# The $E(1,2)$ Cohomology of the Eilenberg-Mac Lane Space $K(\mathbb{Z},3)$

by

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**Abstract**

In this paper, we give new bases of $E_*\left(\underline{BP\langle 1\rangle_4}\right)$ and $E_*\left(\underline{BP\langle 1\rangle_6}\right)$ in order to define a basis of $E^*\left(\underline{BP\langle 1\rangle_4}\right)$ and $E^*\left(\underline{BP\langle 1\rangle_6}\right)$. With these two new bases, we have a short exact sequence of Hopf algebras

$$E^* \longleftarrow E^*\left(K\left(\mathbb{Z},3\right)\right) \xleftarrow{f^*} E^*\left[\left[y_{(i)} : i \geq 0\right]\right] \xleftarrow{v_1^*} E^*\left[\left[x_{(i)} : i \geq 0\right]\right] \longleftarrow E^*.$$

The action of $v_1^*$ is calculated by using formal group laws and used to compute $E^*\left(K\left(\mathbb{Z},3\right)\right)$ as a completion of $E^*\left[z_{(0)}\right]$.

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# Contents

# 1 Introduction

The Eilenberg-Mac Lane Space $K(\mathbb{Z},2)$ is readily recognized as complex projective space $\mathbb{C}P^{\infty}$. Given any multiplicative complex-oriented generalized cohomology theory $E^*(-)$, a classical result of Dold states that

$$E^*(\mathbb{C}P^{\infty}) = E^*[[x]],$$

the graded ring of formal power series over the coefficient ring $E^*$ of $E$, where $x$ denotes the Chern class of the Hopf line bundle over $\mathbb{C}P^{\infty}$.

This paper studies the $E$-cohomology of the next Eilenberg-Mac Lane space $K(\mathbb{Z},3)$, which may be viewed as the classifying space of $K(\mathbb{Z},2)$, for certain complex-oriented $E$.

Note that, for any space $X$, a cohomology class in $E^k(X)$ corresponds to a map $X \longrightarrow \underline{E}_k$, defined up to homotopy, where $\underline{E}_k$ comes from the Brown Representation Theorem, which can be stated as: for any cohomology theory $E$, for all $k$, there exists a classifying space $\underline{E}_k$ such that $E^k(X) \cong [X, \underline{E}_k]$, the set of homotopy classes of unbased maps $X \longrightarrow \underline{E}_k$.

For a prime $p$, the Johnson-Wilson spectrum $BP\langle q\rangle$ is a $BP$-module spectrum with $\pi_*(BP\langle q\rangle) = \mathbb{Z}_{(p)}[v_1, \cdots, v_q]$. For $q > 0$, there is a stable cofibre sequence

$$\Sigma^{2(p^q-1)}BP\langle q\rangle \xrightarrow{v_q} BP\langle q\rangle \longrightarrow BP\langle q-1\rangle \longrightarrow \Sigma^{2p^q-1}BP\langle q\rangle.$$

Unstably, the boundary map is

$$\underline{BP\langle k-1\rangle}_j \longrightarrow \underline{BP\langle k\rangle}_{j+2p^k-1}$$

and the iteration is

$$K\left(\mathbb{Z}/p^i, q+1\right) \longrightarrow K\left(\mathbb{Z}_{(p)}, q+2\right) \longrightarrow \underline{BP\langle 1\rangle}_{q+2p+1} \longrightarrow \cdots \longrightarrow \underline{BP\langle q\rangle}_{g(q)}$$

where $g(q) = 2(p^q + p^{q-1} + \cdots + p + 1)$ and $\underline{BP\langle 0\rangle}_{q+2} = K\left(\mathbb{Z}_{(p)}, q+2\right)$. In [RWY], there is an exact sequence in the category of $K(n)_*$-Hopf algebras:

$$\begin{aligned} K(n)_* &\longrightarrow K(n)_*\left(K\left(\mathbb{Z}_{(p)}, q+2\right)\right) \\ &\longrightarrow K(n)_*\left(\underline{BP\langle q\rangle}_{g(q)}\right) \xrightarrow{v_{q*}} K(n)_*\left(\underline{BP\langle q\rangle}_{g(q)-|v_q|}\right). \end{aligned}$$

This induces, by Theorem 1.19 in [RWY],

$$BP^*\left(K\left(\mathbb{Z}_{(p)}, q+2\right)\right) \cong \frac{BP^*\left(\underline{BP\langle q\rangle}_{g(q)}\right)}{\left(v_q^*\right)},$$

where $\left(v_q^*\right)$ denotes the ideal generated by the image under $v_q^*$ of the augmentation ideal in $BP^*\left(\underline{BP\langle q\rangle}_{g(q)-|v_q|}\right)$.

In 1999, Tamanoi [T], who was motivated by the above result, created new generators and wrote down a very simple relation in terms of his new generators. Let $\iota_{g(q)+t}^{\langle q\rangle}$ be the inclusion map

$$\iota_{g(q)+t}^{\langle q\rangle} : \underline{BP\langle q\rangle}_{g(q)+t} \longrightarrow \underline{BP}_{g(q)+t}$$

for $t \leq 0$. Recall that by Wilson's Splitting Theorem [Wil75], $\underline{BP\langle q\rangle}_{g(q)+t}$ is a factor space of $\underline{BP}_{g(q)+t}$. Thus, $\iota_{g(q)+t}^{\langle q\rangle}$ exists. This map defines a cohomology class

$$\iota_{g(q)+t}^{\langle q\rangle} \in BP^{g(q)+t}\left(\underline{BP\langle q\rangle}_{g(q)+t}\right),$$

namely the $BP$ fundamental class for $BP^{g(q)+t}\left(\underline{BP\langle q\rangle}_{g(q)+t}\right)$. Then, Tamanoi defined the $v_q$-series

$$[v_q] = v_q^*\left(\iota_{g(q)}^{\langle q\rangle}\right) \in BP^{g(q)-|v_q|}\left(\underline{BP\langle q\rangle}_{g(q)}\right),$$

which comes from the composition map

$$\underline{BP\langle q\rangle}_{g(q)} \xrightarrow{v_q} \underline{BP\langle q\rangle}_{g(q)-|v_q|} \xrightarrow{\iota_{g(q)-|v_q|}^{\langle q\rangle}} \underline{BP}_{g(q)-|v_q|}.$$

Also, he showed that

$$[v_q] = v_q\theta_q^{\langle q\rangle} + v_{q+1}\theta_{q+1}^{\langle q\rangle} + \cdots + v_{q+j}\theta_{q+j}^{\langle q\rangle} + \cdots,$$

where the class $\theta_{q+j}^{\langle q\rangle}$ with $j \geq 1$ is defined as the following composition:

$$\begin{aligned}\underline{BP\langle q\rangle}_{g(q)} &\longrightarrow \underline{BP}_{g(q)} \xrightarrow{v_q} \underline{BP}_{g(q)-|v_q|} \xrightarrow{\text{proj}\langle q+j\rangle} \underline{BP\langle q+j\rangle}_{g(q)+2p^{q+j}-2p^q} \\ &\longrightarrow \underline{BP}_{g(q)+2p^{q+j}-2p^q}\end{aligned}$$

and the projection is provided by Wilson's theorem.

We take $p=2$. By the short exact sequence of Hopf algebras

$$BP^* \longleftarrow BP^*\left(K\left(\mathbb{Z},3\right)\right) \stackrel{f^*}{\longleftarrow} BP^*\left(\underline{BP\left\langle 1\right\rangle}_6\right) \stackrel{v_1^*}{\longleftarrow} BP^*\left(\underline{BP\left\langle 1\right\rangle}_4\right) \longleftarrow BP^*,$$

we have

$$BP^*\left(K\left(\mathbb{Z},3\right)\right) \cong \frac{BP^*\left(\underline{BP\left\langle 1\right\rangle}_6\right)}{\left(v_1^*\right)},$$

where $\left(v_1^*\right)$ denotes the image under $v_1^*$ of the augmentation ideal of $BP^*\left(\underline{BP\left\langle 1\right\rangle}_4\right)$.

The relation Tamanoi found in this case is

$$v_1^*\left(x_{(0)}\right) = v_1 y_{(0)} + v_2 y_{(1)} + \cdots + v_i y_{(i-1)} + \cdots.$$

By tensoring with $E\left(1,2\right)^*$, we have $E\left(1,2\right)^*\left(K\left(\mathbb{Z},3\right)\right) \cong E\left(1,2\right)^*\left(\underline{BP\left\langle 1\right\rangle}_6\right) \diagup \left(v_1^*\right)$ where $v_1^*\left(x_{(0)}\right) = v_1 y_{(0)} + v_2 y_{(1)}$.

We use the formal group law to create a new set of generators $y'_{(i)}$, for which the relations in $E\left(1,2\right)^*\left(\underline{BP\left\langle 1\right\rangle}_6\right)$ take the simplified form

$$v_1^*\left(x_{(i)}\right) = v_1^{2^i} y'_{(i)} + v_2^{2^i} y'_{(i+1)} + D_{(i)}$$

for $i \geq 0$, where $D_{(i)}$ is decomposable.

This paper is organized as follows: In section 2, we collect some basic notation and theorems for the cohomology theories $BP$ and $E\left(k,n\right)$. We calculate some formal group laws in section 3. In section 4, a basis of $BP_*\left(\underline{BP\left\langle 1\right\rangle}_4\right)$ and $BP_*\left(\underline{BP\left\langle 1\right\rangle}_6\right)$ will be given so that we can write down a dual basis for $BP^*\left(\underline{BP\left\langle 1\right\rangle}_4\right)$ and $BP^*\left(\underline{BP\left\langle 1\right\rangle}_6\right)$. These are used to prove the main results.

# 2 Preliminary

The coefficient ring for Brown-Peterson cohomology is

$$BP^* \cong \mathbb{Z}_{(p)}\left[v_1, v_2, \cdots\right]$$

where the degree of $v_n$ is $-2\left(p^n-1\right)$. In homology, we often write the coefficient ring as $BP_*$, with $BP_n = BP^{-n}$ (see [RW77]).

We have the Hopf algebroid $\left(BP_*, BP_*BP\right)$ which satisfies:

1. $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$, where $\deg v_n = 2(p^n - 1)$.

2. $BP_*BP \cong BP_*[t_1, t_2, \cdots]$, where $\deg t_n = 2(p^n - 1)$.

3. $\eta_R : BP_* \to BP_*BP$ is a ring homomorphism, where $\eta_R v$ is written as $[v]$.

Let $I_n$ be the ideal $(p, v_1, v_2, \cdots, v_{n-1})$. And, $v_0 = p$.
Let $BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n]$ and $\left\{\underline{BP\langle n\rangle}_*\right\}$ be the $\Omega$-spectrum of $BP\langle n\rangle$.
Let $P(n)$ be the theory with coefficient ring

$$P(n)^* \cong BP^*/I_n.$$

And,

$$P(0) = BP \text{ or } BP_p^{\hat{}}$$

where $BP_p^{\hat{}}$ is the $p$-adic completion of $BP$. We have stable cofibrations

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \longrightarrow P(n+1)$$

which lead to long exact sequences in cohomology.

There are spectra $E(k,n)$ with coefficient rings

$$E(k,n)^* \cong v_n^{-1}BP\langle n\rangle^*/I_k$$

with similar stable cofibrations

$$\Sigma^{2(p^k-1)}E(k,n) \xrightarrow{v_k} E(k,n) \longrightarrow E(k+1,n)$$

and long exact sequences. In particular, for $1 \le k \le n$,

$$E(k,n)^* \cong v_n^{-1}BP\langle n\rangle^*/I_k = \mathbb{F}_p\left[v_k, v_{k+1}, \cdots, v_n, v_n^{-1}\right].$$

A special case, when $k = n > 0$, is the $n^{\text{th}}$ Morava $K$-theory, $K(n)^*(X)$, with

$$K(n)^* \cong \mathbb{Z}/p\left[v_n, v_n^{-1}\right].$$

Also, when $k = 0$, $E(n)^* = E(0,n)^* \cong v_n^{-1}BP\langle n\rangle^* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}]$.

Let $E$ denote any of $BP_p^{\hat{}}$, $P(n)$ or $E(k,n)$ where $0 \le k \le n$ and $0 < n$.

## 2.1 Formal Group Laws

We summarize the properties of formal group laws. We are working with the Brown-Peterson spectrum with prime 2.

**Definition 2.1** *A formal group law is a power series*

$$F(y,z) = \sum_{i,j\geq 0} a_{ij} y^i z^j$$

*such that*

1. $F(y,0) = y$ *and* $F(0,z) = z$,
2. $F(y, F(z,w)) = F(F(y,z), w)$,

*where* $a_{ij}$ *is in a commutative ring with unit. Moreover, we say a formal group law is commutative if* $F(y,z) = F(z,y)$*. We write* $y +_F z$ *for* $F(y,z)$*.*

**Example 2.2** *We have two trivial examples of commutative formal group laws:*

$$F(y,z) = y + z$$

*and*

$$F(y,z) = y + z + yz.$$

By the first condition in the formal group law definition, we have

$$a_{i0} = \begin{cases} 1 & \text{, if } i = 1 \\ 0 & \text{, if } i \neq 1 \end{cases}$$

and

$$a_{0j} = \begin{cases} 1 & \text{, if } j = 1 \\ 0 & \text{, if } j \neq 1 \end{cases}.$$

Therefore, we may write our formal group law in the following form

$$F(y,z) = y + z + \sum_{i,j\geq 1} a_{ij} y^i z^j$$

Let $x \in BP^2(\mathbb{C}P^\infty)$ be the Conner-Floyd Chern class.

**Proposition 2.3** (In [RW77] Lemma 3.3 on Page 255) *We have*

1. $BP^*(\mathbb{C}P^\infty) \cong BP^*[[x]]$, *the ring of formal power series on* $x = x^{\mathbb{C}P^\infty}$ *over* $BP^*$.

2. $BP^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong BP^*(\mathbb{C}P^\infty)\,\widehat{\otimes}_{BP^*} BP^*(\mathbb{C}P^\infty)$.

3. $BP_*(\mathbb{C}P^\infty)$ *is* $BP_*$*-free on* $\beta_i \in BP_{2i}(\mathbb{C}P^\infty)$ *for* $i \geq 0$, *dual to* $\left(x^{\mathbb{C}P^\infty}\right)^i$, *i.e.* $\left\langle \left(x^{\mathbb{C}P^\infty}\right)^i, \beta_j \right\rangle = \delta_{ij}$.

4. $BP_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq BP_*(\mathbb{C}P^\infty) \otimes_{BP_*} BP_*(\mathbb{C}P^\infty)$.

5. *The diagonal* $\mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ *induces a coproduct* $\psi$ *on* $BP_*(\mathbb{C}P^\infty)$ *with* $\psi(\beta_n) = \sum_{i=0}^{n} \beta_i \otimes \beta_{n-i}$.

6. *The* $H$*-space product* $m_{\mathbb{C}P^\infty} : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ *induces a coproduct* $m^*_{\mathbb{C}P^\infty}$ *on* $BP^*(\mathbb{C}P^\infty)$ *with* $m^*_{\mathbb{C}P^\infty}\left(x^{\mathbb{C}P^\infty}\right) = \sum_{i,j\geq 0} a_{ij} \left(x^{\mathbb{C}P^\infty}\right)^i \otimes \left(x^{\mathbb{C}P^\infty}\right)^j$ *where* $a_{ij} \in BP^{-2(i+j-1)} = BP_{2(i+j-1)}$.

7. $F(y,z) = y +_{F_{BP}} z = y +_F z = \sum_{i,j\geq 0} a_{ij} y^i z^j$ *is a commutative associative formal group law over* $BP^*$.

Let $\beta(s) = \sum_{i\geq 0} \beta_i s^i$ where $s$ is a formal indeterminate.

**Proposition 2.4** (In [RW77] Theorem 3.4 on Page 255) *In the power series ring* $BP_*(\mathbb{C}P^\infty)[[s,t]]$, *we have*

$$\beta(s)\,\beta(t) = \beta(s +_F t)$$

*where the product is that induced by the* $H$*-space structure of* $\mathbb{C}P^\infty$. *Moreover, for a natural number* $n$,

$$\beta(s)^n = \beta\left([n]_F(s)\right),$$

*where* $[n]_F(s) = s +_F s +_F + \cdots +_F s$ *for* $n > 0$.

Consider $x \in BP^2(\mathbb{C}P^\infty) = [\mathbb{C}P^\infty, \underline{BP}_2]$ as a map

$$\mathbb{C}P^\infty \xrightarrow{x} \underline{BP}_2.$$

This map induces a map $x_* : BP_*(\mathbb{C}P^\infty) \to BP_*(\underline{BP}_2)$.

**Definition 2.5** *We define*

$$b_i = x_* (\beta_i) \in BP_{2i} (\underline{BP}_2) .$$

As with $\beta(s)$, we have

$$x_* (\beta (s)) = x_* \left( \sum_{i \geq 0} \beta_i s^i \right) = \sum_{i \geq 0} x_* (\beta_i) s^i = \sum_{i \geq 0} b_i s^i .$$

Therefore, we define $b(s) = \sum_{i \geq 0} b_i s^i \in BP_* (\underline{BP}_2) [[s]]$, that is, $b(s) = x_* (\beta (s))$.

The $H$-space product $m : \underline{BP}_2 \times \underline{BP}_2 \to \underline{BP}_2$ induces a coproduct $m^*$ on $BP^* (\underline{BP})$ with $m^* (x) = \sum_{i,j \geq 0} a_{ij} x^i \otimes x^j$ where $a_{ij} \in BP^{-2(i+j-1)} = BP_{2(i+j-1)}$.

According to [RW80], $BP_* (\underline{BP}_*)$ is a Hopf ring, where the maps of coalgebras

$$* : BP_* (\underline{BP}_k) \otimes BP_* (\underline{BP}_k) \cong BP_* (\underline{BP}_k \times \underline{BP}_k) \to BP_* (\underline{BP}_k)$$

are induced by the maps $\underline{BP}_k \times \underline{BP}_k \to \underline{BP}_k$ that represent addition in cohomology and the maps of coalgebras

$$\circ : BP_* (\underline{BP}_k) \otimes BP_* (\underline{BP}_m) \cong BP_* (\underline{BP}_k \times \underline{BP}_m) \to BP_* \left(\underline{BP}_{k+m}\right)$$

are induced by the maps $\underline{BP}_k \times \underline{BP}_m \to \underline{BP}_{k+m}$ that represent multiplication in cohomology.

**Definition 2.6** *In $BP_* (\underline{BP}_*) [[s,t]]$, we define*

$$b(s) +_{[F]} b(t) = b(s) +_{[F]_{BP}} b(t) = \mathop{*}_{i,j \geq 0} [a_{ij}] \circ b(s)^{\circ i} \circ b(t)^{\circ j} .$$

**Proposition 2.7** (In [RW77] Theorem 3.8 on Page 256) *In $BP_* (\underline{BP}_*) [[s,t]]$, we have relations:*

1. $b(s +_F t) = b(s) +_{[F]} b(t)$,
2. $b([2]_F (s)) = [2]_{[F]} (b(s))$.

We can rewrite the relation $b(s +_F t) = b(s) +_{[F]} b(t)$ as

$$b \left( \sum_{i,j \geq 0} a_{ij} s^i t^j \right) = \mathop{*}_{i,j \geq 0} [a_{ij}] \circ b(s)^{\circ i} \circ b(t)^{\circ j} ,$$

that is,

$$\sum_{n\geq 0} b_n \left(\sum_{i,j\geq 0} a_{ij} s^i t^j\right)^n = \underset{i,j\geq 0}{*} [a_{ij}] \circ \left(\sum_{k\geq 0} b_k s^k\right)^{\circ i} \circ \left(\sum_{l\geq 0} b_l s^l\right)^{\circ j}.$$

**Proposition 2.8** (In [RW77] Theorem 3.11(b) on Page 257) *In $BP$ for the prime 2, we have*

$$[2]_F(t) = \sum\nolimits_{n>0}^{F_{BP}} v_n t^{2^n} \bmod (2)\,.$$

In $BP$ with prime 2, we have, see [RW77],

$$[2]_F(t) = 2t - v_1 t^2 + 2v_1^2 t^3 - \left(7v_2 + 8v_1^3\right) t^4 + \left(30 v_2 v_1 + 26 v_1^4\right) t^5 - \left(111 v_2 v_1^2 + 84 v_1^5\right) t^6 + \cdots.$$

**Remark 2.9** From Lemma 1.12 in [RW77],

1. $a_{10} = a_{01} = 1$ and $a_{i0} = a_{0i} = 0$ if $i \neq 1$,
2. $b_0 = [0_2]$,
3. $b_0 \circ b_i = 0$ for all $i > 0$,
4. $b_0 \circ [a] = [0_{2-2i}]$ where $a \in BP^{-2i}$,
5. $2b_1 = [2] \circ b_1$,
6. $b_1^{*2} = 2b_2 - v_1 b_1 + [v_1] \circ b_1^{\circ 2}$ in $BP_2\left(\underline{BP}_2\right)$ from page 261.

## 2.2 Wilson's Splitting Theorem

We study the commutative diagram of $H$-spaces and canonical $H$-maps

$$\begin{array}{ccccc} & & \underline{BP}_6 & \xrightarrow{v_1} & \underline{BP}_4 \\ & & \downarrow & & \downarrow \\ K\left(\mathbb{Z}_{(2)}, 3\right) & \xrightarrow{f} & \underline{BP\langle 1\rangle}_6 & \xrightarrow{v_1} & \underline{BP\langle 1\rangle}_4 \end{array} \qquad (1)$$

in which the bottom row is a fibration.

By Wilson [Wil73], $\underline{BP\langle 1\rangle}_4$ splits off $\underline{BP}_4$ as an $H$-space. $\underline{BP\langle 1\rangle}_6$ splits off $\underline{BP}_6$, but not as an $H$-space. Both maps $v_1$ represent multiplication by $v_1$ in cohomology.

(The fibre of an $H$-map is automatically an $H$-space, and by uniqueness, the resulting $H$-space structure on $K\left(\mathbb{Z}_{(2)},3\right)$ must be the standard one, up to homotopy.)

We apply homology theories $E_*(-)$ and cohomology theories $E^*(-)$ to this diagram, for various $E = BP$ or $E(1,2)$. Note that since all the $E$ we consider are 2-local, we may replace $E^*\left(K\left(\mathbb{Z}_{(2)},3\right)\right)$ by $E^*(K(\mathbb{Z},3))$. $BP_*(\underline{BP}_k)$ was computed by Ravenel-Wilson [RW77] and is a free $BP_*$-module. Since each $E$ we use comes with a multiplicative map $BP \to E$ that induces a surjection $BP_* \to E_*$, comparison of the Atiyah-Hirzebruch spectral sequences shows that $E_*(\underline{BP}_k)$ and $E_*\left(\underline{BP\langle 1\rangle}_k\right)$ are free $E_*$-modules for $k = 4, 6$. More precisely,

$$E_* \otimes_{BP_*} BP_*(\underline{BP}_k) \cong E_*(\underline{BP}_k).$$

In view of the above splittings, the same holds for $\underline{BP\langle 1\rangle}_k$. By dualizing, we have

$$E^* \widehat{\otimes}_{BP^*} BP^*(\underline{BP}_k) \cong E^*(\underline{BP}_k)$$

in cohomology.

Ravenel-Wilson-Yagita [RWY] established the short exact sequence of bicommutative Hopf algebras

$$E^* \longrightarrow E^*\left(\underline{BP\langle 1\rangle}_4\right) \xrightarrow{v_1^*} E^*\left(\underline{BP\langle 1\rangle}_6\right) \xrightarrow{f^*} E^*(K(\mathbb{Z},3)) \longrightarrow E^*$$

for various $E$. We wish to use this to compute $E(1,2)^*(K(\mathbb{Z},3))$.

## 2.3 The Formal Group Law for $E(1,2)$

Consider the spectrum $E(1,2)$ associated with the prime 2 where the coefficient ring is $E(1,2)^* = \mathbb{Z}/2\left[v_1, v_2, v_2^{-1}\right]$. We study its formal group law $F(y,z)$.

For convenience, we define $A_1 = v_1$ and $A_2 = v_2$. Also, for $k \geq 3$, we define

$$A_k = \sum_{\substack{m,n \geq 1 \\ 2m+2^2n=2k}} a_{mn} v_1^m v_2^n.$$

**Lemma 2.10** $v_1 t^2 +_F v_2 t^4 = \sum_{k\geq 1} A_k t^{2k}.$

**Proof.** By the formal group law, we have

$$v_1t^2 +_F v_2t^4 = v_1t^2 + v_2t^{2^2} + \sum_{m,n\geq 1} a_{mn}v_1^m v_2^n t^{2m+2^2n}.$$

Take the coefficient of $t^{2k}$. ■

A lower degree calculation (2.11) is in the Appendix.

We note that $(a+b)^2 \equiv a^2 + b^2 \bmod 2$. Thus, we have

$$\left(v_1t^2 +_F v_2t^4\right)^{2^q} = \sum_{k\geq 1} A_k^{2^q} t^{2^{q+1}k}.$$

In general, we define $C_{q,k}$ as the coefficient of $t^{2k}$ in the equation

$$\left(v_1t^2 +_F v_2t^4\right)^q = \sum_{k\geq 1} C_{q,k}t^{2k}.$$

It's easy to see that

$$C_{q,k} = 0, \text{ if } k < q$$

and

$$C_{2^q,k} = A_{2^{-q}k}^{2^q}.$$

for all $q \geq 1$ and $k \geq 2^q$, interpreted as 0 if $k$ is not divisible by $2^q$.

## 2.4 Appendix: Low Degree Examples

We collect some precise low degree calculations here for reference.

**Example 2.11** *The first few terms of*

$$\begin{aligned}
& v_1t^2 +_F v_2t^4 \\
= \ & v_1t^2 + v_2t^4 + a_{11}v_1v_2t^6 + a_{21}v_1^2v_2t^8 + \left(a_{31}v_1^3v_2 + a_{12}v_1v_2^2\right)t^{10} \\
& + \left(a_{41}v_1^4v_2 + a_{22}v_1^2v_2^2\right)t^{12} + \left(a_{51}v_1^5v_2 + a_{32}v_1^3v_2^2 + a_{13}v_1v_2^3\right)t^{14} \\
& + \left(a_{61}v_1^6v_2 + a_{42}v_1^4v_2^2 + a_{23}v_1^2v_2^3\right)t^{18} + \cdots \\
= \ & A_1t^2 + A_2t^4 + A_3t^6 + A_4t^8 + A_5t^{10} + A_6t^{12} + A_7t^{14} + A_8t^{16} + \cdots.
\end{aligned}$$

**Example 2.12** *The first few terms of*

$$\begin{aligned}
& \left(v_1t^2 +_F v_2t^4\right)^2 \\
= & \left(\sum_{k\geq 1} A_k t^{2^k}\right)^2 \\
= & \left(A_1t^2 + A_2t^4 + A_3t^6 + A_4t^8 + \cdots\right)\left(A_1t^2 + A_2t^4 + A_3t^6 + A_4t^8 + \cdots\right) \\
= & A_1A_1t^4 + \left(A_1A_2 + A_2A_1\right)t^6 + \left(A_1A_3 + A_2A_2 + A_3A_1\right)t^8 \\
& + \left(A_1A_4 + A_2A_3 + A_3A_2 + A_4A_1\right)t^{10} + \cdots \\
= & A_1^2t^4 + 2A_1A_2t^6 + \left(A_2^2 + 2A_1A_3\right)t^8 + 2\left(A_1A_4 + A_2A_3\right)t^{10} + \cdots.
\end{aligned}$$

*or*

$$\begin{aligned}
& \left(v_1t^2 +_F v_2t^4\right)^2 \\
= & v_1^2t^4 + 2v_1v_2t^6 + \left(v_2^2 + 2a_{11}v_1^2v_2\right)t^8 + 2\left(a_{21}v_1^3v_2 + a_{11}v_1v_2^2\right)t^{10} \\
& + \left(a_{11}^2v_1^2v_2^2 + 2\left(a_{31}v_1^4v_2 + a_{12}v_1^2v_2^2 + a_{21}v_1^2v_2^2\right)\right)t^{12} \\
& +2\left(a_{12}v_1v_2^3 + a_{41}v_1^5v_2 + a_{22}v_1^3v_2^2 + a_{31}v_1^3v_2^2 + a_{11}a_{21}v_1^3v_2^2\right)t^{14} + \left(a_{21}^2v_1^4v_2^2\right)t^{16} \\
& +2\left(a_{51}v_1^6v_2 + a_{13}v_1^2v_2^3 + a_{22}v_1^2v_2^3 + a_{32}v_1^4v_2^2 + a_{41}v_1^4v_2^2 + a_{11}a_{12}v_1^2v_2^3 + a_{11}a_{31}v_1^4v_2^2\right)t^{16} \\
& +2\left(a_{32}v_1^3v_2^3 + a_{51}v_1^5v_2^2 + a_{13}v_1v_2^4 + a_{61}v_1^7v_2 + a_{42}v_1^5v_2^2 + a_{23}v_1^3v_2^3\right)t^{18} \\
& +2\left(a_{11}a_{22}v_1^3v_2^3 + a_{12}a_{21}v_1^3v_2^3 + a_{11}a_{41}v_1^5v_2^2 + a_{21}a_{31}v_1^5v_2^2\right)t^{18} + \cdots
\end{aligned}$$

**Example 2.13** *For $p = 2$, the first few terms of*

$$\begin{aligned}
& \left(v_1t^2 +_F v_2t^4\right)^{2^2} \\
= & \left(\sum_{k\geq 1} A_k t^{2^k}\right)^{2^2} \\
= & \left(A_1t^2 + A_2t^4 + A_3t^6 + \cdots\right)^4 \\
= & A_1^4t^8 + 2^2A_1^3A_2t^{10} + 2\left(2A_1^3A_3 + 3A_1^2A_2^2\right)t^{12} \\
& + \left(A_2^4 + 6A_1^2A_3^2 + 12A_1A_2^2A_3 + 12A_1^2A_2A_4 + 4A_1^3A_5\right)t^{16} + \cdots.
\end{aligned}$$

# 3 $BP$ Homology and Cohomology of $\underline{BP\langle 1\rangle}_4$ and $\underline{BP\langle 1\rangle}_6$

We focus on the prime 2. Let $E$ be a 2-local cohomology theory. We borrow the notations and results from [RW77]:

**Definition 3.1** *$[v_i]$ is defined in $E_0\left(\underline{BP}_{-2(2^i-1)}\right)$ using $v_i \in \pi_{2(2^i-1)}(BP)$.*

**Definition 3.2** *$\Delta_k = (0, \cdots, 0, 1, 0, \cdots)$ where 1 is in the $k$-th position.*

**Definition 3.3** *$b_{(m)} = b_{2^m} \in E_{2^{m+1}}(\underline{BP}_2)$.*

**Definition 3.4** *In the ring of indecomposables $QE_*(\underline{BP}_*)$, define*

$$\left[v^I\right] b^J = \left[v_1^{i_1} v_2^{i_2} \cdots\right] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots$$

*where $I = (i_1, i_2, \cdots)$ and $J = (j_0, j_1, \cdots)$ are sequences of non-negative integers almost all zero. Note that Proposition 2.7 implies that the other $b_n$ are redundant.*

**Definition 3.5** *For $J = (j_0, j_1, \cdots)$, we define the length of $J$ as $\ell(J) = j_0 + j_1 + \cdots$.*

**Definition 3.6** *We call $\left[v^I\right] b^J$ **allowable** if*

$$J = 2\Delta_{k_1} + 2^2\Delta_{k_2} + \cdots + 2^n\Delta_{k_n} + J',$$

*where $k_1 \leq k_2 \leq \cdots \leq k_n$ and $J'$ is non-negative, implies $i_n = 0$.*

**Proposition 3.7** *The allowable $\left[v^I\right] b^J$ $(J \neq 0)$ form a basis of the free $BP_*$-module $QBP_*\left(\underline{BP}'_*\right)$, where $\underline{BP}'_*$ denotes the base components of $\underline{BP}_*$. ($\underline{BP}'_k = \underline{BP}_k$ for $k > 0$.)*

**Proposition 3.8** *The $\left[v^I\right] b^J \circ b_{(0)}$ with $\left[v^I\right] b^J$ allowable ($J$ possibly zero) form a basis of the submodule of primitives $PBP_*\left(\underline{BP}'_*\right)$.*

**Remark 3.9** A basis of $QBP_{2m}(\underline{BP}_{2n}), n > 0$ is given by all allowable $\left[v^I\right] b^J$ with $m = \sum_{k \geq 0} j_k 2^k$ and $n = \sum_{k \geq 0} j_k - \sum_{k > 0} i_k \left(2^k - 1\right)$.

We need to compute $v_1^* : E^*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow E^*\left(\underline{BP\langle 1\rangle}_6\right)$. These are power series rings. We compute on the indecomposables, $Qv_1^* : QE^*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow QE^*\left(\underline{BP\langle 1\rangle}_6\right)$, which is dual to $Pv_{1*} : PE_*\left(\underline{BP\langle 1\rangle}_6\right) \longrightarrow PE_*\left(\underline{BP\langle 1\rangle}_4\right)$. The latter is simply multiplication by $[v_1]$.

The double suspension factors as

$$b_{(0)} \circ - : E_*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow QE_*\left(\underline{BP\langle 1\rangle}_4\right) \xrightarrow{\cong} PE_*\left(\underline{BP\langle 1\rangle}_6\right) \hookrightarrow E_*\left(\underline{BP\langle 1\rangle}_6\right)$$

and similarly for $PE_*\left(\underline{BP\langle 1\rangle}_4\right)$. We therefore compute the homomorphism of free $E_*$-modules

$$v_{1*} : QE_*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow QE_*\left(\underline{BP\langle 1\rangle}_2\right).$$

We temporarily omit $\circ$ here, as it is the only multiplication in $QE_*\left(\underline{BP\langle 1\rangle}_*\right)$.

The Ravenel-Wilson basis is not the most useful here.

**Theorem 3.10** *The elements $b_i$ for $i > 0$ form a basis of $QE_*\left(\underline{BP\langle 1\rangle}_2\right)$ as a free $E_*$-module.*

To show that a set of elements is a basis, it is sufficient to work $\mathrm{mod}\, I_\infty = (2, v_1, v_2, \cdots)$ and check that we have a basis over $\mathbb{F}_p$, according to the Nakayama Lemma. We introduce the Bendersky generators $h_i$ of $QE_*\left(\underline{BP}_*\right)$ (see [B82]). They may be defined by

$$b(x) = h_0 x +_{[F]} h_1 x^2 +_{[F]} h_2 x^4 +_{[F]} \cdots \tag{2}$$

where $[F]$ denotes the formal group law using the right $BP_*$-action by the elements $[v_i]$. If we replace each $b_{(i)}$ by $h_i$ in the Ravenel-Wilson basis of $QE_*\left(\underline{BP}_*\right)$, we obtain another basis (since $h_i \equiv b_{(i)} \,\mathrm{mod}\, I_\infty$, see below.) When we map to $\underline{BP\langle 1\rangle}_2$, the resulting basis of $E_*\left(\underline{BP\langle 1\rangle}_2\right)$ consists of the elements

$$\left[v_1^k\right] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k}$$

where $k \geq 0$ and $i_0 < i_1 < i_2 < \cdots < i_k$.

The theorem follows from the following lemma.

**Lemma 3.11** *Given $n > 0$, write $n$ in binary form as $n = 2^{i_0} + 2^{i_1} + \cdots + 2^{i_k}$, with $i_0 < i_1 < \cdots < i_k$. Then in $QE_*\left(\underline{BP}_2\right)$,*

$$b_n \equiv \left[v_1^k\right] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k} \bmod \left(I_\infty + \left(\left[v_2\right], \left[v_3\right], \cdots\right)\right).$$

**Proof.** The main relation (see Proposition 2.7) reduces to

$$0 \equiv b\left(\left[2\right]\left(x\right)\right) \equiv \left[v_1\right] b\left(x\right)^2 \equiv \sum_i \left[v_1\right] b_i^2 x^{2i}.$$

The coefficient of $x^{2^{i+1}}$ yields $\left[v_1\right] b_{(i)}^2 \equiv 0$.

We deduce by induction on $i$ that $\left[v_1\right] h_i^2 \equiv 0$, starting from $b_{(0)} = h_0$. When we expand (2) and take the coefficients of $x^{2^i}$, we get $b_{(i)} = h_i +$terms with a factor $\left[v_1\right] h_j^2$ for some $j < i$. Hence $b_{(i)} \equiv h_i$ and $\left[v_1\right] h_i^2 \equiv 0$.

If $\left[v_1\right] x^2 \equiv 0$ and $\left[v_1\right] y^2 \equiv 0$, the right formal group law takes the form $x +_{[F]} y \equiv x + y + \left[v_1\right] xy$. Then the coefficient of $x^n$ in (2) yields $b_n$ on the left, and the only surviving term on the right is $\left[v_1^k\right] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k}$. ∎

**Lemma 3.12** *In $QE_*\left(\underline{BP\left\langle 1\right\rangle}_2\right)$, each $b_n$ can be expressed as a polynomial in the elements $b_{(i)}$ and $\left[v_1\right]$, with no $v_1$ or $v_2$.*

**Proof.** The coefficient of $x^n$ in (2) expresses $b_n$ as a polynomial in the $h_i$ and $\left[v_1\right]$. If $n = 2^i$, we can use $b_{(i)} = h_i + \cdots$ to express $h_i$ as a polynomial in $b_{(i)}$, the $h_j$ for $j < i$, and $\left[v_1\right]$, and hence inductively as a polynomial in the $b_{(j)}$ and $\left[v_1\right]$. ∎

**Remark 3.13** If $n$ is not a power of 2, this polynomial is divisible by $\left[v_1\right]$.

Next, we consider $QE_*\left(\underline{BP\left\langle 1\right\rangle}_4\right)$.

**Lemma 3.14** *If $n$ is not a power of 2, there is an element $\hat{b}_n \in QE_*\left(\underline{BP\left\langle 1\right\rangle}_4\right)$ such that $\left[v_1\right] \hat{b}_n = b_n$ in $QE_*\left(\underline{BP\left\langle 1\right\rangle}_2\right)$.*

**Proof.** When we expand (2) and take the coefficient of $x^n$ to find $b_n$, every term on the right (working $\bmod \left[v_2\right], \left[v_3\right], \cdots$) contains a factor $\left[v_1\right]$. ∎

In particular, by Lemma 3.11,

$$\hat{b}_n \equiv \left[v_1^{k-1}\right] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k} \bmod I_\infty.$$

The only missing elements from the modified Ravenel-Wilson basis of $QE_*\left(\underline{BP\langle 1\rangle}_4\right)$ are the elements $h_i^2 \equiv b_{(i)}^2$ for $i \geq 0$. We therefore take as a basis of $QE_*\left(\underline{BP\langle 1\rangle}_4\right)$, the elements $\hat{b}_n$ for $n$ not a power of 2 and the elements $b_{(i)}^2$. We now compute $v_{1*}$. The main relation, mod 2, is

$$\sum_q b_q\left([2](x)\right)^q = b\left([2](x)\right) \equiv [v_1]\, b(x)^2 \equiv \sum_i [v_1]\, b_i^2 x^{2i}.$$

Recall that $([2](x))^q = \sum_{k\geq 1} C_{q,k} x^{2k}$. Take coefficients of $x^{2^{i+1}}$. Then

$$\begin{cases} v_{1*} b_{(i)}^2 = [v_1]\, b_{(i)}^2 = \sum_{q\leq 2^i} C_{q,2^i} b_q, & \text{for } i \geq 0; \\ v_{1*}\hat{b}_n = [v_1]\,\hat{b}_n = b_n, & \text{for } n \text{ not a power of 2} \end{cases} \tag{3}$$

gives $v_{1*}$ precisely in terms of our bases.

We summarize the relationship between $E_*$, $QE_*$ and $PE_*$ in a diagram:

$$\begin{array}{ccccc}
E_*\left(\underline{BP\langle 1\rangle}_4\right) & \xrightarrow{v_{1*}} & E_*\left(\underline{BP\langle 1\rangle}_2\right) & & \\
\downarrow & & \downarrow & & \\
QE_*\left(\underline{BP\langle 1\rangle}_4\right) & \xrightarrow{Qv_{1*}} & QE_*\left(\underline{BP\langle 1\rangle}_2\right) & & \\
\cong\downarrow - \circ b_{(0)} & & \cong\downarrow & & \\
PE_*\left(\underline{BP\langle 1\rangle}_6\right) & \xrightarrow{Pv_{1*}} & PE_*\left(\underline{BP\langle 1\rangle}_4\right) & & \\
\cap & & \cap & & \\
E_*\left(\underline{BP\langle 1\rangle}_6\right) & \xrightarrow{v_{1*}} & E_*\left(\underline{BP\langle 1\rangle}_4\right) & \xrightarrow{\langle x_{(i)}, -\rangle} & E_* \\
\downarrow & & \downarrow & & \\
QE_*\left(\underline{BP\langle 1\rangle}_6\right) & \xrightarrow{Qv_{1*}} & QE_*\left(\underline{BP\langle 1\rangle}_4\right) & &
\end{array} \tag{4}$$

## 3.1 Generators of Cohomology

We return to the primitives. Our basis of $PE_*\left(\underline{BP\langle 1\rangle}_4\right) = PM_4 \oplus PR_4$ consists of two parts:

1. Elements $b_{(0)} \circ b_n$ for $n$ not a power of 2, which span the submodule $PM_4$.

2. Elements $b_{(0)} \circ b_{(i)}$ for $i \geq 0$, which span the submodule $PR_4$.

Our basis of $PE_*\left(\underline{BP\langle 1\rangle}_6\right) = PM_6 \oplus PR_6$ consists of two parts:

1. Elements $b_{(0)} \circ \hat{b}_n$ for $n$ not a power of 2, which span the submodule $PM_6$.

2. Elements $b_{(0)} \circ b_{(i)}^{\circ 2}$ for $i \geq 0$, which span the submodule $PR_6$.

In terms of these bases, $v_{1*}$ is given by

$$\begin{cases} v_{1*}\left(b_{(0)} \circ \hat{b}_n\right) = [v_1] \circ b_{(0)} \circ \hat{b}_n = b_{(0)} \circ b_n, & \text{for } n \text{ not a power of 2} \\ v_{1*}\left(b_{(0)} \circ b_{(i)}^{\circ 2}\right) = [v_1] \circ b_{(0)} \circ b_{(i)}^{\circ 2} = \sum_{q \leq 2^i} C_{q,2^i} b_{(0)} \circ b_q. \end{cases} \tag{5}$$

Note that $v_{1*}\left(b_{(0)} \circ b_{(i)}^{\circ 2}\right)$ contains terms in $PM_4$.

Now, we can consider the duals $QE^*\left(\underline{BP\langle 1\rangle}_4\right)$ and $QE^*\left(\underline{BP\langle 1\rangle}_6\right)$, the indecomposables of the algebras $E^*\left(\underline{BP\langle 1\rangle}_4\right)$ and $E^*\left(\underline{BP\langle 1\rangle}_6\right)$.

**Proposition 3.15** (In [RWY] Page 6) *Since, for $k \leq 2\left(2^q + 2^{q-1} + \cdots + 2 + 1\right)$, the spaces $\underline{BP\langle q\rangle}_k$ are all torsion-free for ordinary homology, we know that they are $BP_*$-free and the Brown-Peterson cohomology is just the $BP^*$-dual. Likewise, the induced cohomology homomorphisms are just the dual homomorphisms.*

**Proposition 3.16** (In [RWY] Section 8.5 at Page 39) *Both $BP^*\left(\underline{BP\langle 1\rangle}_6\right)$ and $BP^*\left(\underline{BP\langle 1\rangle}_4\right)$ are power series rings on generators dual to the primitives in the $BP$-homology.*

Dualizing to cohomology, we write $E^*\left(\underline{BP\langle 1\rangle}_4\right) = A\widehat{\otimes}M$, where $A$ and $M$ are power series rings generated by the duals of $PR_4$ and $PM_4$, respectively, and $E^*\left(\underline{BP\langle 1\rangle}_6\right) = B\widehat{\otimes}N$, where $B$ and $N$ are power series rings generated by the duals of $PR_6$ and $PM_6$, respectively. Since $v_{1*} : PM_6 \cong PM_4$ is an isomorphism, so is $Qv_1^* : QM \to QN$ (considered as quotient modules). So

$$\operatorname{coker}\left[Qv_1^* : QA \oplus QM \to QB \oplus QN\right] \cong \operatorname{coker}\left[Qv_1^* : QA \to QB\right].$$

We therefore limit attention to $A$ and $B$. Define generators $x_{(i)} \in A$ dual to $b_{(0)} \circ b_{(i)}$ and $y_{(i)} \in B$ dual to $b_{(0)} \circ b_{(i)}^{\circ 2}$, so that

$$A = E^*\left[\left[x_{(i)} : i \geq 0\right]\right]$$

and

$$B = E^* \left[\left[y_{(i)} : i \geq 0\right]\right].$$

It is to be understood that all such rings allow any infinite linear combination of monomials with coefficients in $E^*$ of appropriate degrees. Note that the degree of $x_{(i)}$ is $2(1+2^i)$ and the degree of $y_{(i)}$ is $2(1+2^{i+1})$. Then dualizing (5), in $Qv_1^* : QE^*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow QE^*\left(\underline{BP\langle 1\rangle}_6\right)$, we have

$$v_1^*\left(x_{(i)}\right) = \sum_{j \geq i} C_{2^i,2^j} y_{(j)}.$$

Recall that $C_{2^i,2^j} = A_{2^{j-i}}^{2^i} = A_{(j-i)}^{2^i}$, where we write $A_{(i)} = A_{2^i}$, and that $A_{(0)} = A_1 = v_1$ and $A_{(1)} = A_2 = v_2$. Then, for $i \geq 0$, $Qv_1^* : QA \longrightarrow QB$ is given by

$$v_1^*\left(x_{(i)}\right) = v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{k \geq 2} A_{(k)}^{2^i} y_{(i+k)}. \tag{6}$$

Also,

$$v_1^*\left(u_n\right) = w_n + \sum_j C_{n,2^j} y_{(j)},$$

where $u_n$ is dual to $b_{(0)} \circ b_n$ and $w_n$ is dual to $b_{(0)} \circ \hat{b}_n$.

## 3.2 The Module $QE^*(K(\mathbb{Z},3))$

We consider the indecomposables of the algebra $E^*(K(\mathbb{Z},3))$, where $E = E(1,2)$. Put $z_{(i)} = f^*\left(y_{(i)}\right) \in Q = QE^*(K(\mathbb{Z},3))$; these elements generate the $E^*$-module topologically, with respect to the $z$-filtration in which $F^n = F^nQ$ consists of all elements $\sum_{k \geq n} d_k z_{(k)}$, with $d_k \in E^*$. From (6), we have the relations

$$v_1^{2^i} z_{(i)} + v_2^{2^i} z_{(i+1)} + \sum_{k \geq 2} A_{(k)}^{2^i} z_{(i+k)} = 0$$

for $i \geq 0$. Note that for $k \geq 2$, $A_{(k)}$ is divisible by $v_1 v_2$; write $A_{(k)} = v_1 v_2 A'_{(k)}$.

We also have the $v_1$-filtration, defined by the submodules $v_1^k Q$. Since $v_2$ is invertible, we may rewrite the above relation as

$$z_{(i+1)} = v_1^{2^i}\left(v_2^{-2^i} z_{(i)} + \sum_{k \geq 2}\left(A'_{(k)}\right)^{2^i} z_{(i+k)}\right).$$

**Lemma 3.17** *$Q/v_1^{2^n}Q$ is a free $E^*/\left(v_1^{2^n}\right)$-module generated by $z_{(0)}$.*

**Proof.** It is easy to see that $z_{(0)}$ generates $Q/v_1^{2^n}Q$. We have $Q = E^*z_{(0)} + v_1Q$. Multiply by $v_1^i$; then

$$E^*z_{(0)} + v_1^iQ = E^*z_{(0)} + v_1^iE^*z_{(0)} + v_1^{i+1}Q = E^*z_{(0)} + v_1^{i+1}Q.$$

By induction,

$$Q = E^*z_{(0)} + v_1Q = E^*z_{(0)} + v_1^{2^n}Q.$$

To see that there are no relations, let $F$ be the free $R$-module on generators $y_{(0)}$, $y_{(1)}$, $\cdots$, $y_{(n)}$, considered as a quotient module of $QE^*\left(\underline{BP\langle 1\rangle}_6\right)$ via $q : QE^*\left(\underline{BP\langle 1\rangle}_6\right) \longrightarrow F$, where for convenience we write $R = E^*/\left(v_1^{2^n}\right)$. We note that $Q/v_1^{2^n}Q$ is a quotient of $F$. Suppose $cz_{(0)} = 0$ in $Q/v_1^{2^n}Q$, where $c \in R$. Since $qv_1^*\left(x_{(i)}\right) = 0$ for $i \geq n$, this lifts to a relation

$$cy_{(0)} = \sum_{i=0}^{n-1} c_i qv_1^*\left(x_{(i)}\right) = \sum_{i=0}^{n-1} c_i \left( v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{k=2}^{n-i} v_1^{2^i} v_2^{2^i} \left(A'_{(k)}\right)^{2^i} y_{(i+k)} \right)$$

in $F$, for some coefficients $c_i \in R$. The coefficient of $y_{(0)}$ gives $c = v_1c_0$. The coefficient of $y_{(1)}$ gives $0 = v_1^2c_1 + v_2c_0$, or $c_0 = v_1^2v_2^{-1}c_1$. We show by induction on $i$ that $c_{i-1} = v_1^{2^i}v_2^{-2^{i-1}}\left(1 + a_i\right)c_i$ for $0 < i \leq n-1$, for some $a_i \in (v_1) \subset R$. Assume this holds for $i < m$. The coefficient of $y_{(m)}$, where $0 < m \leq n-1$, in the above relation gives

$$\begin{aligned} 0 &= v_1^{2^m}c_m + v_2^{2^{m-1}}c_{m-1} + \sum_{k=2}^{m} v_1^{2^{m-k}}v_2^{2^{m-k}}\left(A'_{(k)}\right)^{2^{m-k}} c_{m-k} \\ &= v_1^{2^m}c_m + v_2^{2^{m-1}}\left(1 + v_1A\right)c_{m-1} \end{aligned}$$

where by the induction hypothesis we express each $c_{m-k}$ in terms of $c_{m-1}$ and $A \in R$. Now $1 + v_1A$ is invertible in $R$ by $\left(1 + v_1A\right)^{-1} = 1 + a_m$, where $a_m = \sum_{i=1}^{2^n-1} v_1^iA^i$, so that $c_{m-1} = v_1^{2^m}v_2^{-2^{m-1}}\left(1 + a_m\right)c_m$.

The same argument applies when $m = n$, except that there is no $c_n$, to yield $c_{n-1} = 0$. Then $c_i = 0$ for all $i$ and $c = v_1c_0 = 0$. ∎

**Theorem 3.18** *The $z$-filtration and the $v_1$-filtration define the same topology on $Q = QE^*\left(K\left(\mathbb{Z}, 3\right)\right)$. $Q$ is the free $\mathbb{F}_2\left[v_2, v_2^{-1}\right][[v_1]]$-module generated by $z_{(0)}$.*

**Proof.** We have already seen that $z_{(i+1)} \in v_1^{2^i} Q$ for all $i \geq 0$ so that $F^n \subset v_1^{2^{n-1}} Q$. Conversely, $v_1^{2^n-1} Q \subset F^n$. This follows from $v_1^{2^n - 2^i} z_{(i)} \in F^n$ for $0 \leq i \leq n$, by downward induction on $i$, starting from $i = n$. For the induction step, we multiply the above relation by $v_1^{2^n - 2^{i+1}}$.

We know the $z$-filtration is complete, being a quotient of the complete filtered $E^*$-module $QE_* \left( \underline{BP \langle 1 \rangle}_6 \right)$. By the above, the $v_1$-filtration is also complete, and

$$Q = \lim_n Q \big/ v_1^{2^n} Q \cong \lim_n \mathbb{F}_2 \left[ v_2, v_2^{-1} \right] [v_1] \big/ \left( v_1^{2^n} \right) = \mathbb{F}_2 \left[ v_2, v_2^{-1} \right] [[v_1]] .$$

■

We would like to extend this result to the whole algebra $E^* \left( K \left( \mathbb{Z}, 3 \right) \right)$.

## 3.3 Bases of Homology and Cohomology

We need to extend bases of $PE_* \left( \underline{BP \langle 1 \rangle}_6 \right)$ and $PE_* \left( \underline{BP \langle 1 \rangle}_4 \right)$ to $E_* \left( \underline{BP \langle 1 \rangle}_6 \right)$ and $E_* \left( \underline{BP \langle 1 \rangle}_4 \right)$ in order to define $x_{(i)}$ and $y_{(i)}$ as elements of $E^* \left( \underline{BP \langle 1 \rangle}_4 \right)$ and $E^* \left( \underline{BP \langle 1 \rangle}_6 \right)$. There is a standard way to do this.

Given a monomial $c \in E_* \left( \underline{BP \langle 1 \rangle}_r \right)$ in the elements $[v_1]$, $b_{(i)}$ and $b_n$, define $Vc$ by replacing each $b_{(i)}$ by $b_{(i-1)}$ and each $b_n$ by $b_{n/2}$, taking $Vc = 0$ if any $i = 0$ or any $n$ is odd. (This depends on how $c$ is written as a monomial, but the choice does not matter; the Verschiebung $V$ is only defined naturally mod $I_\infty$.)

**Lemma 3.19** *For any $y \in E^* \left( \underline{BP \langle 1 \rangle}_r \right)$ and monomial $c$ as above,*

$$\left\langle y^2, c \right\rangle = \langle y, Vc \rangle^2 .$$

**Proof.** $\langle y^2, c \rangle = \langle \psi^* \left( y \otimes y \right), c \rangle = \langle y \otimes y, \psi c \rangle = \sum \langle y, c' \rangle \langle y, c'' \rangle$, where $\psi c = \Sigma c' \otimes c''$. Suppose that

$$c = [v_1^m] \circ b_{i_1} \circ b_{i_2} \circ \cdots \circ b_{i_k}.$$

Then

$$\psi c = \sum [v_1^m] \circ b_{j_1} \circ b_{j_2} \circ \cdots \circ b_{j_k} \otimes [v_1^m] \circ b_{i_1 - j_1} \circ b_{i_2 - j_2} \circ \cdots \circ b_{i_k - j_k},$$

summing over all $j_1, j_2, \cdots, j_k$, $0 \leq j_r \leq i_r$, and

$$\left\langle y^2, c \right\rangle = \sum \langle y, [v_1^m] \circ b_{j_1} \circ b_{j_2} \circ \cdots \circ b_{j_k} \rangle \langle y, [v_1^m] \circ b_{i_1 - j_1} \circ b_{i_2 - j_2} \circ \cdots \circ b_{i_k - j_k} \rangle .$$

Since we are working mod 2, all terms cancel in pairs except for the desired middle term $\langle y, Vc\rangle \langle y, Vc\rangle$. The same proof works for $*$-products of such monomials $c$. ■

We construct a basis of $E_*\left(\underline{BP\langle 1\rangle}_6\right)$ as follows. We have a basis of $PE_*\left(\underline{BP\langle 1\rangle}_6\right)$ consisting of the elements $b_{(0)} \circ b_{(i)}^{\circ 2}$ (for $i \geq 0$) and $b_{(0)} \circ \hat{b}_n$ (for $n$ not a power of 2). We adjoin new basis elements $b_{(k)} \circ b_{(k+i)}^{\circ 2}$ (for $i \geq 0$ and $k > 0$) and $b_{(k)} \circ \hat{b}_{n,k}$ (for $n$ not a power of 2 and $k > 0$), where $V^k \hat{b}_{n,k} = \hat{b}_n$. The desired basis of $E^*\left(\underline{BP\langle 1\rangle}_6\right)$ consists of all $*$-products of these basis elements (including the empty product 1), without repetition of factors.

We define $y_{(i)}$ as dual to $b_{(0)} \circ b_{(i)}^{\circ 2}$ (for $i \geq 0$). The lemma tells us that, for all $i, k \geq 0$, $y_{(i)}^{2^k}$ is dual to $b_{(k)} \circ b_{(k+i)}^{\circ 2}$. (The duality for other monomials in the $y$'s is far more obscure.) We also define $w_n$ as dual to $b_{(0)} \circ \hat{b}_n$ (for $n$ not a power of 2). Then we have $E^*\left(\underline{BP\langle 1\rangle}_6\right) = B\widehat{\otimes}N$, where $B = E^*\left[\left[y_{(i)} : i \geq 0\right]\right]$ and $N = E^*\left[\left[w_n : n \neq 2^r\right]\right]$.

A similar process extends the basis of $PE_*\left(\underline{BP\langle 1\rangle}_4\right)$ to a basis of $E_*\left(\underline{BP\langle 1\rangle}_4\right)$. We define $x_{(i)}$ as dual to $b_{(0)} \circ b_{(i)}$ and $u_n$ as dual to $b_{(0)} \circ b_n$. Then we have $E^*\left(\underline{BP\langle 1\rangle}_4\right) = A\widehat{\otimes}M$, where $A = E^*\left[\left[x_{(i)} : i \geq 0\right]\right]$ and $M = E^*\left[\left[u_n : n \neq 2^r\right]\right]$.

**Proposition 3.20** *We have a short exact sequence of (completed) Hopf algebras*

$$E^* \longleftarrow E^*(K(\mathbb{Z},3)) \overset{f^*}{\longleftarrow} E^*\left[\left[y_{(i)} : i \geq 0\right]\right] \overset{v_1^*}{\longleftarrow} E^*\left[\left[x_{(i)} : i \geq 0\right]\right] \longleftarrow E^*.$$

# 4 $E(1,2)^*(K(\mathbb{Z},3))$

In this chapter, we assume that $E$ is $E(1,2)$.

We wish to compute $v_1^* : A\widehat{\otimes}M \longrightarrow B\widehat{\otimes}N$. This is more complicated than $Qv_1^*$ because when we take decomposables into account, $A$ no longer maps into $B$. We restate (6), this time including the decomposables.

**Lemma 4.1** *In* $E(1,2)^*\left(\underline{BP\langle 1\rangle}_6\right)$,

$$v_1^*\left(x_{(i)}\right) = v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{k \geq 2} A_{(k)}^{2^i} y_{(i+k)} + D_{(i)}$$

*and*

$$v_1^*(u_n) = w_n + \sum_j C_{n,2^j} y_{(j)} + D_n$$

*where $D_{(i)}$ and $D_n$ are decomposable.* ■

It is easy to simplify the second equation. Because we are working in formal power series algebras, to make a valid change of generators, it is only necessary to check that it is invertible on the indecomposables. Here we simply replace $w_n$ by $w'_n = v_1^*(u_n)$, so that the second equation becomes $v_1^*(u_n) = w'_n$. Then in the quotient algebra $E(1,2)^*(K(\mathbb{Z},3))$, any occurrence of $w'_n$ in $D_{(i)}$ is replaced by zero.

First, we easily recover a well-known result.

**Proposition 4.2** $K(2)^*(K(\mathbb{Z},3)) \cong K(2)^*\left[\left[y_{(0)}\right]\right]$.

**Proof.** In this case, (6) simplifies to $v_1^*\left(x_{(i)}\right) = v_2^{2^i} y_{(i+1)}$. But, $v_2$ is invertible. ■

## 4.1 The Bockstein Spectral Sequence

We have stable cofibrations

$$E(1,2) \xrightarrow{v_1} E(1,2) \longrightarrow K(2).$$

The induced long exact sequence is

$$\begin{aligned} \cdots \quad \longrightarrow \quad & E(1,2)^*(K(\mathbb{Z},3)) \xrightarrow{v_1} E(1,2)^*(K(\mathbb{Z},3)) \\ \longrightarrow \quad & K(2)^*(K(\mathbb{Z},3)) \xrightarrow{\partial} E(1,2)^*(K(\mathbb{Z},3)) \longrightarrow \cdots. \end{aligned}$$

**Proposition 4.3** (In [RWY] Lemma 5.1 at Page 174) *Let $X$ be a space. Let $0 \leq k \leq n$ and $n > 0$. If $K(n)^*(X)$ is even-dimensional, then $E(k,n)^*(X)$ is even-dimensional and has no $v_k$ torsion ($v_0 = p$).*

According to the above section, we know that (ignoring the invertible element $v_2$)

$$\overline{K(2)}^*(K(\mathbb{Z},3)) \cong \mathbb{F}_2\left[\left[y_{(0)}\right]\right]$$

is even-dimensional. This implies $K(2)^*(K(\mathbb{Z},3))$ is even-dimensional. Therefore, $E(1,2)^*(K(\mathbb{Z},3))$ is even-dimensional and has no $v_1$ torsion. Hence, the long exact sequence breaks into the short exact sequence of graded groups,

$$0 \longrightarrow E(1,2)^*(K(\mathbb{Z},3)) \xrightarrow{v_1} E(1,2)^*(K(\mathbb{Z},3)) \longrightarrow K(2)^*(K(\mathbb{Z},3)) \xrightarrow{\partial} 0.$$

The Bockstein spectral sequence degenerates.

The filtration of the Bockstein spectral sequence provides a natural topology. We have the ideal in $E(1,2)^*(K(\mathbb{Z},3))$

$$\left(v_1^i\right) = \mathrm{im}\left(E(1,2)^*(K(\mathbb{Z},3)) \xrightarrow{v_1} E(1,2)^*(K(\mathbb{Z},3)) \xrightarrow{v_1} \cdots \xrightarrow{v_1} E(1,2)^*(K(\mathbb{Z},3))\right)$$

where we multiply by $v_1^i$. This gives us a decreasing filtration by ideals

$$E(1,2)^*(K(\mathbb{Z},3)) = \left(v_1^0\right) \supseteq \left(v_1^1\right) \supseteq \left(v_1^2\right) \supseteq \cdots,$$

where each quotient is a copy of $K(2)^*(K(\mathbb{Z},3)) = K(2)^*\left[\left[y_{(0)}\right]\right]$. In effect, this gives a lower bound for $E(1,2)^*(K(\mathbb{Z},3))$. It gives no information on $(v_1^\infty) = \bigcap_n (v_1^n)$.

**Definition 4.4** *The topology on $E(1,2)^*(K(\mathbb{Z},3))$ from the Bockstein spectral sequence has a basis consisting of the cosets $x + \left(v_1^i\right)$ for all $x \in E(1,2)^*(K(\mathbb{Z},3))$ and all $i$.*

## 4.2 Topology of Cohomology in $E(1,2)^*(K(\mathbb{Z},3))$

We take $E = E(1,2)$ in this section.

**Definition 4.5** *Define $z_{(i)} = f^*\left(y_{(i)}\right)$ in $E^*(K(\mathbb{Z},3))$ for all $i \geq 0$.*

We treat $E^*(K(\mathbb{Z},3))$ as the cokernel (in the sense of algebras) of the algebra homomorphism $v_1^* : A \longrightarrow B$, considered as a quotient of $v_1^* : E^*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow E^*\left(\underline{BP\langle 1\rangle}_6\right)$. Thus, the elements $z_{(i)}$ generate $E^*(K(\mathbb{Z},3))$ topologically. The formula for $v_1^*\left(x_{(i)}\right)$ provides the relation

$$v_1^{2^i} z_{(i)} + v_2^{2^i} z_{(i+1)} + \sum_{j \geq i+2} A_{(j-i)}^{2^i} z_{(j)} = D_{(i)},$$

where $D_{(i)}$ is decomposable.

We have two topologies on $E^*(K(\mathbb{Z},3))$. The first, from the filtration by powers of the ideal $(v_1)$, was discussed in Section 4.1. This is the $v_1$-topology. We have good information on the quotients by $(v_1^n)$, but no information on $(v_1^\infty)$. The second is inherited from the topology on $E^*\left(\underline{BP\langle 1\rangle}_6\right)$ as dual to the algebra $E_*\left(\underline{BP\langle 1\rangle}_6\right)$.

In the quotient by a basic neighborhood of 0, we retain only finitely many of the monomials in the $z_{(i)}$. We call this the $z$-topology.

We compare these two topologies on $E^* (K (\mathbb{Z}, 3))$. They are not the same. Let $Z$ be the augmentation ideal generated topologically by all the $z_{(i)}$. Below, we display schematically the $v_1$-filtration and the filtration by (completed) powers of $Z$, where we treat $\mathbb{F}_2 \left[v_2, v_2^{-1}\right]$ as the graded ground field and suppress it.

| | | | | |
|---|---|---|---|---|
| $\cdots$ | $z^3_{(0)}$ | $z^2_{(0)}$ | $z_{(0)}$ | $1$ |
| | | | $z_{(1)}$ | $v_1$ |
| | | $z^2_{(1)}$ | | $v_1^2$ |
| | $z^3_{(1)}$ | | $z_{(2)}$ | $v_1^3$ |
| | | | $\vdots$ | $\vdots$ |

As with any path-connected space, there is a canonical $E^*$-module decomposition $E^* (K (\mathbb{Z}, 3)) \cong E^* 1 \oplus Z$. It is obvious that no power of $v_1$ lies in $Z$, and Section 4.1 shows that no power of $z_{(0)}$ lies in $(v_1)$.

We need another coarser filtration that we call the $z_+$-filtration. We filter $E^* (K (\mathbb{Z}, 3))$ by ideals generated topologically by all except finitely many monomials in the $z_{(i)}$ for $i > 0$. This is again a complete filtration, being a quotient of the corresponding $y_+$-filtration on $B$ indicated by writing $B = \left(E^* \left[\left[y_{(0)}\right]\right]\right) \left[\left[y_{(i)} : i > 0\right]\right]$.

**Lemma 4.6** *The $v_1$-filtration and the $z_+$-filtration define the same topology on the augmentation ideal $Z$.*

**Proof.** A basic neighborhood $U$ if 0 in the $z_+$-filtration is the ideal generated (topologically) by the elements $z_{(i)}$ for $i \geq k$ and $Z^m$, for some $k$ and $m$. We saw already in the proof of Theorem 3.18 that $v_1^{2^k-1} z_{(i)} \in U + Z^2$ for all $i$, i.e., $v_1^{2^k-1} Z \subset U + Z^2$. Then $v_1^{2^k-1} Z^j \subset U + Z^{j+1}$ for all $j$, and

$$
\begin{aligned}
v_1^{(m-1)\left(2^k-1\right)} Z &\subset U + v_1^{(m-2)\left(2^k-1\right)} Z^2 \subset U + v_1^{(m-3)\left(2^k-1\right)} Z^3 \subset \cdots \\
&\subset U + v_1^{2^k-1} Z^{m-1} \subset U + Z^m = U.
\end{aligned}
$$

Conversely, given $n$, we have seen that $z_{(i)} \in (v_1^n)$ if $2^{i-1} \geq n$ and that $z_{(i)}^n \in (v_1^n)$ for all $i > 0$. Thus the $z_+$-neighborhood $U$ lies in $(v_1^n)$ if we take $m = n$ and choose $k$ such that $2^{k-1} \geq n$. ■

**Remark 4.7** The $v_1$-filtration is clearly not complete in the summand $E^*1$. Both the $z$- and $z_+$-filtrations are complete, but make the $E^*1$ summand discrete.

**Corollary 4.8** *The $z$-filtration and the filtration consisting of the ideals $((v_1^i) \cap Z) + \left(z_{(0)}^j\right)$ for $i, j \geq 0$ define the same topology on $E^*(K(\mathbb{Z}, 3))$.*

**Proof.** A basic $z$-neighborhood of 0 has the form $U + \left(z_{(0)}^j\right)$, where $U$ is a $z_+$-neighborhood of 0. Choose $i$ such that $(v_1^i) \cap Z \subset U$. Then $((v_1^i) \cap Z) + \left(z_{(0)}^j\right) \subset U + \left(z_{(0)}^j\right)$.

Conversely, given an ideal $((v_1^i) \cap Z) + \left(z_{(0)}^j\right)$, we can find a $z_+$-neighborhood $U$ of 0 such that $U \subset (v_1^i) \cap Z$. Then we have a $z$-neighborhood $U + \left(z_{(0)}^j\right) \subset ((v_1^i) \cap Z) + \left(z_{(0)}^j\right)$. ■

**Theorem 4.9** *As a topological ring, $E(1,2)^*(K(\mathbb{Z},3)) \cong E(1,2)^*\left[\left[z_{(0)}\right]\right]^\wedge$, where the completion indicates $v_1$-completion except on the $E^*$-module summand $E^*1$ generated by 1. Precisely,*

$$E(1,2)^*(K(\mathbb{Z},3)) \cong E^*1 \oplus \mathbb{F}_2\left[v_2, v_2^{-1}\right]\left[\left[v_1, z_{(0)}\right]\right] z_{(0)}.$$

**Proof.** By Lemma 4.6, $Z$ is complete under the $v_1$-filtration. Section 4.1 shows that $E^*(K(\mathbb{Z},3)) / (v_1^n) \cong \mathbb{F}_2\left[v_2, v_2^{-1}\right]\left[\left[z_{(0)}\right]\right][v_1] / (v_1^n)$. Hence

$$\begin{aligned} Z &= \lim_n Z / v_1^n Z = \left(\lim_n \mathbb{F}_2\left[v_2, v_2^{-1}\right]\left[\left[z_{(0)}\right]\right][v_1] / (v_1^n)\right) z_{(0)} \\ &= \mathbb{F}_2\left[v_2, v_2^{-1}\right]\left[\left[z_{(0)}\right]\right][[v_1]] z_{(0)}. \end{aligned}$$

The topology on $E^*(K(\mathbb{Z},3))$ is given by the $z$-filtration, or, in view of Corollary 4.8, by the ideals $((v_1^i) \cap Z) + \left(z_{(0)}^j\right)$. We can now rewrite $(v_1^i) \cap Z$ as $\left(v_1^i z_{(0)}\right)$. The summand $E^*1$ is simply a copy of $E^*$, with no completion. ■

## 4.3 Relations in $E(1,2)^*(K(\mathbb{Z},3))$

Furthermore, we try to generalize Tamanoi's result to higher degree. Therefore, we change our generators a little to get a better looking formula. We note that we do not use the invertibility of $v_2$. Again, we write $E = E(1,2)$.

Earlier, we showed that in $E^*\left(\underline{BP\langle 1\rangle}_6\right)$,

$$v_1^*\left(x_{(i)}\right) = v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{j\geq i+2} A_{(j-i)}^{2^i} y_{(j)} + D_{(i)}$$

for $i \geq 0$, with $D_{(i)}$ decomposable. It would be desirable to change generators to eliminate the unwanted terms. We can do the following:

**Theorem 4.10** *There are alternate generators $y'_{(i)}$ of the algebra $B$ such that $B = E^*\left[\left[y'_{(i)} : i \geq 0\right]\right]$ and*

$$v_1^*\left(x_{(i)}\right) = v_1^{2^i} y'_{(i)} + v_2^{2^i} y'_{(i+1)} + D_{(i)}$$

*for all $i \geq 0$, where $D_{(i)}$ is decomposable.*

Instead of changing generators directly, it is more convenient to change the basis of $QE_*\left(\underline{BP\langle 1\rangle}_6\right)$ before dualizing; then the change of generators will automatically be continuous. First, we need to improve a previous observation.

**Lemma 4.11** *For $k \geq 2$, $A_{(k)}$ is divisible by $v_1^2$.*

**Proof.** Recall from Lemma 2.10 that $A_{(k)}$ is the coefficient of $x^{2^{k+1}}$ in

$$[2]\,(x) = v_1 x^2 +_F v_2 x^4 = v_1 x^2 + v_2 x^4 + \sum_{i,j\geq 1} a_{ij} v_1^i v_2^j x^{2i+4j}.$$

Modulo $v_1^2$, the only terms that survive are $v_1 x^2 + v_2 x^4 + \sum_{j\geq 1} a_{1j} v_1 v_2^j x^{4j+2}$. There is no term in $x^{2^n}$ for $n \geq 3$. ■

We consider $Qv_{1*} : QE_*\left(\underline{BP\langle 1\rangle}_4\right) \longrightarrow QE_*\left(\underline{BP\langle 1\rangle}_2\right)$, as before. Again, we suppress $\circ$ here, as it is the only multiplication.

**Lemma 4.12** *In $QE_*\left(\underline{BP\langle 1\rangle}_4\right)$, there are elements $d_{(i)}$ and $e_{(i)}$ such that:*

*(a) $v_{1*}\left(d_{(i)}\right) = v_1^{2^{i+1}-1} b_{(i)}$ for all $i \geq 0$.*

*(b) The elements $e_{(i)}$ for all $i \geq 0$ form a basis, and satisfy*

$$v_{1*}\left(e_{(i)}\right) = v_1^{2^i} b_{(i)} + v_2^{2^{i-1}} b_{(i-1)}.$$

**Proof.** We construct $d_{(i)}$ and $e_{(i)}$ inductively, starting from $d_{(0)} = e_{(0)} = b_{(0)}^2$.

Suppose we have $d_{(i)}$ for $i < k$. We recall from (6) that

$$v_{1*}\left(b_{(k)}^2\right) = v_1^{2^k} b_{(k)} + v_2^{2^{k-1}} y_{(k-1)} + \sum_{i \geq 2} A_{(i)}^{2^{k-i}} y_{(k-i)}.$$

By induction,

$$v_{1*}\left(\left(v_1^{-2^{k-i+1}+1} A_{(i)}^{2^{k-i}}\right) d_{(k-i)}\right) = A_{(i)}^{2^{k-i}} b_{(k-i)},$$

where we note that the coefficient of $d_{(k-i)}$ lies in $E_*$. We set

$$e_{(k)} = b_{(k)}^2 + \sum_{i \geq 2} \left(v_1^{-2^{k-i+1}+1} A_{(i)}^{2^{k-i}}\right) d_{(k-i)},$$

to cancel out the unwanted terms. To remove the $b_{(k-1)}$ term also, we multiply by a further $v_1^{2^k-1}$ and set $d_{(k)} = v_1^{2^{k-1}} e_{(k)} + v_2^{2^{k-1}} d_{(k-1)}$, so that

$$v_{1*}\left(d_{(k)}\right) = v_1^{2^k-1+2^k} b_{(k)} + v_1^{2^k-1} v_2^{2^{k-1}} b_{(k-1)} + v_1^{2^k-1} v_2^{2^{k-1}} b_{(k-1)} = v_1^{2^{k+1}-1} b_{(k)}.$$

■

**Remark 4.13** Closer examination of the proof shows that the elements $e_{(i)}$ are uniquely determined by property (b).

**Proof of Theorem.** We proceed as before. We use the elements $b_{(0)} \circ e_{(i)}$ as a basis of $PE_*\left(\underline{BP\langle 1\rangle_6}\right)$, instead of the $b_{(0)} \circ b_{(i)}^{\circ 2}$, and extend to a basis of the whole of $E_*\left(\underline{BP\langle 1\rangle_6}\right)$. We define $y'_{(i)}$ as dual to $b_{(0)} \circ e_{(i)}$. The simplified formula for $v_{1*}\left(e_{(i)}\right)$ leads to the simplified formula for $v_1^*\left(y'_{(i)}\right)$. ■

**Remark 4.14** This leads to a more transparent interpretation of Theorem 4.9. Put $z'_{(i)} = f^*\left(y'_{(i)}\right)$ for each $i$. In terms of these generators of $E^*(K(\mathbb{Z},3))$, the relations become

$$v_1^{2^i} z'_{(i)} + v_2^{2^i} z'_{(i+1)} = D_{(i)}.$$

Modulo decomposables, $z'_{(i+1)} \equiv v_1^{2^i} v_2^{-2^i} z'_{(i)}$, so by induction

$$z'_{(n)} \equiv v_1^{2^n-1} v_2^{-2^n+1} z'_{(0)}$$

for all $n$.

**Remark 4.15** It would be desirable to find a nonlinear change of generators that would remove the terms $D_{(i)}$ as well, but this appears far more difficult.

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## Vita

Hsin-hao Su was born in November 1972 and was raised in Taipei, Taiwan. In 1995, he received a Bachelor of Arts and Science degree in applied mathematics from Feng Chia University, Taichung, Taiwan. In 1998, he received a Master of Science degree in mathematics from National Tsing Hua University, Hsinchu, Taiwan. He enrolled in the graduate program at Johns Hopkins University in the fall of 2000. He defended this thesis on March 21, 2006.