is a uniform upper bound. In Section 5.2, we obtain an important intermediate result. We show that, by putting certain restrictions on N, there is a uniform upper bound for the diameter of a nonorientable stable compact embedded minimal surface $M \subset N$. Our result is the following.

Theorem 5.1. *Let N be an element of the class of compact 3-manifolds with Ric $\geq \kappa > 0$, $|sec| \leq k$, and inj $\geq i_0$, where Ric is Ricci curvature, sec is sectional curvature, and inj is injectivity radius. Let M be a nonorientable stable compact embedded minimal surface in N. Then, there exists a constant*

$$D = D(\kappa, k, i_0)$$

such that diam$(M) \leq D$.

We conjecture that there is a uniform upper bound for diam(M) for the class of 3-manifolds N subject only to the restriction Ric $\geq \kappa > 0$. This would be analogous to the uniform upper bound for Area(M), the Ross Area Bound. At this time, it is unclear whether the conjecture is true.

In Section 5.3, we will use Theorem 5.1 to obtain a uniform upper bound for genus(M), given certain restrictions on N. Our ultimate goal here in Section 5 is to prove our second main result, Theorem 1.2, which says that if N is an element of the class of compact 3-manifolds with $\text{Ric} \geq \kappa > 0$, $|\text{sec}| \leq k$, and $\text{inj} \geq i_0$, and $M \subset N$ is a stable compact embedded minimal surface, then there exists a constant G, depending only on κ, k, and i_0, such that $\text{genus}(M) \leq G$.

An important consequence of Theorem 1.2 is that, given a compact 3-manifold N with $\text{Ric} \geq \kappa > 0$, $|\text{sec}| \leq k$, and $\text{inj} \geq i_0$, the space of nonorientable stable compact embedded minimal surfaces in N is compact in the C^m-topology for $m \geq 2$. To see this, let $\{M_n\}$ be a sequence of nonorientable stable compact embedded minimal surfaces in N. By Theorem 1.2, there exists a subsequence of $\{M_n\}$, which we also call $\{M_n\}$, such that each M_n has the same fixed genus. Then, by [4, Thm. 1], there exists a subsequence of $\{M_n\}$ which, for any $m \geq 2$, converges in the C^m-topology to a compact embedded minimal surface $M \subset N$. M clearly must be nonorientable and stable.

5.1 A Nonuniform Upper Bound for Genus(M)

In this section, we show that, given a compact 3-manifold N of positive Ricci curvature, there is an upper bound for the genus of a nonorientable stable compact embedded minimal surface $M \subset N$. However, this upper bound depends on the choice of N, and in fact any positive integer γ can arise as the genus of some M in some N. Let $\gamma \geq 1$ be an integer. Then, Ross (see [24]) uses the following argument of H. Rubinstein to show that there is some lens space L containing a nonorientable stable compact embedded minimal surface $M \subset L$ of genus γ. Let p and q be positive integers with $q \leq p$ and $\gcd(2p, q) = 1$. Let L be the lens space $L(2p, q)$, and let $\widehat{M} \subset L$ be a nonorientable compact embedded minimal surface of smallest possible genus. By [3], $\widehat{M}$ has genus $\Gamma(2p, q)$, which is defined recursively by

$$\begin{aligned} \Gamma(2,1) &= 1 \\ \Gamma(2p,q) &= \Gamma(2(p-q), q') + 1, \end{aligned}$$

where $q' \equiv \pm q \pmod{2(p-q)}$ and $1 \leq q' \leq p - q$. $\widehat{M}$ is incompressible (by [18, Lemma 6.3]), so by [20, Thm. 1], $\widehat{M}$ is isotopic to a nonorientable stable compact embedded minimal surface $M \subset L$, also of genus $\Gamma(2p, q)$. Bredon and Wood (see [3]) have shown that by varying p and q, $\Gamma(2p, q)$ can take any positive integer value γ. Note that this example demonstrates that the assumption $\text{inj} \geq i_0$ in Theorem 1.2 is necessary, as the injectivity radii of the $L(2p, q)$ approach 0.

We have shown that there is no uniform upper bound for the genus of a nonorientable stable compact embedded minimal surface in some compact 3-manifold with positive Ricci curvature. However, if we fix such a 3-manifold N, there is an upper bound.

Theorem 5.2. *Let N be a compact 3-manifold with positive Ricci curvature. Then, the set of positive integers γ such that there exists a nonorientable stable compact embedded minimal surface $M \subset N$ of genus γ has an upper bound (depending on N).*

We conjecture that a similar result holds for compact embedded minimal surfaces $M \subset N$ of higher index. Theorem 5.2 is the case where $\text{index}(M) = 0$.

Proof of Theorem 5.2. Suppose not. Then, there exist nonorientable stable compact embedded minimal surfaces $M_n \subset N$ of genus γ_n with $\gamma_n \to \infty$. Let κ be the minimum of Ric_N. Then, $\kappa > 0$, and by the Ross Area Bound (Theorem 4.2),

$$\text{Area}(M_n) \leq \frac{4\pi}{\kappa}$$

for all n, since the M_n are stable. For small enough $r > 0$ (depending on the lower bound for $\text{inj}(N)$), $B_r \cap M_n$ is orientable and stable for all n and all balls $B_r \subset N$, and so by the Schoen Curvature Estimate (Theorem 2.3), there exists a constant K such that, for small enough r (depending on k and the lower bound for $\text{inj}(N)$) and all $\sigma \in (0, r]$,

$$\sup_{B_{r-\sigma}} |A_n|^2 \leq \frac{K}{\sigma^2}$$

for all n and all balls $B_{r-\sigma} \subset N$, where A_n is the second fundamental form of M_n. Since the M_n are without boundary, we get a uniform curvature estimate on all of M_n, instead of just on balls. Therefore, by Lemma 3.4, a subsequence of $\{M_n\}$ converges to a compact embedded minimal surface $M \subset N$. By construction, M has infinite genus, contradicting the compactness of N. $\square$

5.2 A Uniform Upper Bound for Diam(M)

Before we proceed to showing that, for a certain class of compact 3-manifolds N, there exists a uniform upper bound for genus(M), we must first establish a similar upper bound for the diameter of M, which we have previously stated as Theorem 5.1.

Proof of Theorem 5.1. Let $\{B_r(y_j)\}$ be a finite covering of N such that $\{B_{r/4}(y_j)\}$ is still a covering of N, where r is small enough so that the Schoen Curvature Estimate holds. In particular, r is small enough so that each $B_r \cap M$ is orientable and stable. So, r depends on k and i_0. Then, $\{B_{r/4}(y_j) \cap M\}$ covers M, and, by Schoen, each $B_{r/2}(y_j) \cap M$ is a union of embedded discs.

We claim that the number of discs in each $B_{r/2}(y_j) \cap M$ which intersect $B_{r/4}(y_j) \cap M$ has an upper bound which is independent of j and the choice of M. Let D_j be any such disc. Then, there exists $x_j \in D_j \cap \partial B_{r/4}(y_j)$. So, $B_{r/4}(x_j) \subset B_{r/2}(y_j)$, and by monotonicity of area, there exists a constant $C > 0$, depending on k and i_0, such that

$$\text{Area}(D_j \cap B_{r/4}(x_j)) \geq C \left(\frac{r}{4}\right)^2.$$

So, each D_j is of at least some fixed positive area, and so by the Ross Area Bound, we see that the number of such discs is bounded above by some constant. This constant depends on κ, C, and r, and so it depends only on κ, k, and i_0.

Since there are finitely many balls $B_{r/4}(y_j)$ and finitely many discs in each $B_{r/4}(y_j) \cap M$, there must be a finite total number of discs which cover M. Then, diam(M) is bounded above by the product

$$\text{(total number of discs)} \times \text{(maximum diameter of any of the discs)}.$$

The maximum diameter of any of the discs depends on the Schoen Curvature Estimate, and hence is bounded above by a constant depending only on k. The total number of discs has an upper bound which depends on κ, k, i_0, and the number of balls $B_{r/4}(y_j)$ needed to cover N. By [23, Thm. 9.1.2], $\text{diam}(N) \leq \pi\sqrt{\frac{2}{\kappa}}$, and so the number of $B_{r/4}(y_j)$ is bounded above by a constant depending on κ and r, and hence depending only on κ, k, and i_0. Therefore, diam(M) is bounded above by a constant depending only on κ, k, and i_0. ∎

5.3 A Uniform Upper Bound for Genus(M)

In this section, we prove that, for a certain class of compact 3-manifolds N, there is a uniform upper bound for the genus of a nonorientable stable compact embedded minimal surface $M \subset N$. Namely, we prove our second main result, Theorem 1.2.

Proof of Theorem 1.2. By the Schoen Curvature Estimate, and since M is minimal,

$$\sec_M = \sec_N - \frac{1}{2}|A|^2 \geq -k - \frac{1}{2}\frac{K}{r^2}$$

for sufficiently small r (depending on k and i_0). Since K depends only on k, we have shown that $\sec_M$ has a uniform lower bound depending only on k and i_0. By Theorem 5.1, diam(M) has a uniform upper bound depending only on κ, k, and i_0. Let $\chi(M)$ be the Euler characteristic of M. Then, by [23, p. 350], the sum of the Betti numbers of M has a uniform upper bound depending only on κ, k, and i_0, and so $\chi(M)$ has a uniform lower bound depending only on κ, k, and i_0. Our result follows since genus is a decreasing function of Euler characteristic. □

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Vita

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