ENERGIES OF ZEROS OF RANDOM SECTIONS ON RIEMANN SURFACES

by

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ABSTRACT

The purpose of this thesis is to determine the asymptotic of the average energy of a configuration of N zeros of a system of random polynomials of degree N as $N \to \infty$ and more generally the zeros of random holomorphic sections of a line bundle $L \to M$ over any Riemann surface M. We also compare our results to the well-known minimum of energies.

Readers:Professor Steve Zelditch (Advisor)

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DEDICATION

I dedicate this dissertation to my parents Guangen and Chunyan and my wife Mohan without whose unwavering love and support it would never have been completed.

Contents

1 Introduction

This thesis is concerned with the asymptotic of the average energy of the configuration of zeros of N-degree random polynomials as $N \to \infty$ and more generally the zeros of random holomorphic sections of a line bundle $L \to M$ over any compact Riemann surface without boundary. The energy of a configuration of points $\{z_1, \ldots, z_N\}$ on a surface M equipped with a Riemannian metric g is defined by

$$\mathcal{E}_{G_g}^N = \sum_{i \neq j} G_g(z_i, z_j), \tag{1.1}$$

where G_g is the Green's function for g, the kernel of inverse Laplacian on the orthogonal complement of the constant function (see §2.5); other energies will also be studied. Electrons moving freely on a surface distribute themselves in a minimal energy configuration, and many articles have been devoted to finding the minimal energy configurations and the asymptotic of the minimal energy.

The question studied in this thesis is the extent to which zeros of random polynomials of degree N tend to resemble minimal energy configurations of N points. Zeros of random polynomials in complex dimension one repel and like minimal energy configurations tend to stay $1/\sqrt{N}$ apart. Our main results show that if the Gaussian ensemble is defined by the same metric as the Green's function, the average energy of such random zeros is of the same order of magnitude as that of minimal energy configurations, and both energies have the same leading constants.

To state our results, we need some notation. Throughout the thesis, we identify polynomials of degree N with holomorphic sections $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ of the N-th power of the hyperplane section bundle over the complex projective line $\mathbb{CP}^1$. Our methods apply equally to holomorphic sections $H^0(M, L^N)$ of powers of a positive holomorphic line bundle $L \to M$ over any compact Riemann surface. Thus, in addition to studying zeros of polynomials, we study zeros of random theta functions over a Riemann surface of genus one, and zeros of random holomorphic k-differentials over a surface of higher genus. Moreover, our results apply to general Kähler metrics g on these Riemann surfaces.

As we recall in §2.1, a choice of Hermitian metrics on L induces inner products on $H^0(M, L^N)$ and then a Gaussian measure μ_{h^N} on thess spaces. Roughly speaking, a random section S_c is expressed in terms of an orthonormal basis S_j of $H^0(M, L^N)$ as $S_c = \sum c_j S_j$ where the $c_j' s$ are independent complex Gaussian random variables. Also, we define the Riemannian form $dV_M = \omega_h = \frac{i}{2}\Theta_h$. Here, Θ_h is the curvature form (see §2.1).

We call $(H^0(M, L^N), \mu_{h^N})$ the Hermitian Gaussian ensemble of random sections. Let $\mathbf{E}_{\mu_{h^N}}$ be the associated expected (average) value. We define $\mathbf{E}_{\mu_{h^N}} \mathcal{E}^N_{G_g}$, the expected (average) value of the energy of the zeros of Gaussian random sections chosen from the ensemble $(H^0(M, L^N), \mu_{h^N})$, by

$$\mathbf{E}_{\mu_{h^N}} \mathcal{E}_{G_g}^N = \int_{H^0(M,L^N)} \mathcal{E}_{G_g}(Z_s)\, d\mu_{h^N}(s) = \int_{\mathbb{C}^{d_N}} \mathcal{E}_{G_g}(Z_s) \frac{e^{-|c|^2}}{\pi^{d_N}}\, dc, \qquad (1.2)$$

where

$$\mathcal{E}_{G_g}^N(Z_s) = \mathcal{E}_{G_g}^N(z_1, \ldots, z_N) = \sum_{\substack{z_i \neq z_j \\ z_i, z_j \in Z_s}} G_g(z_i, z_j), \qquad (1.3)$$

$Z_s = \{z_1, \ldots, z_N\}$ is the set of zeros of s, and $d_N = \dim H^0(M, L^N)$.

Note that if s has double zeros, the energy is infinite, but this occurs with zero probability.

We assume the Chern class of L, $c_1(L) = 1$. Before stating our main result, we should point out that our methods apply to any Riemannian metric g on M and Hermitian metric h on L. They have a special form when g is associated with h in the sense that

$$\omega_h = \frac{\sqrt{-1}}{2} g_{1\bar{1}}\, dz \wedge d\bar{z},$$

where $g_{1\bar{1}}$ is the local representation of g. In this case, the expected (average) energy satisfies:

Theorem 1.1 *Let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle on a compact Riemann surface M with $\frac{i}{2}\Theta_h = \omega_h$, where ω_h gives M a Riemannian form. Let $(H^0(M, L^N), \mu_{h^N})$ be the associated Hermitian Gaussian ensemble. If the metric g_h for Green's function G_g is associated* *with h, then the expected average of Green's function energy of zeros of random sections of L^N is given by:*

$$\mathbf{E}_{\mu_{h^N}}\mathcal{E}^N_{G_{g_h}} = -\frac{1}{4\pi}N\log N - \frac{1}{4\pi}N - N\int_M F_{g_h}(z,z)\frac{\omega_h(z)}{\pi} + o(N), \quad (1.4)$$

where $\int_M F_{g_h}(z,z)\frac{\omega_h(z)}{\pi}$ is the Robin constant.

If we do not assume that g is associated with h, then the asymptotics change:

Theorem 1.2 *With the same notation as in Theorem 1.1, let us consider a general pair h and g, then the expected average of the Green's function energy of zeros of random sections is given by:*

$$\mathbf{E}_{\mu_{h^N}}\mathcal{E}^N_{G_g} = C_g N^2 - \frac{1}{4\pi}N\log N + \frac{N}{2\pi}\int_M \log\lambda(z)\frac{\omega_h(z)}{\pi} - \frac{N}{4\pi} + o(N), \quad (1.5)$$

where $C_g = \int_{M\times M} G_g(z,w)\frac{\omega_h(z)}{\pi}\wedge\frac{\omega_h(w)}{\pi} \geq 0$ and $\lambda(z)$ is a positive function given in the proof.

Remark 1.1

- If g is derived from h, then $C_g = 0$.
- In the future, we simply use μ_h and $\mathbf{E}$ instead of μ_{h^N} and $\mathbf{E}_{\mu_{h^N}}$ respectively.

In [10], Hriljac mentioned that Elkies proved that

$$\min_{z_1,\dots,z_N}\left(\sum_{i\neq j} G_g(z_i,z_j)\right) \geq -\frac{1}{4\pi}N\log N - \frac{11}{6\pi}N + o(N), \quad (1.6)$$

for any Riemannian metric.

Remark 1.2 1. We see that the leading order term in equation (1.6) is the same as the one in equation (1.4). This means that the probability of the energy above the minimum goes to zero as $N \to \infty$ by Chebychev's inequality, i.e.

$$P(s : \mathcal{E}_{G_g}(s) \geq a + \epsilon N \log N) \leq \frac{o(N)}{\epsilon},$$

where $a = -\frac{1}{4\pi} N \log N - \frac{11}{6\pi} N + o(N)$ which is the minimum of the energy.

2. The expected Green's function energy is scale invariant. That is, if we rescale the metric $g \to rg$, then our result (1.4) doesn't change. When $g \to rg$, Δ_g becomes $\frac{1}{r}\Delta_g$. As we discuss in §2.5, $G_g(z, w)$ is the kernel of $(-\Delta^{-1})$, $G_g(z, w)\, dV_g \to rG_g(z, w)\, dV_g$ as $g \to rg$. On the other hand, $dV_g \to r\, dV_g$ as $g \to rg$. Therefore, $G_g(z, w)$ doesn't change as $g \to rg$.

3. When $C_g = 0$, then the leading order term is independent of h; we will explain this in the remark of §3.1.

4. We define the Green's function G_g to be positive near the diagonal. Since the average of energy $\mathbf{E}\mathcal{E}^N_{G_g}$ is negative when g is derived from h, the distant pairs (z_i, z_j), not neighboring pairs, dominate the energy.

In the case of $\mathbb{CP}^1$, we also consider the energies $\mathcal{E}_s$ defined by

$$\mathcal{E}_s(z_1, \ldots, z_N) = \sum_{i \neq j} \frac{1}{[z_i, z_j]^s}. \tag{1.7}$$

In the case $s = 0$, the s-energy is defined to be the logarithmic energy

$$\mathcal{E}_0(z_1, \ldots, z_N) = \sum_{i \neq j} -\log[z_i, z_j]. \tag{1.8}$$

Here $[z, w]$ is the chordal distance between two points on S^2, where z and w are the two points on $\mathbb{CP}^1$ corresponding to some points on S^2. If r is the geodesic distance on S^2, then the relation between $[\cdot, \cdot]$ and r is

$$[a, b] = \sqrt{2(1 - \cos r(a, b))} \tag{1.9}$$

where, a and b are points on S^2. $-\log[z, w]$ is so-called elliptic Green's function which differs by a constant with the Green's function G_g, i.e. $G_g[z, w] = -\log[z, w] - C$.

Theorem 1.3 *1. Consider $\mathbb{CP}^1$ with the Fubini-Study metric g. Let us consider the hyperplane line bunlde $(L = \mathcal{O}(1), h) \to \mathbb{CP}^1$ with $\frac{i}{2}\Theta_h = \omega_g$, where Θ_h is the curvature form. we recall equation (1.7) and have expected s-energy:*

- *when* $s = 2$

$$\mathbf{E}\mathcal{E}_2^N = \frac{1}{4}N^2 \log N + \frac{3N^2}{4}\log(\log N) + \frac{N^2}{2}\log 2 + \frac{1}{2}N^2(\frac{M}{L})^2 - 2N^2 \log\frac{M}{L} + o(N^2). \tag{1.10}$$

Here, M and L are constants which will be explained in the proof of the Theorem.

- *When* $s < 2$

$$\mathbf{E}\mathcal{E}_{s<2}^N = \frac{2^{1-s}}{2-s}N^2 + \frac{1}{(2(2-s))}N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}} + o(N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}}) \tag{1.11}$$

- *When* $2 < s < 4$

$$\mathbf{E}\mathcal{E}_{2<s<4}^N = C\frac{N^{1+\frac{s}{2}}}{4-s} + O(N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}}). \tag{1.12}$$

We will discuss the constant C in the remark at the end of this section.

2. *Under the same conditions as above, we recall equation (1.8) and have expected logarithmic energy:*

$$\mathbf{E}\mathcal{E}_0^N = -(\log 2 - \frac{1}{2})N^2 + \frac{N}{2}\log^2 N - \frac{1}{2}N\log(\log N)\log N + \frac{1}{2}N\log N + \frac{1}{2}(\log 2 + 1)N + o(N). \tag{1.13}$$

Let us compare our results on average energy to the prior results on minimal energy. For the s-energy case, Saff-Kuijilaars in [12] identified $\mathbb{CP}^1$ as $S^2 \in \mathbb{R}^3$ and considered the energy

$$\mathcal{E}'_s(x_1, \dots, x_N) = \sum_{i<j} \frac{1}{|x_i - x_j|^s}$$

where x_i are the points on $S^2 \subset \mathbb{R}^3$ not on $\mathbb{CP}^1$ and $|x - y|$ is the chordal distance of S^2 which is $r(x, y)$ in the equation (1.9). They investigated the energy $\mathcal{E}'_s$. Moreover, they define the minimal s-energy for N points on the sphere

$$E_s(N) := \min_{\{x_1,\dots,x_N\}} \{\mathcal{E}'_s\}.$$

It was proved by Saff-Kuijlaars that when $s = 2$,

$$E_2(S^2, N) \sim \frac{1}{8}\, N^2 \log N. \tag{1.14}$$

When $s > 2$,

$$C_1 N^{1+s/2} \leq E_s(N) \leq C_2 N^{1+s/2}, \tag{1.15}$$

$C_1, C_2 > 0$. When $s < 2$,

$$E_2(S^2, N) \leq \frac{1}{2} V_2(s) N^2 - C N^{1+\frac{s}{2}}, \tag{1.16}$$

where, $C > 0$ and $V_2(s) = \frac{\Gamma(\frac{3}{2})\Gamma(2-s)}{\Gamma((2-s+1)/2)\Gamma(2-\frac{s}{2})}$.

B. Bergersen, D. Boal and P. Palffy-Muboray in [2] identified $\mathbb{CP}^1$ as

$S^2 \in \mathbb{R}^3$ and considered the energy

$$\mathcal{E}_0'(x_1, \ldots, x_N) = \sum_{i<j} -\log|x_i - x_j|.$$

They investigated the ground-state energy of the logarithm energy of N points $\{x_1, ..., x_N\}$, which is the minimal energy of $\mathcal{E}_0'$ for large N:

$$\min_{\{x_1,\ldots,x_N\}} \{\mathcal{E}_0'\} = -(\frac{1}{2}\log 2 - \frac{1}{4})N^2 - \frac{N}{4}\log N + \cdots^1 \tag{1.17}$$

and in that paper, gave a formula for the ground-state energy E.

Remark 1.3

- Our s-energy is twice the s-energy in [12], $\mathcal{E}_s = 2\mathcal{E}_s'$. So by equations (1.10) and (1.14), we see that when $s = 2$ the leading order term of the expected average of energy is the same as the one in minimum energy. The 0-energy has the same property.

- In equation (1.12), we can't figure out the constant precisely. Actually, it is a conjecture in [12]. Since in the Green's function energy, 2-energy and 0-energy, all the leading order terms of expected average are the same as the one in minimum energy, this paper probably offers a method to solve the conjecture. It will be discussed more after the proof of Theorem 1.3(1).

An additional motivation for studying energies of random zeros is that it gives an example of numerical integration over the Riemann surface. In

[1] We use E_o to be consistent with our notation, in [2], they used a different notation.

the Monte Carlo method of numerical integration, one integrates a function with respect to a probability measure μ by generating N random points from the ensemble (M, μ) and averaging over the points. In this thesis, we generate N random points from (M, ω_h) by taking the zeros of a random polynomial. The same numerical integration procedure is used in the recent paper [6] to numerically integrate quantities over Calabi-Yau threefolds. In this thesis we use a more elementary numerical integrations to show the speed of convergence of the integration procedure.

2 Background

We begin with some notations and basic properties of sections of holomorphic line bundles, Gaussian measures and the relationship between polynomials and sections. The notations are the same as in [3] and [16]. Here we only deal with the complex dimension one case, and [3] discuss the general case.

2.1 Complex Geometry

2.1.1 Positive Holomorphic Line Bundles

- Holomorphic line bundles

Definition 2.1 *A complex manifold M is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate map $\psi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic on $\psi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β.*

A one-dimensional complex manifold is called a Riemann surface. Let M be a Riemann surface.

Definition 2.2 *A holomorphic line bundle L on M consists of a family $\{L_z\}_{z\in M}$ of complex lines parametrized by M, together with a holomorphic structure on $L = \bigcup_{z\in M} L_z$ such that*

1. *The projection map $\pi : L \to M$ taking L_z to z is holomorphic, and*

2. *For every $z \in M$, there exists an open set U in M containing z and a biholomorphism*

$$\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}$$

taking the vector space L_z isomorphically onto $\{z\} \times \mathbb{C}$ for each $z \in U$; φ_U is called a trivialization of L over U.

L_z is called the fiber of L at z. We denote by L^N the line bundle as N's tensor powers of L with the fiber $L_z \otimes \cdots \otimes L_z$, for each $z \in M$.

For example: let $T'_z(M)$ be the holomorphic tangent space to M at z. So $T'_z(M) = \mathbb{C}\{\frac{\partial}{\partial z}\}$. $T(M)$ is the tangent bundle to M, and $T_z(M) = T'_z(M) \oplus T"_z(M)$ where $T"_z(M)$ is the antiholomorphic tangent space to M at z. The dual bundle of TM is called cotangent bundle T^*M, whose fiber is cotangent space to M. $T^*_z(M) = \mathbb{C}\{dz\} \oplus \mathbb{C}\{d\bar{z}\}$.

Definition 2.3 *A Riemannian metric g on M is a correspondence with associates to each point $z \in M$ an inner product $\langle\ ,\ \rangle_z$ on the tangent space $T_z(M)$.*

Locally $g = g_{1\bar{1}} dz \otimes d\bar{z}$ and $g_{1\bar{1}}$ is called the local representation of the Riemannian metric g.

Definition 2.4 *A 1-1 form φ can be written as $\varphi = f(z, \bar{z}) dz \wedge d\bar{z}$.*

- Holomorphic sections

Definition 2.5 *A section s of the vector bundle $L \to M$ is a map*

$$s : M \to L$$

such that $s(z) \in L_z$ for all $z \in M$.

If s is a holomorphic map, then s is called a holomorphic section. Let $H^0(M, L)$ be the space of holomorphic sections, then $H^0(M, L)$ has a finite complex dimension because of compactness of M. Let e_L be a local non-vanishing section for L over $U \subset M$, then s can be written as $s(z) = g(z)e_L(z)$ for any $z \in U$, where $g(z)$ is a holomorphic function. The zeros of $s(z)$ are the those of $g(z)$.

- Hermitian metrics

Definition 2.6 *A Hermitian metric on L is a Hermitian inner product on each fiber L_z of L, varying smoothly with $z \in M$, i.e., such that if e_L is a local non-vanishing section for L, then the functions*

$$h(z) = (e_L(z), e_L(z))$$

are smooth.

The curvature form corresponding to h is given by

$$\Theta_h = -\partial\bar{\partial} \log \|e_L\|_h^2,$$

where, $\|e_L\|_h = h(e_L, e_L)^{1/2}$. $\omega_h = \frac{\sqrt{-1}}{2}\Theta_h$ is the Kähler form corresponding to h. The Chern class $C_1(L)$ is defined by

$$C_1(L) = \frac{\sqrt{-1}}{2\pi}\int_M \Theta_h.$$

We give $H^0(M, L)$ the Hermitian inner product

$$\langle s_1, s_2\rangle = \int_M h(s_1, s_2)\,\omega_h \qquad (s_1, s_2 \in H^0(M, L)).$$

Definition 2.7 *A Szegö kernel* $\Pi(z, w)$ *of* $(L, h) \to M$ *is defined by*

$$P(s(z)) = \int_M \Pi(z, w)s(w)\omega_h,$$

where, $P : L^2(M, L) \to H^0(M, L)$ *is the orthogonal projection onto* $H^0(M, L)$ *and* $s \in L^2(M, L)$ *the space of* L^2 *sections.*

Actually, the Szegö kernel is the Schwartz kernel of P.

- Positivity

Definition 2.8 *A holomorphic line bundle* $L \to M$ *is positive if there exists a metric on* L *with curvature form* Θ *such that* $(\sqrt{-1}/2\pi)\Theta$ *is a positive* $(1, 1)$ *form, i.e. for every holomorphic tangent vector* $v \in T'_z(M)$,

$$\sqrt{-1}\langle(\sqrt{-1}/2\pi)\Theta, v \wedge \bar{v}\rangle > 0.$$

2.1.2 Zeros Currents And The Poincaré Lelong Formula

We denote by $(L, h) \to M$ a holomorphic line bundle with smooth Hermitian metric h whose curvature form Θ_h is a positive $(1,1)$-form. Here, e_L is a local non-vanishing holomorphic section of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ is the h-norm of e_L. As in [3], we give M the Hermitian metric corresponding to the Kähler form $\omega_h = \frac{\sqrt{-1}}{2}\Theta_h$ and we can use ω_h as a Riemannian volume form for M.

Moreover,

$$\begin{aligned} \omega_h(z) &= -\frac{i}{2}\partial\bar{\partial}\log\|e_L\|_h^2 \\ &= i\partial_z\bar{\partial}_z\varphi(z), \end{aligned} \tag{2.1}$$

where $\|e_L\|_h = e^{-\varphi(z)}$, φ is the Kähler potential and $\partial_z = \frac{\partial}{\partial z}\,dz$. We denote by $H^0(M, L^N)$ the space of holomorphic sections of $L^N = L \otimes \cdots \otimes L$. The metric h induces Hermitian metrics h^N on L^N given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. We give $H^0(M, L^N)$ the Hermitian inner product

$$\langle s_1, s_2\rangle = \int_M h^N(s_1, s_2)\,\omega_h \qquad (s_1, s_2 \in H^0(M, L^N)), \tag{2.2}$$

and we write $\|s\| = \langle s, s\rangle^{1/2}$. As definition (2.7), we can define the Szegö projection P_N and Szegö kernel Π_N as follow:

$$P_N(s(z)) = \int_M \Pi_N(z,w)s(w)\omega_h,$$

where, $P_N : L^2(M, L^N) \to H^0(M, L^N)$ is the orthogonal projection onto $H^0(M, L^N)$ and $s \in L^2(M, L^N)$ the space of L^2 sections.

For a holomorphic section $s \in H^0(M, L^N)$, we let Z_s denote the current of integration over the zero divisor of s:

$$(Z_s, \varphi) = \int_{Z_s} \varphi, \quad \varphi \in \mathcal{D}^{0,0}(M),$$

where $\mathcal{D}^{0,0}(\Omega)$ is the set of compactly supported $(0,0)$ forms (compactly supported smooth functions) on M. A current is an element of the dual space $\mathcal{D}^{0,0}(M)'$.

The Poincaré-Lelong formula (see e.g. [7]) states that

$$Z_s = \frac{i}{\pi}\partial\bar{\partial}\log|g| = \frac{i}{\pi}\partial\bar{\partial}\log||s||_{h^N} + N\frac{\omega_h}{\pi}. \tag{2.3}$$

We also denote by $|Z_s|$ the Riemannian $0-$volume, i.e. a Riemannian function along the regular points of Z_s, regarded as a measure on M:

$$(|Z_s|, \varphi) = \sum_{s(z)=0} \varphi(z); \tag{2.4}$$

2.2 Random sections and Gaussian measures

We now give $H^0(M, L^N)$ the complex Gaussian probability measure

$$d\mu_h(s) = \frac{1}{\pi^{d_N}} e^{-|c|^2} dc, \quad s = \sum_{j=1}^{d_N} c_j S_j^N, \tag{2.5}$$

where $\{S_j^N : 1 \leq j \leq d_N\}$ is an orthonormal basis for $H^0(M, L^N)$ with respect to the equation (2.2) and dc is $2d_N-$dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2d_N$ real variables $\mathcal{R}c_j, \mathcal{I}c_j$ $(j = 1, ..., d_N)$ are independent random variables with mean 0 and variance $\frac{1}{2}$; i.e.,

$$\mathbf{E}c_j = 0, \quad \mathbf{E}c_j c_k = 0, \quad \mathbf{E}c_j \bar{c}_k = \delta_{jk}.$$

Here and throughout this thesis, $\mathbf{E}$ denotes expectation: $\mathbf{E}\varphi = \int \varphi \, d\mu$.

We then regard the currents Z_s (resp. measures $|Z_s|$), as current-valued (resp. measure-valued) random variables on the probability space ($H^0(M, L^N)$, $d\mu$); i.e., for each test form (resp. function) φ, $(|Z_s|, \varphi)$ (resp. $(|Z_s|, \varphi)$) is a complex-valued random variable.

Since the zero current Z_s is unchanged when s is multiplied by an element of $\mathbb{C}^*$, our results are the same if we instead regard Z_s as a random variable on the unit sphere $SH^0(M, L^N)$ with Haar probability measure. We prefer to use Gaussian measures in order to facilitate computations.

2.3 Correlation currents and measures

The $n-$point correlation current of the zeros is the current on $M^n = \underbrace{M \times \cdots \times M}_{n}$ given by

$$K_n^N(z^1, ..., z^n) := \mathbf{E}(Z_s(z^1) \otimes Z_s(z^2) \otimes \cdots \otimes Z_s(z^n)) \qquad (2.6)$$

in the sense that for any test form $\varphi_1(z^1)\otimes\cdots\otimes\varphi_n(z^n)\in\mathcal{D}^{0,0}(M)\otimes\cdots\otimes\mathcal{D}^{0,0}(M)$,

$$(K_n^N(z^1,...,z^n),\varphi_1(z^1)\otimes\cdots\otimes\varphi_n(z^n))=\mathbf{E}[(Z_s,\varphi_1)(Z_s,\varphi_2)\cdots(Z_s,\varphi_n)], \tag{2.7}$$

where

$$\begin{aligned}&(K_n^N(z^1,...,z^n),\varphi_1(z^1)\otimes\cdots\otimes\varphi_n(z^n))\\=&\int_{M\times\cdots\times M}\varphi_1(z^1)\cdots\varphi_n(z^n)K_n^N(z^1,...,z^n).\end{aligned}$$

When $n=2$, the correlation measures take the form

$$K_2^N(z,w)=[\Delta]\wedge(K_1^N(z)\otimes 1)+\kappa^N(z,w)\omega_z\otimes\omega_w \quad (N\gg 0), \tag{2.8}$$

where $[\Delta]$ denotes the current of integration along the diagonal $\Delta=(z,z)\subset M\times M$, i.e.

$$\int\int_{M\times M}f(z,w)[\Delta]\wedge(K_1^N(z)\otimes 1)=\int_M f(z,z)K_1^N(z), \tag{2.9}$$

and $\kappa^N\in C^\infty(M\times M)$. In [16], Bernard Shiffman and Steve Zelditch proved:

Theorem 2.1 (Shiffman-Zelditch)

$$K_2^N(z,w) - K_1^N(z) \wedge K_1^N(w) = O(N^{-k}) \quad \textit{uniformly for } r_h(z,w) \geq b\sqrt{\frac{\log N}{N}}, \tag{2.10}$$

for all positive k.

2.4 Relationship between polynomials and sections

Let $\mathbb{CP}^1$ denote the set of lines through the origin in $\mathbb{C}^2$. Let X_0, X_1 denote the Euclidean coordinates on $\mathbb{C}^2$ and also the corresponding homogeneous coordinates on $\mathbb{CP}^1$. The hyperplane bundle $\mathcal{O}(1) \to \mathbb{CP}^1$ is the bundle whose fiber over $X \in \mathbb{CP}^1$ corresponds to the space of linear functionals on the line λX. $\mathcal{O}(N) \to \mathbb{CP}^1$ is the N time tensor power bundle of $\mathcal{O}(1)$. The fiber of a power $\mathcal{O}(N)$ over a point X corresponds to the space of N-linear forms on the line $\lambda X \subset \mathbb{C}^2$, and any N-linear form F on $\mathbb{C}^2$ induces by restriction a global section

$$\sigma_F(X) = F|_{\lambda X}$$

of $\mathcal{O}(N)$. Since we are restricting F to one line at a time, we see that $\sigma_F = 0$ if F is alternating in any two factors, and so we have a map

$$Sym^N(\mathbb{C}^{2*}) \to H^0(\mathbb{CP}^1, \mathcal{O}(N))$$

from the space of symmetric N-linear forms on $\mathbb{C}^2$ − that is, homogeneous polynomials $F(X_0, X_1)$ of degree N in X_0, X_1 − to the space of global sections of $\mathcal{O}(N)$. According to [7], We know this map is an isomorphism. By homogenizing, we may identify the space of polynomials of degree N in one complex variable with the space $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ of holomorphic sections of the Nth power of the hyperplane bundle over $\mathbb{CP}^1$. This space carries a natural $SU(2)$-invariant inner product and associated Gaussian measure $d\mu$. We associate to a degree N polynomial p, the zero set $Z_p = \{p(z) = 0\}$, which is almost always discrete, and thus obtain a random point process on $\mathbb{CP}^1$.

2.5 Green's function on Riemann surfaces

In this section, we discuss Green's functions on Riemann surfaces M with the Riemannian metric g. The Laplacian operator on M corresponding to g is defined by

$$\Delta_g = 4g_{1\bar{1}}^{-1} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}},$$

where, $g_{1\bar{1}}$ is the local representation for g. The Green's function is the kernel of $(-\Delta_g)^{-1}$ i.e.

$$-\Delta_g^{-1} f(z) = \int_M G_g(z, w) f(w) \, dV_g,$$

which is orthogonal to the constant functions, that is

$$\int_M G_g(z,w)\,dV_g = 0. \tag{2.11}$$

Here, $-\Delta_g$ is the Laplacian operator. Let φ_j be the orthnormal eigenfunctions of $-\Delta$, then

$$G_g(z,w) = \sum_{j\neq 0} \frac{\varphi_j(z)\varphi_j(w)}{\lambda_j^2}, \tag{2.12}$$

where $-\Delta_g\varphi_j = \lambda_j^2\varphi_j$, and $\lambda_0 = 0$. So

$$-\Delta_g G_g(z,w) = \sum_{j\neq 0} \varphi_j(z)\varphi_j(w) = \delta_z(w) - \frac{1}{\mathrm{vol}(M,g)}. \tag{2.13}$$

It is well-known that $G_g(z,w)$ on a Riemann surface has the following formula (see [8]):

$$G_g(z,w) = -\frac{1}{2\pi}\chi(z,w)\log r_g(z,w) + F_g(z,w) \tag{2.14}$$

here, $F_g \in C^\infty(M\times M)$, r_g is the corresponding distance function and $\chi(z,w)$ is a cut-off function which equals 1 on $r_g(z,w) \leq C_1$ and 0 on $r_g(z,w) \geq C_2$, $0 < C_1 < C_2$.

3 Proof of Theorem 1.1

In the proof of this theorem, we will use g instead of g_h without confusion. By the assumption of the theorem, we conclude that the geodesic distance r_h derived from h equals r_g.

Lemma 3.1 $\int_{M\times M} G_g(z,w)K_1^N(z)\wedge K_1^N(w) = O(N^{-2})$.

Proof In [16], the authors proved that

$$K_1^N(z) = \frac{i}{\pi}\partial\bar{\partial}\log \Pi_N(z,z) + \frac{N}{\pi}\omega_h(z), \tag{3.1}$$

where $\Pi_N(z,z)$ is the Szegö kernel. We have the following asymptotic (the Tian-Yau-Zelditch Theorem):

$$\Pi_N(z,z) = N(1+\frac{S(z)}{N} + O(N^{-2})), \tag{3.2}$$

where $S(z)$ is the scalar curvature of $\omega_h(z)$ defined by

$$S(z) = g_{1\bar{1}}^{-2}\frac{\partial^2 g_{1\bar{1}}}{\partial z\partial\bar{z}},$$

where $g_{1\bar{1}}$ has such property $\omega_h = \frac{\sqrt{-1}}{2}g_{1\bar{1}}\,dz\wedge d\bar{z}$ (see [13]). So we get

$$\log\Pi_N(z,z) = \log N + \frac{S(z)}{N} + O(N^{-2}), \tag{3.3}$$

and by (3.1)

$$K_1^N(z) = \frac{N}{\pi}\omega_h(z) + \frac{i}{\pi N}\partial\bar{\partial}S(z) + O(N^{-2}). \tag{3.4}$$

Therefore,

$$\begin{aligned} &\int_{M\times M} G_g(z,w) K_1^N(z) \wedge K_1^N(w) \\ =& \int_{M\times M} G_g(z,w)(\frac{N}{\pi}\omega_h(z) + \frac{i}{\pi N}\partial\bar{\partial}S(z) + O(N^{-2})) \qquad (3.5)\\ &\wedge (\frac{N}{\pi}\omega_h(w) + \frac{i}{\pi N}\partial\bar{\partial}S(w) + O(N^{-2})). \qquad (3.6)\end{aligned}$$

By the equation (2.11) and $dV_g = \omega_h$, the above term becomes

$$-\frac{1}{(\pi N)^2}\int_{M\times M} G_g(z,w)\partial\bar{\partial}S(z) \wedge \partial\bar{\partial}S(w) + o(N^{-2}). \tag{3.7}$$

Hence

$$\int_{M\times M} G_g(z,w) K_1^N(z) \wedge K_1^N(w) = O(N^{-2}).$$

Lemma 3.2 *If* $w = z + \frac{u}{\sqrt{N}}$*, we have:*

$$\mathbf{E}\mathcal{E}_{G_g}^N = N\int_M \int_{|u|\le b\sqrt{\log N}} G_g(z, z+\frac{u}{\sqrt{N}})(H(\frac{1}{2}|u|^2)-1)\omega_h(z) \otimes \frac{i}{2\pi}\partial\bar{\partial}|u|^2 + O(N^{-1/2}),$$

where H *is the Hannay's function defined by*

$$H(t) = \frac{(\sinh^2 t + t^2)\cosh t - 2t\sinh t}{\sinh^3 t}, \tag{3.8}$$

and $|u| = \sqrt{N} r_g(z,w)$.

Proof According to our discussion in section 2.3, we know that

$$\mathbf{E}(G_g(z,w), Z_s \otimes Z_s - [\Delta] \wedge (Z_s \otimes 1)) = (G_g(z,w), K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)),$$

then we get:

$$\begin{aligned}
\mathbf{E}\mathcal{E}_{G_g}^N &= \int_{H^0(M,L^N)} \mathcal{E}_{G_g}^N(Z_s) d\mu_h(s) \\
&= \mathbf{E}(G_g(z,w), Z_s \otimes Z_s - [\Delta] \wedge (Z_s \otimes 1)) \\
&= (G_g(z,w), K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)) \\
&= \int_{M \times M} G_g(z,w)(K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)) \\
&= \int_M \int_{r_g(z,w) \leq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w) K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1) \\
&+ \int_M \int_{r_g(z,w) \geq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w) K_2^N(z,w).
\end{aligned}$$

By the lemma 3.1, we have:

$$
\begin{aligned}
\mathbf{E}\mathcal{E}^N_{G_g} &= \int_M \int_{r_g(z,w)\leq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)K^N_2(z,w) - [\Delta]\wedge (K^N_1(z)\otimes 1) \\
&+ \int_M \int_{r_g(z,w)\geq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)K^N_2(z,w) - \int_{M\times M} G(z,w)K^N_1(z)\wedge K^N_1(w) \\
&+O(N^{-2}) \\
&= \int_M \int_{r_g(z,w)\leq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K^N_2(z,w) - [\Delta]\wedge (K^N_1(z)\otimes 1) - K^N_1(z)\wedge K^N_1(w)) \\
&+ \int_M \int_{r_g(z,w)\geq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K^N_2(z,w) - K^N_1(z)\wedge K^N_1(w)) + O(\frac{1}{N^2}) \\
&:= I + II.
\end{aligned}
$$

Since $G_g(z,w) = -\frac{1}{2\pi}\chi(z,w)\log r_g(x,y) + F(z,w)$ and F is bounded because of the compactness of M, when $r_g(z,w)\geq b\sqrt{\frac{\log N}{N}}$, $|G_g(z,w)|$ is bounded by $\log N$. By equation (2.10),the II term becomes:

$$
\int_M \int_{r_g(z,w)\geq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K^N_2(z,w) - K^N_1(z)\wedge K^N_1(w)) = O(N^{-k}),
$$

for all $k > 0$, so we get

$$
\begin{aligned}
\mathbf{E}\mathcal{E}^N_{G_g} = &\int_M \int_{r_g(z,w)\leq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K^N_2(z,w) - [\Delta]\wedge (K^N_1(z)\otimes 1) \\
&- K^N_1(z)\wedge K^N_1(w)) + O(N^{-2}). \qquad (3.9)
\end{aligned}
$$

By Theorem 4.1 of [5] we have:

$$K_2^N(z_0 + \frac{z}{\sqrt{N}}, z_0 + \frac{w}{\sqrt{N}}) = K_2^\infty(z,w) + O(\frac{1}{\sqrt{N}}), \tag{3.10}$$

where $K_2^\infty(z,w) = [\pi\delta_0(z-w)) + H(\frac{1}{2}|z-w|^2)] \cdot \frac{i}{2\pi}\partial\bar\partial|z|^2 \wedge \frac{i}{2\pi}\partial\bar\partial|w|^2$.

According to the equation (3.8), we know when $t \to 0$, $H(t) = t - \frac{2}{9}t^3 + O(t^5)$ and when $t \to \infty$, $H(t) = 1 + O(e^{-t})$. Here is the graph of $H(t) - 1$ (Figure 1):

We change variables

$$w = z + \frac{u}{\sqrt{N}}.$$

Combining (2.8) and (3.10), we obtain

$$\begin{aligned} K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1) =& K_2^N(z, z + \frac{u}{\sqrt{N}}) - [\Delta] \wedge (K_1^N(z) \otimes 1) \\ =& NH(\frac{1}{2}|u|^2) \cdot \frac{\omega_h(z)}{\pi} \wedge \frac{i}{2\pi}\partial\bar\partial|u|^2 + O(N^{-1/2}) \end{aligned} \tag{3.11}$$

since

$$\begin{aligned} K_1^N(z) \wedge K_1^N(w) =& (\frac{N}{\pi}\omega_h(z) + \frac{i}{\pi N}\partial\bar\partial S(z) + O(N^{-2})) \\ & \wedge (\frac{N}{\pi}\omega_h(w) + \frac{i}{\pi N}\partial\bar\partial S(w) + O(N^{-2})) \\ =& \frac{N}{\pi}\omega_h(z) \wedge \frac{N}{\pi}\omega_h(w) + O(1). \end{aligned} \tag{3.12}$$

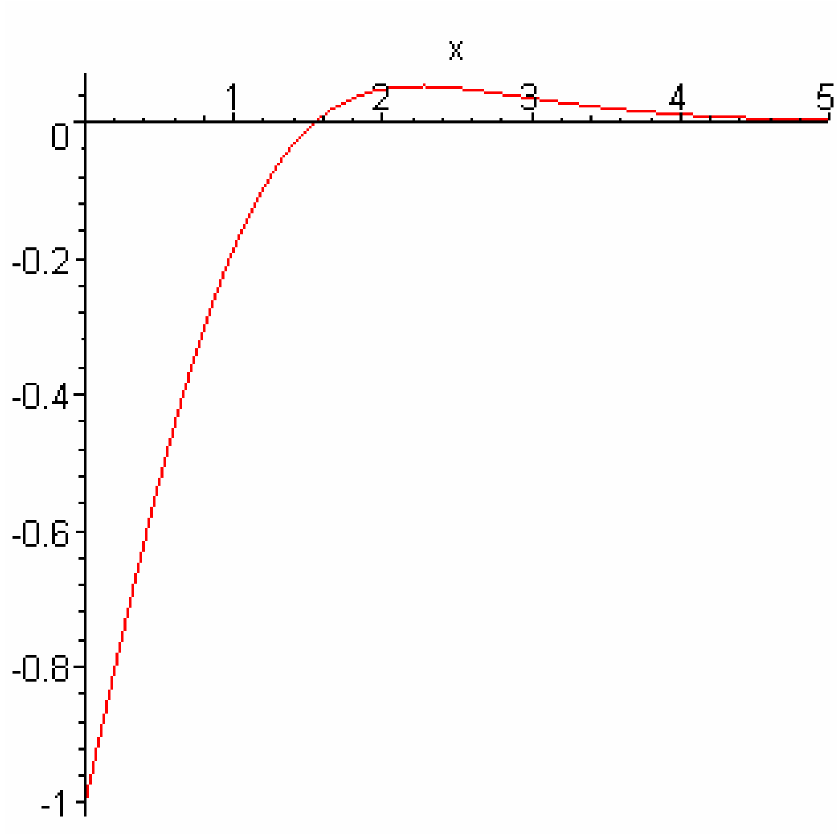

Figure 1: $H(t) - 1$

According to (2.1), we can choose e_L so that the Taylor expansion is:

$$\varphi(z + \frac{u}{\sqrt{N}}) = \frac{|u|^2}{2N} + O(\frac{|u|^3}{N^{\frac{3}{2}}}).$$

Since $u = \sqrt{N}(w - z)$, we have

$$\partial_w \bar{\partial}_w \varphi(w) = \partial_u \bar{\partial}_u \varphi(z + \frac{u}{\sqrt{N}})$$
$$= \partial_u \bar{\partial}_u \frac{|u|^2}{2N} + O(N^{-\frac{3}{2}}).$$

Therefore,

$$\frac{N}{\pi}\omega_h(w) = N\frac{i}{\pi}\partial_u \bar{\partial}_u \frac{|u|^2}{2N} + O(N^{-\frac{1}{2}})$$
$$= \frac{i}{2\pi}\partial_u \bar{\partial}_u |u|^2 + O(N^{-\frac{1}{2}})$$

and we have

$$K_1^N(z) \wedge K_1^N(z + \frac{u}{\sqrt{N}}) = \frac{N}{\pi}\omega_h(z) \wedge \frac{i}{2\pi}\partial\bar{\partial}|u|^2 + O(N^{-\frac{1}{2}}), \tag{3.13}$$

on $|u| \leq b\sqrt{\log N}$.

Since $r_g(z, z + \frac{u}{\sqrt{N}}) = \frac{|u|}{\sqrt{N}} + O(N^{-\frac{3}{2}})$ in normal coordinates, we combine equations (3.11) and (3.13) to get

$$\begin{aligned}
\mathbf{E}\mathcal{E}^N_{G_g} = & \int_M \int_{r_g(z,w)\le \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K_2^N(z,w) - [\Delta] \wedge (K_1^N(z)\otimes 1) \\
& - K_1^N(z)\wedge K_1^N(w)) + O(N^{-2}) \\
= & N \int_M \int_{|u|\le b\sqrt{\log N}} G_g(z, z + \frac{u}{\sqrt{N}})(H(\frac{1}{2}u^2) - 1)\frac{\omega_h(z)}{\pi} \otimes \frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
& + O(N^{-1/2}).
\end{aligned}$$

Now we complete the proof of Theorem 1.1 :

Proof According to §2.5

$$G_g(z,w) = -\frac{1}{2\pi}\chi(z,w)\log r_g(z,w) + F_g(z,w).$$

u is the local coordinate for z, therefore, if $r_g(z, z + \frac{u}{\sqrt{N}}) = \frac{|u|}{\sqrt{N}} + O(N^{-\frac{3}{2}})$, then

$$\log r_g(z,w) = \log(\frac{|u|}{\sqrt{N}} + O(N^{-\frac{3}{2}})),$$

and

$$F_g(z, z + \frac{u}{\sqrt{N}}) = F_g(z,z) + O(\frac{u}{\sqrt{N}}).$$

We have:

$$
\begin{aligned}
&\mathbf{E}\mathcal{E}_{G_g}^N \\
&= N\int_M\int_{0\le|u|\le b\sqrt{\log N}}[-\frac{1}{2\pi}\log(\frac{|u|}{\sqrt{N}}+O(N^{-\frac{3}{2}}))][H(\frac{1}{2}|u|^2)-1]\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad + N\int_M\int_{0\le|u|\le b\sqrt{\log N}}[F_g(z,z)+O(\frac{u}{\sqrt{N}})][H(\frac{1}{2}|u|^2)-1]\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad + O(N^{-1/2}) \\
&= \frac{1}{2\pi}N\log\sqrt{N}\int_M\frac{\omega_h(z)}{\pi}\int_{0\le|u|\le b\sqrt{\log N}}[H(\frac{1}{2}|u|^2)-1]\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad -\frac{N}{2\pi}\int_M\int_{0\le|u|\le b\sqrt{\log N}}\log|u|(H(\frac{1}{2}|u|^2)-1)\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad + N\int_M\int_{0\le|u|\le b\sqrt{\log N}}[F_g(z,z)+O(\frac{u}{\sqrt{N}})][H(\frac{1}{2}|u|^2)-1]\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad + o(N) \\
&\sim \frac{1}{4\pi}N\log N\int_{0\le|u|<\infty}[H(\frac{1}{2}|u|^2)-1]\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad -\frac{N}{2\pi}\int_M\int_{0\le|u|<\infty}\log|u|(H(\frac{1}{2}|u|^2)-1)\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad + N\int_M\int_{0\le|u|<\infty}[F_g(z,z)+O(\frac{u}{\sqrt{N}})][H(\frac{1}{2}|u|^2)-1]\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&\quad + o(N) \\
&:= I + II + III + o(N)
\end{aligned}
$$

We have

$$\int_M\frac{\omega_h(z)}{\pi}=1,$$

because we assumed the Chern class of L, $c_1(L)=1$. Using normal coordinates, we have

$$\begin{aligned}
&\int_{0\le|u|<\infty}\left[H(\frac{1}{2}|u|^2)-1\right]\frac{i}{2\pi}\partial\bar{\partial}|u|^2\\
&=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\left[H(\frac{1}{2}r^2)-1\right]r\,drd\theta\\
&=2\int_0^{\infty}\left[H(\frac{1}{2}r^2)-1\right]r\,dr\\
&=-1
\end{aligned}$$

We calculate the integration directly from the property of correlation currents:

$$\begin{aligned}
&\int_M\frac{\omega_h(z)}{\pi}\int_{0\le|u|<\infty}\left[H(\frac{1}{2}|u|^2)-1\right]\frac{i}{2\pi}\partial\bar{\partial}|u|^2\\
=&\frac{1}{N}\left(\int_M\int_{0\le|u|<\infty}(K^2(z,z+\frac{u}{\sqrt{N}})-[\Delta]\wedge K_1^N(z+\frac{u}{\sqrt{N}})\right)\\
&-\frac{1}{N}\left(\int_M\int_{0\le|u|<\infty}K_1^N(z)\wedge K_1^N(z+\frac{u}{\sqrt{N}}))\right)\\
=&\frac{1}{N}(N^2-N-N^2)\\
=&-1
\end{aligned}$$

Therefore, $I = -\frac{1}{4\pi} N \log N$, and $III = -N \int_M F_g(z,z) \frac{\omega_h(z)}{\pi}$. We have

$$
\begin{aligned}
II &= -\frac{N}{2\pi} \int_0^{2\pi} \frac{1}{\pi} \int_0^{\infty} \log r [H(\frac{1}{2} r^2) - 1] r \, dr d\theta \\
&= -\frac{N}{2\pi} 2 \int_0^{\infty} \log r [H(\frac{1}{2} r^2) - 1] \, d\frac{r^2}{2} \\
&= -\frac{N}{2\pi} \int_0^{\infty} \log \frac{r^2}{2} [H(\frac{1}{2} r^2) - 1] \, d\frac{r^2}{2} - \frac{N}{2\pi} \int_0^{\infty} \log 2 [H(\frac{1}{2} r^2) - 1] d\frac{r^2}{2} \\
&= -\frac{N}{2\pi} (\frac{1}{2} + \frac{1}{2} \log 2) + \frac{N}{2\pi} (\frac{1}{2} \log 2) \\
&= -\frac{N}{4\pi}
\end{aligned}
$$

So we have

$$
\mathbf{E} \mathcal{E}_{G_g}^N = -\frac{1}{4\pi} N \log N - \frac{N}{4\pi} - N \int_M F_g(z,z) \frac{\omega_h(z)}{\pi} + o(N)
$$

Remark 3.4 The leading constant $-\frac{1}{4} N \log N$ for $\mathbf{E}\mathcal{E}_{G_g}^N$ comes from I, which is independent of h, because $G_g(z,w) = -\frac{1}{2\pi} \chi(z,w) \log r_g(z,w) + F_g(z,w)$, where $r_g = r_h(z, z + \frac{u}{\sqrt{N}}) = \frac{|u|}{\sqrt{N}} + O(N^{-\frac{3}{2}})$.

4 Proof of Theorem 1.2

Proof If g is independent on h, then $dV_g \neq \omega_h$ and the equation (2.11) doesn't hold, therefore we don't have lemma 3.1. We have

$$
\begin{aligned}
&\int_{M\times M} G_g(z,w)K_1^N(z)\wedge K_1^N(w)\\
=&\int_{M\times M} G_g(z,w)(\frac{N}{\pi}\omega_h(z)+\frac{i}{\pi N}\partial\bar{\partial}s(z)+O(N^{-2}))\\
&\wedge(\frac{N}{\pi}\omega_h(w)+\frac{i}{\pi N}\partial\bar{\partial}s(w)+O(N^{-2}))\\
=&-\frac{1}{(\pi N)^2}\int_{M\times M} G_g(z,w)\partial\bar{\partial}s(z)\wedge\partial\bar{\partial}s(w)+O(1)\\
=&C_gN^2+O(1).
\end{aligned}
$$

Here $C_g = \int_{M\times M} G_g(z,w)\,\frac{1}{\pi}\omega_h(z)\wedge\frac{1}{\pi}\omega_h(w)$. Let us recall the equation (2.12), we obtain $C_g = \int_{M\times M} G_g(z,w)\,\frac{1}{\pi}\omega_h(z)\wedge\frac{1}{\pi}\omega_h(w) = \sum_{i\neq 0}(\int_M \frac{\varphi_i}{\lambda_i}\frac{\omega_h}{\pi})^2 \geq 0$. So we get:

$$
\begin{aligned}
\mathbf{E}\mathcal{E}^N_{G_g} &= \int_{H^0(M,L^N)} \mathcal{E}^N_{G_g}(Z_s) d\mu_h(s) \\
&= \mathbf{E}(G_g(z,w), Z_s \otimes Z_s - [\Delta] \wedge (Z_s \otimes 1)) \\
&= (G_g(z,w), K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)) \\
&= \int_{M\times M} G_g(z,w)(K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)) \\
&\quad - \int_{M\times M} G(z,w) K_1^N(z) \wedge K_1^N(w) + \int_{M\times M} G(z,w) K_1^N(z) \wedge K_1^N(w) \\
&= \int_{M\times M} G_g(z,w)(K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)) \\
&\quad - \int_{M\times M} G(z,w) K_1^N(z) \wedge K_1^N(w) + C_g N^2 + O(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{M\times M} G_g(z,w)(K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1)) - \int_{M\times M} G(z,w) K_1^N(z) \wedge K_1^N(w) \\
&= \int_M \int_{r_h(z,w) \leq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K_2^N(z,w) - [\Delta] \wedge (K_1^N(z) \otimes 1) - K_1^N(z) \wedge K_1^N(w)) \\
&\quad + \int_M \int_{r_h(z,w) \geq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K_2^N(z,w) - K_1^N(z) \wedge K_1^N(w)).
\end{aligned}
$$

Since $G_g(z,w) = -\frac{1}{2\pi}\chi(z,w)\log r_g(x,y) + F(z,w)$ and F is bounded because of the compactness of M, when $r_h(s,w) \geq b\sqrt{\frac{\log N}{N}}$, $r_g(s,w) \geq b'\sqrt{\frac{\log N}{N}}$ and $|G_g(z,w)|$ is bounded by $\log N$. By equation (2.10),the last term becomes:

$$
\int_M \int_{r_h(z,w) \geq \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K_2^N(z,w) - K_1^N(z) \wedge K_1^N(w)) = O(N^{-k}),
$$

so we get

$$\int_{M\times M} G_g(z,w)(K_2^N(z,w)-[\Delta]\wedge(K_1^N(z)\otimes 1)) - \int_{M\times M} G(z,w)K_1^N(z)\wedge K_1^N(w)$$
$$= \int_M \int_{r_h(z,w)\le \frac{b\sqrt{\log N}}{\sqrt{N}}} G_g(z,w)(K_2^N(z,w)-[\Delta]\wedge(K_1^N(z)\otimes 1) - K_1^N(z)\wedge K_1^N(w))$$
$$+ O(N^{-k}).$$

According to the proof of Lemma 3.2, we have

$$\int_{M\times M} G_g(z,w)(K_2^N(z,w)-[\Delta]\wedge(K_1^N(z)\otimes 1)) - \int_{M\times M} G(z,w)K_1^N(z)\wedge K_1^N(w)$$
$$=N\int_M \int_{|u|\le b\sqrt{\log N}} G_g(z,z+\frac{u}{\sqrt{N}})(H(\frac{1}{2}u^2)-1)\frac{\omega_z}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2 + O(1).$$

According to §3

$$G_g(z,w) = -\frac{1}{2\pi}\chi(z,w)\log r_g(z,w) + F(z,w).$$

u is the local coordinate for z, therefore, if $r_h(z,z+\frac{u}{\sqrt{N}}) = \frac{|u|}{\sqrt{N}} + O(N^{-\frac{3}{2}})$, then $r_g(z,z+\frac{u}{\sqrt{N}}) = \frac{\lambda(z)|u|}{\sqrt{N}} + O(N^{-\frac{3}{2}})$, where $\lambda(z) > 0$, because ω_g and ω_h

are conformal. We now have:

$$
\begin{aligned}
& N\int_M \int_{|u|\le b\sqrt{\log N}} G_g(z, z+\frac{u}{\sqrt{N}})(H(\frac{1}{2}u^2)-1)\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2+O(1)\\
=&N\int_M \int_{0\le|u|\le b\sqrt{\log N}}[-\frac{1}{2\pi}\log(\frac{\lambda(z)|u|}{\sqrt{N}}+O(N^{-\frac{3}{2}}))][H(\frac{1}{2}|u|^2)-1]\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2\\
=&\frac{1}{2\pi}N\log\sqrt{N}\int_M\frac{\omega_h(z)}{\pi}\int_{0\le|u|\le b\sqrt{\log N}}[H(\frac{1}{2}|u|^2)-1]\frac{i}{2\pi}\partial\bar{\partial}|u|^2\\
&-\frac{N}{2\pi}\int_M\int_{0\le|u|\le b\sqrt{\log N}}\log\lambda(z)|u|(H(\frac{1}{2}|u|^2)-1)\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2+o(N)\\
\sim&\frac{1}{4\pi}N\log N\int_{0\le|u|<\infty}[H(\frac{1}{2}|u|^2)-1]\frac{i}{2\pi}\partial\bar{\partial}|u|^2\\
&-\frac{N}{2\pi}\int_M\int_{0\le|u|<\infty}(\log\lambda(z)+\log|u|)(H(\frac{1}{2}|u|^2)-1)\frac{\omega_h(z)}{\pi}\otimes\frac{i}{2\pi}\partial\bar{\partial}|u|^2+o(N)\\
:=&I+II+o(N)
\end{aligned}
$$

$\int_M \frac{\omega_h(z)}{\pi}=1$, because we assumed the Chern class of L, $c_1(L)=1$. Using normal coordinates, we have

$$\int_{0\le|u|<\infty}\left[H(\frac{1}{2}|u|^2)-1\right]\frac{i}{2\pi}\partial\bar{\partial}|u|^2=-1,$$

and

$$\int_{0\le|u|<\infty}\log|u|\left[H(\frac{1}{2}|u|^2)-1\right]\frac{i}{2\pi}\partial\bar{\partial}|u|^2=\frac{1}{2}.$$

Therefore, $I=-\frac{1}{4\pi}N\log N$, and $II=\frac{N}{2\pi}\int_M\log\lambda(z)\frac{\omega_h(z)}{\pi}-\frac{N}{4\pi}$. So we have

$$\mathbf{E}\mathcal{E}^N_{G_g}=C_gN^2-\frac{1}{4\pi}N\log N+\frac{N}{2\pi}\int_M\log\lambda(z)\frac{\omega_h(z)}{\pi}-\frac{N}{4\pi}+o(N).$$

5 Proof of Theorem 1.3

5.1 Proof of Theorem 1.3(1)

Proof

$$
\begin{aligned}
\mathbf{E}\mathcal{E}_s^N &= \int_{\mathbb{CP}^1\times\mathbb{CP}^1} \frac{1}{[z,w]^s}(Z_s\otimes Z_s - [\Delta]\wedge(Z_s\otimes 1)) \\
&= \int_{\mathbb{CP}^1\times\mathbb{CP}^1} \frac{1}{[z,w]^s}(K_2^N(z,w) - [\Delta]\wedge(K_1^N(z)\otimes 1)) \\
&= \int_{\mathbb{CP}^1}\int_{r(z,w)\le\sqrt{\frac{\log N}{N}}} [z,w]^{-s}(K_2^N(z,w) - [\Delta]\wedge(K_1^N(z)\otimes 1)) \\
&+ \int_{\mathbb{CP}^1}\int_{\sqrt{\frac{\log N}{N}}\le r(z,w)\le\pi} [z,w]^{-s}K_2^N(z,w) \\
&= I + II
\end{aligned}
$$

To calculate I, we use the same method as in §3. We change variables

$$
w = z + \frac{u}{\sqrt{N}}.
$$

By equation (1.9) we get

$$
\begin{aligned}
I &= \int_{\mathbb{CP}^1}\frac{\omega_z}{\pi}\int_{0\le|u|\le\sqrt{\log N}} (2(1-\cos\frac{|u|}{\sqrt{N}}))^{-\frac{s}{2}} NH(\frac{1}{2}|u|^2)\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\sqrt{\log N}} (2(1-\cos\frac{r}{\sqrt{N}}))^{-\frac{s}{2}} NH(\frac{1}{2}r^2) r\, dr d\theta \qquad (5.1)
\end{aligned}
$$

Since $2(1-\cos\frac{r}{\sqrt{N}}) \sim \frac{r^2}{N}$, (5.1) becomes

$$
\begin{aligned}
&2N^{1+\frac{s}{2}}\int_0^{\sqrt{\log N}}\frac{1}{r^s}H(\frac{1}{2}r^2)r\,dr\\
=&2N^{1+\frac{s}{2}}\int_{\frac{M}{L}}^{\sqrt{\log N}}H(\frac{1}{2}r^2)r^{1-s}\,dr+2N^{1+\frac{s}{2}}\int_0^{\frac{M}{L}}H(\frac{1}{2}r^2)r^{1-s}\,dr. \qquad (5.2)
\end{aligned}
$$

Here, $\frac{M}{L}<\sqrt{\log N}$, so we can assume $\frac{M}{L}=O(\sqrt[4]{\log N})$, since $H(\frac{1}{2}r^2)\to\frac{1}{2}r^2$ as $r\to 0$. Let $L\to\infty$,

- When $s=2$, the second part of (5.2) is asymptotic to

$$\frac{1}{2}N^2(\frac{M}{L})^2. \qquad (5.3)$$

- When $s<4$ and $s\neq 2$, the second part of (5.2) is asympotitc to

$$\frac{N^{1+\frac{s}{2}}}{4-s}(\frac{M}{L})^{4-s}. \qquad (5.4)$$

And since $H(r)\to 1$ as $r\to\infty$, let $M\to\infty$,

- When $s=2$, the first part of (5.2) is asymptotic to

$$N^2\log(\log N)-2N^2\log(M/L). \qquad (5.5)$$

- When $s<4$ and $s\neq 2$, the first part of (5.2) is asymptotic to

$$\frac{N^{1+\frac{s}{2}}}{2-s}(\log N)^{1-\frac{s}{2}}+N^{1+\frac{s}{2}}(\frac{M}{L})^{2-s}. \qquad (5.6)$$

So

- When $s = 2$,

$$I = N^2 \log(\log N) + \frac{1}{2} N^2 (\frac{M}{L})^2 - 2N^2 \log(M/L). \tag{5.7}$$

- When $s < 4$ and $s \neq 2$,

$$I = \frac{N^{1+\frac{s}{2}}}{4-s} (\frac{M}{L})^{4-s} + \frac{N^{1+\frac{s}{2}}}{2-s} (\log N)^{1-\frac{s}{2}} + N^{1+\frac{s}{2}} (\frac{M}{L})^{2-s}. \tag{5.8}$$

To calculate II, we use equation (2.10)

$$K_2^N(z,w) - K_1^N(z) \wedge K_1^N(w) = O(N^{-k}) \text{ uniformly for } r(z,w) \geq \sqrt{\frac{\log N}{N}},$$

and equation (3.12)

$$K_1^N(z) \wedge K_1^N(w) = \frac{N}{\pi} \omega_z \wedge \frac{N}{\pi} \omega_w + O(1).$$

to get

$$\begin{aligned} II &= \int_{\mathbb{CP}^1} \int_{\sqrt{\frac{\log N}{N}} \leq r(z,w) \leq 2} [z,w]^{-s} K_1^N(z) \wedge K_1^N(w) + O(N^{-k}) \\ &= N^2 \int_{\mathbb{CP}^1} \frac{\omega_z}{\pi} \int_{\sqrt{\frac{\log N}{N}} \leq r(z,w) \leq 2} [z,w]^{-s} \frac{\omega_w}{\pi} + O(1) \end{aligned} \tag{5.9}$$

Since $\int_{\mathbb{CP}^1} \frac{\omega}{\pi} = 1$, if we use the azimuthal angle φ, we get $\int_{S^2} \sin\varphi \, d\varphi d\theta = 4\pi$. For the standard unit sphere, $\varphi = r$, where r is the round distance. Then we have:

$$
\begin{aligned}
II &= \frac{N^2}{4\pi}\int_0^{2\pi}\int_{\sqrt{\frac{\log N}{N}}}^{\pi} \frac{1}{(2(1-\cos\varphi))^{\frac{s}{2}}}\sin\varphi\, d\varphi d\theta \\
&= \frac{N^2}{2}\int_{\sqrt{\frac{\log N}{N}}}^{\pi} \frac{1}{(2(1-\cos\varphi))^{\frac{s}{2}}}\sin\varphi\, d\varphi \\
&= \frac{N^2}{4}\int_{\sqrt{\frac{\log N}{N}}}^{\pi} \frac{1}{(2(1-\cos\varphi))^{\frac{s}{2}}}\, d(2(1-\cos\varphi))
\end{aligned}
\tag{5.10}
$$

- When $s = 2$,

$$
\begin{aligned}
II =& \frac{N^2}{4}\log(2(1-\cos\varphi))|^{\pi}_{\sqrt{\frac{\log N}{N}}} \\
=& \frac{N^2}{2}\log 2 - \frac{N^2}{4}\log\left(2(1-\cos\sqrt{\frac{\log N}{N}})\right) \\
\sim& \frac{N^2}{4}\log N - \frac{N^2}{4}\log(\log N) + \frac{N^2}{2}\log 2.
\end{aligned}
\tag{5.11}
$$

- When $s < 4$ and $s \neq 2$

$$
\begin{aligned}
II =& \frac{2}{2-s}\frac{N^2}{4}(2(1-\cos\varphi))^{1-\frac{s}{2}}|^{\pi}_{\sqrt{\frac{\log N}{N}}} \\
=& \frac{2^{1-s}N^2}{2-s} - \frac{N^2}{2(2-s)}(2(1-\cos\sqrt{\frac{\log N}{N}}))^{1-\frac{s}{2}} \\
\sim& \frac{2^{1-s}N^2}{2-s} - \frac{N^2}{2(2-s)}(\frac{\log N}{N})^{1-\frac{s}{2}} \\
=& \frac{2^{1-s}N^2}{2-s} - \frac{1}{2(2-s)}N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}}.
\end{aligned}
\tag{5.12}
$$

- When $s = 2$, since $\frac{M}{L} < \sqrt{\log N}$,

$$\begin{aligned}\mathbf{E}\mathcal{E}_2^N =& \frac{1}{4}N^2 \log N + \frac{3N^2}{4}\log(\log N) + \frac{N^2}{2}\log 2 \\ &+ \frac{1}{2}N^2(\frac{M}{L})^2 - 2N^2 \log \frac{M}{L} + o(N^2).\end{aligned}$$

- When $s < 2$,

$$\mathbf{E}\mathcal{E}_{s<2}^N = \frac{2^{1-s}}{2-s}N^2 + \frac{1}{(2(2-s))}N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}} + o(N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}}).$$

- When $2 < s < 4$, the leading order term is $\frac{N^{1+\frac{s}{2}}}{4-s}(\frac{M}{L})^{4-s}$ in (5.8), however it is hard to figure out what $\frac{M}{L}$ is.

$$\mathbf{E}\mathcal{E}_{2<s<4}^N = C\frac{N^{1+\frac{s}{2}}}{4-s} + O(N^{1+\frac{s}{2}}(\log N)^{1-\frac{s}{2}}).$$

5.2 Proof of Theorem 1.3(2)

Proof

$$\begin{aligned}
\mathbf{E}\mathcal{E}_0^N &= \int_{\mathbb{CP}^1\times\mathbb{CP}^1} -\log[z,w](Z_s\otimes Z_s - [\Delta]\wedge(Z_s\otimes 1)) \\
&= \int_{\mathbb{CP}^1\times\mathbb{CP}^1} -\log[z,w](K_2^N(z,w) - [\Delta]\wedge(K_1^N(z)\otimes 1)) \\
&= \int_{\mathbb{CP}^1}\int_{r(z,w)\leq\frac{\sqrt{\log N}}{\sqrt{N}}} -\log[z,w](K_2^N(z,w) - [\Delta]\wedge(K_1^N(z)\otimes 1)) \\
&+ \int_{\mathbb{CP}^1}\int_{\frac{\sqrt{\log N}}{\sqrt{N}}\leq r(z,w)\leq\pi} -\log[z,w]K_2^N(z,w) \\
&= I + II
\end{aligned} \tag{5.13}$$

As in §4.1, we change variables

$$w = z + \frac{u}{\sqrt{N}}$$

and by equation (1.9), we get

$$\begin{aligned}
I &= \int_{\mathbb{CP}^1}\frac{\omega_z}{\pi}\int_{0\leq|u|\leq\sqrt{\log N}} -\log\sqrt{2(1-\cos\frac{|u|}{\sqrt{N}})}NH(\frac{1}{2}|u|^2)\frac{i}{2\pi}\partial\bar{\partial}|u|^2 \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\sqrt{\log N}} -\log\sqrt{2(1-\cos\frac{r}{\sqrt{N}})}NH(\frac{1}{2}r^2)r\,drd\theta \\
&= -N\int_0^{\sqrt{\log N}}(\log 2)H(\frac{1}{2}r^2)r\,dr - N\int_0^{\sqrt{\log N}}\log(1-\cos\frac{r}{\sqrt{N}})H(\frac{1}{2}r^2)r\,dr
\end{aligned} \tag{5.14}$$

Since $H(r)\to r$ as $r\to 0$ and $H(r)\to 1$ as $r\to\infty$, we get:

$$I \sim -\frac{\log 2}{2} N \log N - N \int_0^{\sqrt{\log N}} \log(1 - \cos\frac{r}{\sqrt{N}}) H(\frac{1}{2}r^2) r\, dr.$$

Since $1 - \cos\frac{r}{\sqrt{N}} = \frac{r^2}{2N}$

$$\begin{aligned}
&\int_0^{\sqrt{\log N}} \log(1 - \cos\frac{r}{\sqrt{N}}) H(\frac{1}{2}r^2) r\, dr \\
=&\int_0^{\sqrt{\log N}} \log(\frac{r^2}{2N}) H(\frac{1}{2}r^2) r\, dr \\
=&\int_0^{\sqrt{\log N}} (\log\frac{r^2}{2}) H(\frac{1}{2}r^2) d\frac{r^2}{2} - \int_0^{\sqrt{\log N}} \log(N) H(\frac{1}{2}r^2) r\, dr \\
=&\frac{1}{2}\log(\log N)\log N - \frac{1}{2}(\log 2 + 1)\log N - \frac{1}{2}(\log 2 + 1) - \frac{\log^2 N}{2}
\end{aligned}$$

So $I = \frac{N}{2}\log^2 N - \frac{1}{2}N\log(\log N)\log N + \frac{1}{2}N\log N + \frac{1}{2}(\log 2 + 1)N$

To calculate II, we use the same method as in §5.1 and get

$$\begin{aligned}
II =& \int_{\mathbb{CP}^1} \int_{\sqrt{\frac{\log N}{N}} \le r(z,w) \le \pi} - \log[z,w] K_1^N(z) \wedge K_1^N(w) + O(N^{-k}) \\
=& N^2 \int_{\mathbb{CP}^1} \frac{\omega_z}{\pi} \int_{\sqrt{\frac{\log N}{N}} \le r(z,w) \le \pi} - \log[z,w] \frac{\omega_w}{\pi} + O(1). \qquad (5.15)
\end{aligned}$$

Since $\int_{\mathbb{CP}^1} \frac{\omega}{\pi} = 1$, if we use the azimuthal angle φ, we get $\int_{S^2} \sin\varphi\, d\varphi d\theta = 4\pi$. For the standard unit sphere, $\varphi = r$, where r is the round distance. Then

(5.15) becomes

$$\frac{N^2}{4\pi}\int_0^{2\pi}\int_{\sqrt{\frac{\log N}{N}}\le\varphi\le\pi} -\log\sqrt{2(1-\cos\varphi)}\sin\varphi\,d\varphi d\theta \tag{5.16}$$

$$= -\frac{N^2}{4}\int_{\sqrt{\frac{\log N}{N}}}^{\pi}\log 2(1-\cos\varphi)\sin\varphi\,d\varphi$$

$$= -(\log 2-\frac{1}{2})N^2+O(\log N). \tag{5.17}$$

In the end, we get

$$\mathbf{E}\mathcal{E}_0^N = -(\log 2-\frac{1}{2})N^2+\frac{N}{2}\log^2 N-\frac{1}{2}N\log(\log N)\log N$$
$$+\frac{1}{2}N\log N+\frac{1}{2}(\log 2+1)N+o(N).$$

6 Appendix

In this appendix, we give a picture which describes the distribution of random zeros of a given random polynomial. Let

$$p(z) = \sum_{i=1}^{50} c_i \binom{N}{j}^{\frac{1}{2}} z^i,$$

where $\mathbf{E}(c_i) = 0$ and $\mathbf{E}(|c_i|^2) = 1$. According to the picture, we know that

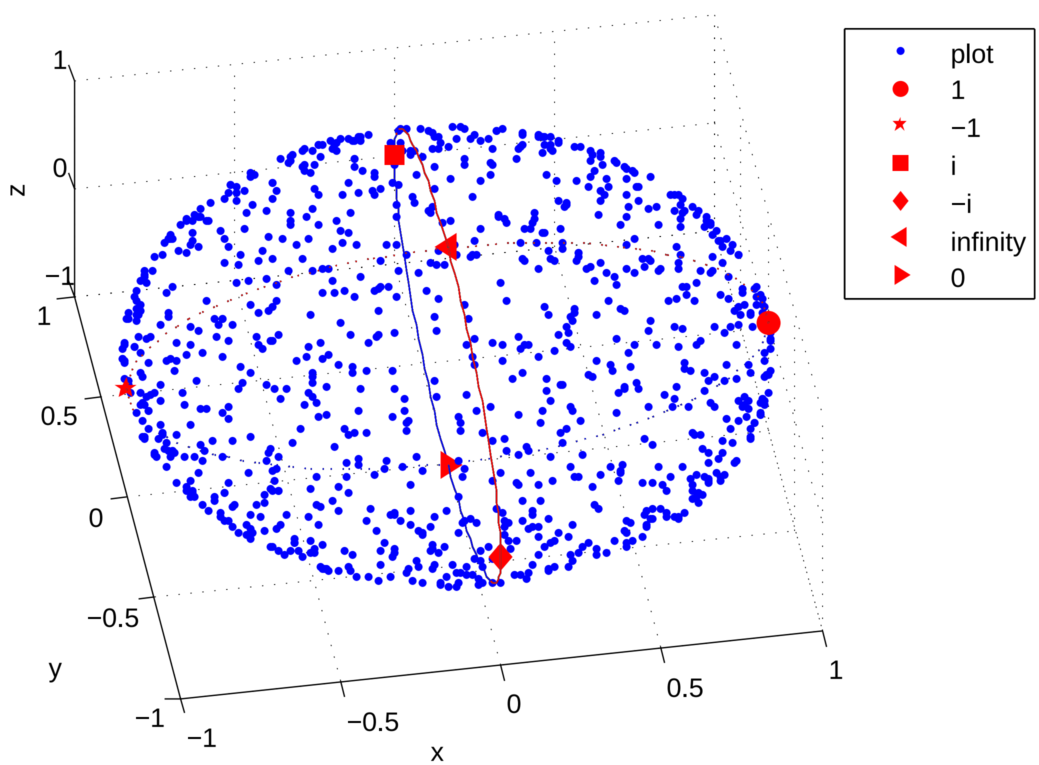

Figure 2: Distribution of zeros

the zeros repel each others which make us understand our result intuitively.

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Vitae

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