ONE DIMENSIONAL FORMAL GROUP LAWS OF HEIGHT N AND N–1

by

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Abstract

By Morava's point of view on the stable homotopy category, the quotient in some sense associated to the filtration related to the height of formal group laws is studied by the category of modules over the function ring of the deformation space of the Honda group law of height n with the lift of the action of the automorphism group on the closed fibre through the Adams-Novikov spectral sequence. The next step to understand the stable homotopy category may be to solve the "extensions". It may be necessary to know the relation between the formal group laws of height n and $n-1$.

In this thesis we study a certain formal group law over a complete discrete valuation ring which is of height n over the closed fibre and of height $n-1$ over the generic fibre. We show that there is a Galois extension of the quotient field of the discrete valuation ring with the Galois group isomorphic to the Morava stabilizer group S_{n-1} of height $n-1$. The action of the Morava stabilizer group S_n of height n on the quotient field lifts to the action on the Galois extension which commutes with the action of the Galois group. Furthermore, there is an $S_n \times S_{n-1}$ equivariant morphism from the lift of the formal group law over the Galois extension to the Honda group law of height $n-1$ on which S_n acts trivially. Then, by a kind of correspondence, we construct a ring homomorphism from the cohomology of S_{n-1} to the cohomology of S_n with the coefficients in the quotient field.

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Contents

1 Introduction

The Hopf algebroid MU_*MU is interpreted in terms of the one dimensional formal group laws. In [11] Morava investigated the category $\mathbf{C}$ of p-local comodules over MU_*MU by using the filtration

$$\mathbf{C} = \mathbf{C}_0 \supset \mathbf{C}_1 \supset \cdots \supset \mathbf{C}_n \supset \cdots$$

where n corresponds to the height of formal group laws over p-local ring. Then he related the quotient category $\mathbf{C}_n/\mathbf{C}_{n+1}$ to the category of modules over the function ring of the deformation space of the standard formal group law H_n of height n with the action of the automorphism group S_n of H_n. Motivated by Morava's work, Miller, Ravenel and Wilson [10] established the frame work on organizing systematically the periodic phenomena on the E_2-term of the Adams-Novikov spectral sequence based on the cobordism theory MU. Then Ravenel [12] formulated his conjectures on the reflection of the algebraic structure on the Adams-Novikov E_2-term on the actual stable homotopy category. Devinatz, Hopkins and Smith [4, 6] solved all the Ravenel conjectures except for the telescope conjecture. By these works, we get a filtration of full subcategories in the stable homotopy category $\mathcal{C}$ of p-local finite spectra

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n \supset \cdots$$

where n is related to the height of formal group laws. By Morava's point of view on the stable homotopy category, the quotient in some sense associated to the above filtration is studied by the category of modules over the function ring of the deformation space of H_n with the lift of the action of S_n on the closed fibre through the Adams-Novikov spectral sequence. The next step to understand the stable homotopy category may be to solve the "extensions". It may be necessary to know the relation between the formal group laws of height n and $n-1$. In this thesis we study a certain formal group law over a complete discrete valuation ring which is of height n over the closed fibre and of height $n-1$ over the generic fibre.

Let $\mathbf{F}$ be a finite field which contains the finite fields $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$. There is the Honda group law H_n of height n over the field $\mathbf{F}$. The Morava stabilizer group S_n is the automorphism group of H_n. There is a universal deformation F_n of H_n. The formal group law F_n is defined over the formal power series ring $W\mathbf{F}[\![u_1, \dots, u_{n-1}]\!]$ where $W\mathbf{F}$ is the ring of Witt vectors in $\mathbf{F}$. Then the action of $S_n \ltimes \Gamma$ on H_n lifts to the action on F_n which induces a continuous action of $S_n \ltimes \Gamma$ on $W\mathbf{F}[\![u_1, \dots, u_{n-1}]\!]$ where Γ is the Galois group of $\mathbf{F}$ over the prime field $\mathbf{F}_p$. Since the ideal generated by $p, u_1, \dots, u_{n-2}$ is invariant under the action of $S_n \ltimes \Gamma$, there is an induced action of $S_n \ltimes \Gamma$ on the quotient ring $\mathbf{F}[\![u_{n-1}]\!]$. We denote by K the quotient field $\mathbf{F}((u_{n-1}))$. We consider that the formal group law F_n is defined over $\mathbf{F}[\![u_{n-1}]\!]$. Then the formal group law F_n is of height n on the closed fibre $\mathbf{F}$ and of height $n-1$ on the generic fibre K. By the result of Lazard [7], the formal group laws over the separably closed field of characteristic $p > 0$ are classified up to isomorphism by their height. Hence there is an isomorphism between F_n and the Honda group law H_{n-1} of height $n-1$ over the separable closure K^{sep}. In [1] Ando, Morava and Sadofsky showed that there is a unique isomorphism between F_n and H_{n-1} over K^{sep} which satisfies certain conditions motivated from a geometric point of view. We would like to consider the above situation with the action of the Morava stabilizer group S_n.

Let Φ be an isomorphism between F_n and H_{n-1} over the separable closure K^{sep}. Let L be an extension of K obtained by adjoining all the coefficients of Φ. Hence we have a morphism of the formal group laws from (F_n, L) to $(H_{n-1}, \mathbf{F})$. The main theorem of this note is as follows.

Theorem 1.1 (Theorem 4.9). *The group $(S_n \times S_{n-1}) \ltimes \Gamma$ acts on (F_n, L) where the action of $S_n \ltimes \Gamma$ is a lift of the action on (F_n, K) and the subgroup $S_{n-1} \ltimes \Gamma$ is identified with the Galois group of the extension $L/\mathbf{F}_p((u_{n-1}))$. If we consider that the group $(S_n \times S_{n-1}) \ltimes \Gamma$ acts on $(H_{n-1}, \mathbf{F})$ where the subgroup S_n acts trivially, then there is a $(S_n \times S_{n-1}) \ltimes \Gamma$ equivariant morphism from $(H_{n-1}, \mathbf{F})$ to (F_n, L).*

In geometric terms $\mathrm{Spec}(\mathbf{F}[\![u_{n-1}]\!])$ is an S_n invariant 1-dimensional subspace of the formal deformation space of the Honda group law H_n. Let U be the punctured disk $\mathrm{Spec}(K) - \mathrm{Spec}(\mathbf{F})$. Then there is a Galois covering of U with the Galois group isomorphic to S_{n-1}. The action of S_n lifts to the Galois covering which commutes with the action of the Galois group. Furthermore, if we consider that the product group $S_n \times S_{n-1}$ acts on H_{n-1} where the action of S_n is trivial, then there is a $S_n \times S_{n-1}$ equivariant morphism from the lift of F_n on the Galois covering to H_{n-1} on the point $\mathrm{Spec}(\mathbf{F})$.

By a kind of correspondence using Theorem 1.1, we construct a ring homomorphism from the cohomology of S_{n-1} to the cohomology of S_n with the coefficients in $K[u^{\pm 1}]$:

$$H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])^{\Gamma} \longrightarrow H^*(S_n; K[u^{\pm 1}])^{\Gamma}$$

where w satisfies $w^{-(p^{n-1}-1)} = v_{n-1}$ (cf. Theorem 12.3).

2 Deformation of formal group laws

Let R be a ring with a maximal ideal I such that the residue field $k = R/I$ is of characteristic $p > 0$. Let G be a formal group law over k of height $n < \infty$. In this section we recall the deformation theory of formal group laws by Lubin and Tate [8].

For a formal power series $f(X)$ over a ring R and a ring homomorphism $\alpha : R \to S$, we denote by $\alpha^* f(X)$ the formal power series over S obtained by the ring homomorphism α. We say that a local ring A with the maximal ideal $\mathfrak{m}$ is complete if the canonical homomorphism $A \to \varprojlim A/\mathfrak{m}^i$ is an isomorphism. For a local homomorphism α between local rings, we denote by $\overline{\alpha}$ the induced homomorphism on the residue fields.

Let A be a complete Noetherian local R-algebra with the maximal ideal $\mathfrak{m}$ such that $IA \subset \mathfrak{m}$. We denote by ι the canonical inclusion of residue fields $k \subset A/\mathfrak{m}$ induced by the R-algebra structure. A deformation of G to A is a formal group law $\widetilde{G}$ over A such that $\iota^* G = \pi^* \widetilde{G}$ where $\pi : A \to A/\mathfrak{m}$ is a canonical projection. Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be two deformations of G to A. We define a $*$-isomorphism between $\widetilde{G}_1$ and $\widetilde{G}_2$ as an isomorphism $\widetilde{u} : \widetilde{G}_1 \to \widetilde{G}_2$ over A such that $\pi^* \widetilde{u}$ is the identity map between $\pi^* \widetilde{G}_1 = \iota^* G = \pi^* \widetilde{G}_2$.

Lemma 2.1 (cf. [8]). *There is at most one $*$-isomorphism between $\widetilde{G}_1$ and $\widetilde{G}_2$.*

We denote by $\mathbf{D}(R)$ the category of complete Noetherian local R-algebras with morphisms as local R-algebra homomorphisms. Let $\mathrm{DEF}(A)$ be the set of all $*$-isomorphism classes of the deformations of G to A. Then DEF defines a functor from $\mathbf{D}(R)$ to the category of sets.

Let $R[\![t_1, \dots, t_{n-1}]\!]$ be a formal power series ring over R with $n-1$ variables. If R is a complete Noetherian local ring, then $R[\![t_1, \dots, t_{n-1}]\!]$ is an object of $\mathbf{D}(R)$. There is a one-to-one correspondence between a local R-algebra homomorphism from $R[\![t_1, \dots, t_{n-1}]\!]$ to A and an $(n-1)$-tuple $(a_1, \dots, a_{n-1})$ of elements of the maximal ideal $\mathfrak{m}$ of A. Lubin and Tate showed that there is a formal group law $F(t_1, \dots, t_{n-1})$ over $R[\![t_1, \dots, t_{n-1}]\!]$ which satisfies the following conditions.

1. $\pi^* F(0, \dots, 0)(x, y) = G(x, y)$ where $\pi : R \to k$ is the projection.

2. For each i $(1 \le i \le n-1)$,

$$F(0, \dots, 0, t_i, \dots, t_{n-1})(x, y) \equiv x + y + t_i C_{p^i}(x, y) \bmod \deg\ (p^i + 1)$$

   where $C_{p^i}(x, y) = (x^{p^i} + y^{p^i} - (x+y)^{p^i})/p$.

We say that a formal group law $F(t_1, \dots, t_{n-1})$ satisfying the above conditions is a universal deformation of G by the following theorem.

Theorem 2.2 (Lubin and Tate [8]). *Let A be an object of $\mathbf{D}(R)$. For every deformation $\widetilde{G}$ of G to A, there is a unique local R-algebra homomorphism $\alpha : R[\![t_1, \dots, t_{n-1}]\!] \to A$ such that $\alpha^* F(t_1, \dots, t_{n-1})$ is $*$-isomorphic to $\widetilde{G}$. Hence, if R is a Noetherian local ring, the functor* DEF *is represented by $R[\![t_1, \dots, t_{n-1}]\!]$:*

$$\mathrm{DEF}(A) \cong \mathrm{Hom}_{\mathbf{D}(R)}(R[\![t_1, \dots, t_{n-1}]\!], A)$$

and $F(t_1, \dots, t_{n-1})$ is a universal object.

We suppose that R is a complete Noetherian local ring. We abbreviate $R[\![t_1, \dots, t_{n-1}]\!]$ to $R[\![t]\!]$ and $F(t_1, \dots, t_{n-1})$ to $F(t)$. For a formal group law

$F(X,Y)$ and an invertible power series $f(X)$ over a same ring, we denote by $F^f(X,Y)$ the formal group law $f(F(f^{-1}(X), f^{-1}(Y)))$. Let $f(X)$ be an automorphism of G over k. For any power series $\widetilde{f}'(X)$ in $R[\![X]\!]$ such that $\pi^*\widetilde{f}'(X) = u(X)$, the formal group law $F(t)^{\widetilde{f}'}(X,Y)$ is a deformation of G over $R[\![t]\!]$. Hence we obtain a unique local R-algebra homomorphism $\alpha : R[\![t]\!] \to R[\![t]\!]$ such that $\alpha^*F(t)$ is $*$-isomorphic to $F(t)^{\widetilde{f}}$. If $\widetilde{g}'(X)$ is another power series in $R[\![X]\!]$ such that $\pi^*\widetilde{g}'(X) = f(X)$, then we get a local R-algebra homomorphism $\alpha' : R[\![t]\!] \to R[\![t]\!]$ such that $\alpha^*F(t)$ is $*$-isomorphic to $F(t)^{\widetilde{g}'}$. But $\widetilde{g}'(\widetilde{f}'^{-1}(X))$ is a $*$-isomorphism from $F(t)^{\widetilde{f}'}$ to $F(t)^{\widetilde{g}'}$. Hence we see that $\alpha = \alpha'$. Let $\widetilde{f}''$ be a unique $*$-isomorphism from $F(t)^{\widetilde{f}}$ to $\alpha^*F(t)$. Then $\widetilde{f}(X) = \widetilde{f}''(\widetilde{f}'(X))$ is an isomorphism from $F(t)$ to $\alpha^*F(t)$ such that $\pi^*\widetilde{f}(X) = f(X)$. We see that a homomorphism $\widetilde{f}(X)$ from $F(t)$ to $\alpha^*F(t)$ such that $\pi^*\widetilde{f}(X) = f(X)$ is unique since there is at most one $*$-isomorphism between deformations. Hence we obtain the following proposition.

Proposition 2.3. *We suppose that R is a complete Noetherian local ring. For every automorphism $f(X)$ of G over k, there is a unique pair $(\alpha, \widetilde{f})$ of a local R-algebra automorphism α of $R[\![t]\!]$ and an isomorphism $\widetilde{f}(X)$ from $F(t)$ to $\alpha^*F(t)$ such that $\pi^*\widetilde{f}(X) = f(X)$.*

For a ring R, we denote by WR the ring of Witt vectors with coefficients in R. If k is a perfect field of characteristic $p > 0$, then Wk is a complete discrete valuation ring of characteristic 0 with the residue field k. In Wk we can take p as a uniformizer. The ring Wk for a perfect field of characteristic $p > 0$ has the following universal property.

Lemma 2.4 (cf. II. 5. [14]). *Let A be a complete local ring with the maximal ideal $\mathfrak{m}$. For a ring homomorphism α from a perfect field k of charac-*

teristic $p > 0$ to the residue field $A/\mathfrak{m}$, there is a unique local homomorphism from Wk to A such that the induced homomorphism on the residue field coincides with α.

Let G be a formal group law of height $n < \infty$ over the perfect field k of characteristic $p > 0$. We denote by $\mathbf{W}_k$ the category with objects as pairs (A, α) where A is a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and α is a homomorphism from k to the residue field $A/\mathfrak{m}$. The morphisms from (A, α) to (B, β) consist of local homomorphism γ from A to B such that $\overline{\gamma} \circ \alpha = \beta$. For an object (A, α) of $\mathbf{W}_k$, a deformation of G to (A, α) is a formal group law $\widetilde{G}$ over A such that $\alpha^* G = \pi^* \widetilde{G}$ where $\pi : A \to A/\mathfrak{m}$ is a canonical projection. A $*$-isomorphism $f(X)$ between deformations $\widetilde{G}_1$ and $\widetilde{G}_2$ of G to (A, α) is an isomorphism of formal group laws over A such that $\pi^* f(X) = X$. Let $\mathrm{Def}(A, \alpha)$ be the set of all $*$-isomorphism classes of deformations of G to (A, α). Then Def defines a functor from $\mathbf{W}_k$ to the category of sets. By Lemma 2.4, we see that the category $\mathbf{W}_k$ is isomorphic to the category $\mathbf{D}(Wk)$. Then we obtain the following theorem by Theorem 2.2. We abbreviate the object $(Wk[\![t_1, \dots, t_{n-1}]\!], id)$ of $\mathbf{W}_k$ to $Wk[\![t_1, \dots, t_{n-1}]\!]$.

Theorem 2.5. *The functor* Def *is represented by* $Wk[\![t_1, \dots, t_{n-1}]\!]$:

$$\mathrm{Def}(A, \alpha) \cong \mathrm{Hom}_{\mathbf{W}_k}(Wk[\![t_1, \dots, t_{n-1}]\!], (A, \alpha)).$$

Let $F(X, Y)$ be a universal deformation on $Wk[\![t]\!]$. For an automorphism $\alpha : k \to k$ and an isomorphism $f : G \to \alpha^* G$, we take any power series $\widetilde{f}'(X)$ in $Wk[\![X]\!]$ such that $\pi^* \widetilde{f}'(X) = f(X)$. Then $F^{\widetilde{f}'}(X, Y)$ is a deformation of G to $(Wk[\![t]\!], \alpha)$. Hence we get a local homomorphism $\beta : Wk[\![t]\!] \to Wk[\![t]\!]$ such that $\beta^* F(X, Y)$ is $*$-isomorphic to $F^{\widetilde{f}'}(X, Y)$ and $\overline{\beta} = \alpha$. Let $\widetilde{f}''(X)$ be a unique $*$-isomorphism from $F^{\widetilde{f}'}(X,Y)$ to $\beta^*F(X,Y)$. Then $\widetilde{f}''(\widetilde{f}'(X))$ is a unique homomorphism from $F(X,Y)$ to $\beta^*F(X,Y)$ such that $\pi^*\widetilde{f}''(\widetilde{f}'(X)) = f(X)$. Hence we get the following proposition.

Proposition 2.6. *Let G be a height $n < \infty$ formal group law over a perfect field k of characteristic $p > 0$. Let F be a universal deformation on $Wk[\![t]\!]$. For every pair (α, f) of an automorphism of field $\alpha : k \to k$ and an isomorphism $f : G \to \alpha^*G$, there is a unique pair $(\beta, \widetilde{f})$ of a continuous automorphism $\beta : Wk[\![t]\!] \to Wk[\![t]\!]$ and an isomorphism $\widetilde{f} : F \to \beta^*F$ such that $\overline{\beta} = \alpha$ and $\pi^*\widetilde{f}(X) = f(X)$.*

Let $\mathbf{F}$ be a finite field which contains the finite filed $\mathbf{F}_{p^n}$ with p^n elements. We note that $\mathbf{F}$ is perfect. The ring of Witt vectors $W\mathbf{F}$ is an unramified extension of the p-adic integer ring $\mathbf{Z}_p$. We consider the height n Honda formal group law H_n defined over $\mathbf{F}$. The formal group law H_n is p-typical with the p-series

$$[p]^{H_n}(X) = X^{p^n}.$$

Let E_n be a formal power series ring over $W\mathbf{F}$ with $(n-1)$ variables:

$$E_n = W\mathbf{F}[\![u_1, \dots, u_{n-1}]\!].$$

The ring E_n is a complete Noetherian local ring with residue field $\mathbf{F}$. There is a p-typical formal group law F_n defined over E_n with the p-series

$$[p]^{F_n}(X) = pX +_{F_n} u_1X^p +_{F_n} u_2X^{p^2} +_{F_n} \cdots +_{F_n} u_{n-1}X^{p^{n-1}} +_{F_n} X^{p^n}.$$

The formal group law F_n is a deformation of H_n to (E_n, id).

Lemma 2.7. *The formal group law F_n is a universal deformation of H_n.*

Proof. We show that F_n satisfies the conditions of a universal deformation. Let $E_{n,i}$ be the formal power series ring over $W\mathbf{F}$ with variables $u_i, \dots, u_{n-1}$. Let $p_i : E_n \to E_{n,i}$ be a local $W\mathbf{F}$-algebra homomorphism given by $p_i(u_j) = 0$ for $j = 1, \dots, i-1$ and $p_i(u_j) = u_j$ for $j = i, \dots, n-1$. Then $p_i^* F_n$ is a p-typical formal group law over $E_{n,i}$ with the p-series

$$[p]^{p_i^* F_n}(X) \equiv pX + u_i X^{p^i} \mod \deg\ (p^i + 1).$$

If $p_i^* F_n(X,Y) \equiv X + Y + bC_k(X,Y)$ mod deg $k+1$, then we have $[n]^{p_i^* F_n}(X) \equiv nX + b\frac{(n-n^k)}{\lambda} X^k$ mod deg $(k+1)$ where $\lambda = p$ if k is a power of p and $\lambda = 1$ otherwise. In particular, we get

$$[p]^{p_i^* F_n}(X) \equiv pX + b\frac{(p-p^k)}{\lambda} X^k \mod \deg\ (k+1).$$

This implies that $k = p^i$ and $b(1 - p^{p^i - 1}) = u_i$. Let $t_i = u_i/(1 - p^{p^i-1})$. Then we have

$$\begin{aligned} E_n &= W\mathbf{F}[\![t_1, \dots, t_{n-1}]\!] \\ E_{n,i} &= W\mathbf{F}[\![t_i, \dots, t_{n-1}]\!] \end{aligned}$$

The $W\mathbf{F}$-algebra homomorphism $E_n \to E_{n,i}$ given by $t_j \mapsto 0$ for $1 \le j < i$ and $t_j \mapsto t_j$ for $i \le j < n$ is p_i. Hence we get

$$p_i^* F_n = X + Y + t_i C_{p^i}(X,Y) \mod \deg\ (p^i + 1).$$

This completes the proof. $\square$

3 Homomorphisms of formal group laws

In this section we recall a generalization of a homomorphism between formal group laws over possibly different ground rings considered by several authors (cf. [19]).

Let R_1 and R_2 be two commutative rings. Let F_1 (resp. F_2) be a formal group law over R_1 (resp. R_2). We understand that a homomorphism from F_1 to F_2 is a pair (α, f) of a ring homomorphism $\alpha : R_2 \to R_1$ and a homomorphism $f : F_1 \to \alpha^* F_2$ in the usual sense. The composition of two homomorphisms $(\alpha, f) : F_1 \to F_2$ and $(\beta, g) : F_2 \to F_3$ is defined as $(\alpha \circ \beta, \alpha^* g \circ f) : F_1 \to F_3$:

$$F_1 \xrightarrow{f} \alpha^* F_2 \xrightarrow{\alpha^* g} \alpha^*(\beta^* F_3) = (\alpha \circ \beta)^* F_3.$$

A homomorphism $(\alpha, f) : F_1 \to F_2$ is an isomorphism if there exists a homomorphism $(\beta, g) : F_2 \to F_1$ such that $(\alpha, f) \circ (\beta, g) = (id, id)$ and $(\beta, g) \circ (\alpha, f) = (id, id)$. Then a homomorphism $(\alpha, f) : F_1 \to F_2$ is an isomorphism if and only if α is a ring isomorphism and f is an isomorphism in usual sense.

Let $\mathbf{F}$ be a finite field which contains the finite fields $\mathbf{F}_{p^n}$ with p^n elements. Let H_n be the height n Honda formal group law over $\mathbf{F}$. The formal group law H_n is a p-typical formal group law with the p-series

$$[p]^{H_n}(X) = X^{p^n}.$$

Let S_n be the Morava stabilizer group. This is an automorphism group of H_n over $\mathbf{F}$ in usual sense. We denote by $G_n(\mathbf{t})$ the automorphism group of H_n over $\mathbf{F}$ in the generalized sense.

Lemma 3.1. *The automorphism group $G_n(\mathbf{t})$ is isomorphic to the semi-direct product* $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_n$.

Proof. An automorphism of H_n consists of a ring isomorphism $\alpha : \mathbf{F} \to \mathbf{F}$ and an isomorphism of formal group laws $f : H_n \to \alpha^* H_n$. Then $\alpha \in \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. Since H_n is defined over the prime field $\mathbf{F}_p$, $\alpha^* H_n = H_n$. Hence we get $f \in S_n$. We regard S_n as the subset of the power series ring $\mathbf{F}[\![X]\!]$. Then the action of the Galois group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ induces an action on S_n. The semi-direct product $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_n$ with respect to this action is isomorphic to the automorphism group of H_n over $\mathbf{F}$. □

Let R be a complete Noetherian local ring with residue field k of characteristic $p > 0$. Let G be a formal group law over k of height $n < \infty$. By , there is a universal deformation F of G over $R[\![t_1, \dots, t_{n-1}]\!]$. We understand that an automorphism of G is a pair (α, f) of a local R-algebra automorphism and an isomorphism $f : F \to \alpha^* F$ in usual sense. We denote by $\mathrm{Aut}^R(F)$ the automorphism group of F in generalized sense. Let $\mathrm{aut}(G)$ be the automorphism group of G in usual sense. There is a natural homomorphism from $\mathrm{Aut}^R(F)$ to $\mathrm{aut}(G)$.

Lemma 3.2. *The natural homomorphism* $\mathrm{Aut}^R(F) \to \mathrm{aut}(G)$ *is an isomorphism.*

Proof. This follows from Proposition 2.3. □

Let G be a height $n < \infty$ formal group law over a perfect field k of characteristic $p > 0$. There is a universal deformation F over $Wk[\![t_1, \dots, t_n]\!]$. We understand that an automorphism of F is a pair (α, f) of a continuous automorphism of $Wk[\![t]\!]$ and an isomorphism $f : F \to \alpha^* F$ in usual sense. We denote by $\mathrm{Aut}(G)$ (resp. $\mathrm{Aut}(F)$) the automorphism group of G (resp. F) in generalized sense. There is a natural homomorphism from $\mathrm{Aut}(F)$ to $\mathrm{Aut}(G)$.

Lemma 3.3. *The natural homomorphism* $\mathrm{Aut}(F) \to \mathrm{Aut}(G)$ *is an isomorphism.*

Proof. This follows from Proposition 2.6. $\square$

Let E_n be the complete local ring $W\mathbf{F}[\![u_1, \dots, u_{n-1}]\!]$ where $W\mathbf{F}$ is the ring of Witt vectors over $\mathbf{F}$. We take a deformation F_n of H_n to (E_n, id) as the p-typical formal group law with p-series

$$[p]^{F_n}(X) = pX +_{F_n} u_1 X^p +_{F_n} \cdots +_{F_n} u_{n-1} X^{p^{n-1}} +_{F_n} X^{p^n}.$$

By Lemma 2.7, F_n is a universal deformation of H_n. Let $\widetilde{G}_n(\mathbf{t})$ be the automorphism group of F_n and let $\widetilde{G}_n^{W\mathbf{F}}(\mathbf{F})$ be the subgroup of $\widetilde{G}_n(\mathbf{F})$ which consists of the automorphism (α, f) such that α is a $W\mathbf{F}$-algebra homomorphism. We note that there is a natural homomorphism $\widetilde{G}_n(\mathbf{F}) \to G_n(\mathbf{F})$ and this induces a homomorphism $\widetilde{G}_n^{W\mathbf{F}}(\mathbf{F}) \to S_n$.

Corollary 3.4. *The natural homomorphisms* $\widetilde{G}_n(\mathbf{F}) \to G_n(\mathbf{F})$ *and* $\widetilde{G}_n^{W\mathbf{F}}(\mathbf{F}) \to S_n$ *are isomorphisms.*

4 Isomorphisms between F_n and H_{n-1}

In this section we investigate a relation between two formal group laws F_n and H_{n-1} over the field $\mathbf{F}((u_{n-1}))$. In particular, we see that the Morava stabilizer group S_{n-1} is realized as the Galois group of the minimum extension over $\mathbf{F}((u_{n-1}))$ on which an isomorphism between F_n and H_{n-1} is defined.

Let $n \geq 2$. Let $\mathbf{F}$ be a finite field. In this section we assume that $\mathbf{F}$ contains the finite fields $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$. Let $k = \mathbf{F}((u_{n-1}))$ be the quotient field of the formal power series ring $\mathbf{F}[\![u_{n-1}]\!]$. There is a $W\mathbf{F}$-algebra homomorphism $\theta : E_n \to k$ given by $\theta(u_i) = 0$ for $i = 1, \dots, n-2$ and $\theta(u_{n-1}) = u_{n-1}$. Then we get a p-typical formal group law $\theta^* F_n$. We abbreviate $\theta^* F_n$ to F_n. The formal group law F_n is a p-typical with the p-series

$$[p]^{F_n}(X) = u_{n-1}X^{p^{n-1}} +_{F_n} X^{p^n}.$$

Let H_{n-1} be the height $n-1$ Honda formal group law over k. Then H_{n-1} is a p-typical formal group law with the p-series

$$[p]^{H_{n-1}}(X) = X^{p^{n-1}}.$$

Let K be a separable closure of k. Then there is an isomorphism between F_n and H_{n-1} over K, since the height of F_n is $n-1$ (cf. Appendix 2, [13]). We fix an isomorphism Φ from F_n to H_{n-1}. Since Φ is a homomorphism between p-typical formal group laws, Φ has a following form:

$$\Phi(X) = \sum_{i \geq 0}{}^{H_{n-1}} \Phi_i X^{p^i}.$$

Let $L_i = k(\Phi_0, \Phi_1, \dots, \Phi_i)$ for $i \geq -1$ and $L = \cup_{i \geq -1} L_i$.

Lemma 4.1. *The extension L_i/k is totally ramified of degree $(p^{n-1}-1)p^{i(n-1)}$ for $i \geq 0$.*

Corollary 4.2. *The extension L/k is totally ramified.*

Proof. Since $\Phi(X)$ is a homomorphism from F_n to H_{n-1}, we have

$$\Phi([p]^{F_n}(X)) = [p]^{H_{n-1}}(\Phi(X)).$$

The left hand side is

$$\begin{aligned} & \Phi(u_{n-1}X^{p^{n-1}}) +_{H_{n-1}} \Phi(X^{p^n}) \\ = \ & \textstyle\sum_{i \geq 0}{}^{H_{n-1}} \Phi_i {u_{n-1}}^{p^i} X^{p^{n+i-1}} +_{H_{n-1}} \sum_{i \geq 0}{}^{H_{n-1}} \Phi_i X^{p^{n+i}}. \end{aligned}$$

The right hand side is

$$\begin{aligned} & \textstyle\sum_{i\geq 0} {}^{H_{n-1}}[p]^{H_{n-1}}(\Phi_i X^{p^i}) \\ = & \textstyle\sum_{i\geq 0} {}^{H_{n-1}}\Phi_i{}^{p^{n-1}} X^{p^{n+i-1}}. \end{aligned}$$

By comparing the coefficient of $X^{p^{n-1}}$, we obtain $\Phi_0 u_{n-1} = \Phi_0{}^{p^{n-1}}$. Since $\Phi(X)$ is an isomorphism, $\Phi_0 \neq 0$. Hence

$$\Phi_0{}^{p^{n-1}-1} - u_{n-1} = 0.$$

This is an Eisenstein polynomial. Therefore L_0/k is totally ramified of degree $p^{n-1}-1$. In particular, Φ_0 is a prime element of L_0.

We assume that L_{i-1}/k is totally ramified of degree $(p^{n-1}-1)p^{(i-1)(n-1)}$ and Φ_{i-1} is a prime element of L_{i-1}. By comparing the coefficient of $X^{p^{n+i-1}}$, we have

$$\Phi_i u_{n-1}{}^{p^i} + f(\Phi_0, \dots, \Phi_{i-1}) = \Phi_i{}^{p^{n-1}}$$

where $f(\Phi_0, \dots, \Phi_{i-1})$ is an element of the integer ring $\mathcal{O}_{L_{i-1}}$ such that $f \equiv \Phi_{i-1} \bmod (\Phi_{i-1}{}^2)$. Hence this is an Eisenstein polynomial. Therefore L_i/L_{i-1} is totally ramified of degree p^{n-1} and Φ_i is a prime element of L_i. By induction, we get the lemma. □

We recall that $\widetilde{G}_n(\mathbf{F})$ is an automorphism group of F_n over E_n in the generalized sense. For $g \in \widetilde{G}_n(\mathbf{F})$, we denote by $\alpha(g)$ the corresponding continuous automorphism of E_n and by $t(g)$ the corresponding isomorphism from F_n to $\alpha(g)^* F_n$. An automorphism $\alpha(g)$ induces a continuous automorphism of k. We abbreviate the induced automorphism of k by $\alpha(g)$. We note that this is a right action of $\widetilde{G}_n(\mathbf{F})$ on k. For an element $g \in \widetilde{G}_n(\mathbf{F})$, we denote by $\widetilde{g}$ a continuous automorphism of the separable closure K which is any extension of the automorphism $\alpha(g)$.

Lemma 4.3. *For any g and i, $L_i^{\widetilde{g}} = L_i$. In particular, $L_i/\mathbf{F}_p((u_{n-1}))$ is a Galois extension.*

Corollary 4.4. *For any extension $\widetilde{g}$ of $\alpha(g)$, $L^{\widetilde{g}} = L$. In particular, $L/\mathbf{F}_p((u_{n-1}))$ is a Galois extension.*

Proof. We have an isomorphism

$$F_n \xrightarrow{\Phi} H_{n-1}.$$

By applying $\widetilde{g}$, we get an isomorphism

$$\widetilde{g}^* F_n \xrightarrow{\widetilde{g}^*\Phi} \widetilde{g}^* H_{n-1}.$$

Note that $\widetilde{g}^* F_n = \alpha(g)^* F_n$ (resp. $\widetilde{g}^* H_{n-1} = H_{n-1}$), since F_n is defined over k (resp. $\mathbf{F}_p$). By a commutative diagram:

$$\begin{array}{ccc} F_n & \xrightarrow{t(g)} & \alpha(g)^* F_n \\ \Phi \Big\downarrow & & \Big\downarrow \widetilde{g}^*\Phi \\ H_{n-1} & \xrightarrow{h(g,\widetilde{g})} & H_{n-1}, \end{array}$$

we have

$$\Phi^{\widetilde{g}}(t(g)(X)) = h(g, \widetilde{g})(\Phi(X)).$$

Here

$$t(g)(X) \quad = \quad \textstyle\sum_{i \geq 0} {}^{g^* F_n} t_i(g) X^{p^i}.$$

and t_i are continuous functions from S_n to the integer ring $\mathcal{O}_k$ of k for all $i \geq 0$. The automorphism $h(g, \tilde{g}) : H_{n-1} \to H_{n-1}$ is an element of the Morava stabilizer group S_{n-1}. The power series $h(g, \widetilde{g})(X)$ has a following form:

$$h(g, \widetilde{g})(X) = \sum_{i \geq 0} {}^{H_{n-1}} h_i(g, \widetilde{g}) X^{p^i}$$

where $h_i(g, \widetilde{g}) \in \mathbf{F}_{p^{n-1}}$. Then the left hand side is

$$\sum_{i,j\geq 0}{}^{H_{n-1}}\Phi_j{}^{\widetilde{g}} t_i(g)^{p^j} X^{p^{i+j}}.$$

The right hand side is

$$\sum_{i,j\geq 0}{}^{H_{n-1}} h_j(g, \widetilde{g}) \Phi_i{}^{p^j} X^{p^{i+j}}.$$

By comparing the coefficient of X, we obtain

$$\Phi_0{}^{\widetilde{g}} t_0(g) = h_0(g, \widetilde{g})\Phi_0.$$

Since $h_0(g, \widetilde{g}) \in \mathbf{F}_{p^{n-1}}$, we get

$$\Phi_0{}^{\widetilde{g}} = h_0(g, \widetilde{g})\Phi_0 t_0(g)^{-1} \in k(\Phi_0) = L_0.$$

We assume that $\Phi_0{}^{\widetilde{g}}, \dots, \Phi_{i-1}{}^{\widetilde{g}} \in L_{i-1}$. Then by comparing the coefficient of X^{p^i}, we obtain

$$\Phi_i{}^{\widetilde{g}} t_0(g)^{p^i} - h_0(g, \widetilde{g})\Phi_i \in L_{i-1}.$$

Hence we get

$$\Phi_i{}^{\widetilde{g}} \in L_{i-1}(\Phi_i) = L_i.$$

This completes the proof. □

We suppose that a Galois group acts on the field on the right. For $\sigma \in \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1})))$, we consider the following diagram:

$$\begin{array}{ccc} F_n & \overset{id}{=} & F_n{}^{\sigma} \\ \Phi \big\downarrow & & \big\downarrow \Phi^{\sigma} \\ H_{n-1} & \xrightarrow{h'(\sigma)} & H_{n-1}, \end{array}$$

We note that ${F_n}^\sigma = F_n$ since F_n is defined over $\mathbf{F}_p((u_{n-1}))$. This diagram defines a map

$$h' : \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1}))) \to G_{n-1}(\mathbf{F}).$$

Lemma 4.5. *The map* $h' : \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1}))) \to G_{n-1}(\mathbf{F})$ *is a homomorphism.*

Proof. For $\sigma' \in \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1})))$, we have a commutative diagram:

$$\begin{tikzcd} F_n \arrow[r, phantom, "="] \arrow[d, "\Phi^\sigma"'] & F_n \arrow[d, "\Phi^{\sigma'\sigma}"] \\ H_{n-1} \arrow[r, "h'(\sigma')^\sigma"] & H_{n-1}. \end{tikzcd}$$

Then we get a commutative diagram:

$$\begin{tikzcd} F_n \arrow[r, phantom, "="] \arrow[d, "\Phi"'] & F_n \arrow[r, phantom, "="] \arrow[d, "\Phi^\sigma"] & F_n \arrow[d, "\Phi^{\sigma'\sigma}"] \\ H_{n-1} \arrow[r, "h'(\sigma)"] & H_{n-1} \arrow[r, "h'(\sigma')^\sigma"] & H_{n-1}. \end{tikzcd}$$

This means h' is a homomorphism. □

The Morava stabilizer group S_{n-1} is the automorphism group of H_{n-1} over the algebraic closure $\overline{\mathbf{F}}_p$ in usual sense. We denote an element $h \in S_{n-1}$ by $h = h_0 + h_1 T + h_2 T^2 + \cdots$ where $h_i \in W\mathbf{F}_{p^{n-1}}, h_i^{p^{n-1}} = h_i$ for $i \geq 0$ and $h_0 \neq 0$. Then h corresponds to the automorphism

$$h(X) = \sum_{i\geq 0}{}^{H_{n-1}} \pi(h_i) X^{p^i}$$

where $\pi : W\mathbf{F}_{p^{n-1}} \to \mathbf{F}_{p^{n-1}}$ is the projection. Let $S_{n-1}^{(0)} = S_{n-1}$. We define the subgroups $S_{n-1}^{(i)}$ for $i \geq 1$ by

$$S_{n-1}^{(i)} = \{h \in S_{n-1} |\ h_0 = 1, h_1 = 0, \ldots, h_{i-1} = 0\}.$$

Then $S_{n-1}^{(i+1)}$ is a normal subgroup of S_{n-1} and the quotient group $S_{n-1}/S_{n-1}^{(i+1)}$ is finite of order $(p^{n-1}-1)p^{(n-1)i}$ for $i \geq 0$. The canonical homomorphism $S_{n-1} \to \varprojlim S_{n-1}/S_{n-1}^{(i+1)}$ is an isomorphism. Hence S_{n-1} and $G_{n-1}(\mathbf{F}) = \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_{n-1}$ are profinite groups.

Theorem 4.6. *The map* $h' : \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1}))) \to G_{n-1}(\mathbf{F})$ *is an isomorphism.*

Proof. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 \to & \mathrm{Gal}(L/k) & \longrightarrow & \mathrm{Gal}(L/\mathbf{F}_p((u_{n-1}))) & \longrightarrow & \mathrm{Gal}(k/\mathbf{F}_p((u_{n-1}))) & \to 1 \\ & \Big\downarrow & & \Big\downarrow{\scriptstyle h'} & & \Big\downarrow & \\ 1 \to & S_{n-1} & \longrightarrow & G_{n-1}(\mathbf{F}) & \longrightarrow & \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) & \to 1. \end{array}$$

Since $k/\mathbf{F}_p((u_{n-1}))$ is an unramified extension, the right vertical arrow is an isomorphism. Hence it is sufficient to show that the left vertical arrow $h' : \mathrm{Gal}(L/k) \to S_{n-1}$ is an isomorphism. If $\sigma \in \mathrm{Gal}(L/L_i)$, then $\Phi^\sigma(X) \equiv \Phi(X)$ mod degree p^{i+1}. Hence $h'(\sigma)(X) \equiv X$ mod degree p^{i+1}. This shows that $h'(\mathrm{Gal}(L/L_i)) \subset S_{n-1}^{(i+1)}$. Then h' induces a homomorphism:

$$\overline{h}' : \mathrm{Gal}(L_i/k) \longrightarrow S_{n-1}/S_{n-1}^{(i+1)}.$$

If $\overline{h}'(\sigma) = e$ for $\sigma \in \mathrm{Gal}(L/k)$, then we have $h'(\sigma)(X) \equiv X$ mod degree p^{i+1}. This implies $\Phi^\sigma(X) \equiv X$ mod degree p^{i+1}. Therefore $\overline{h}'$ is a monomorphism. Since the degree of L_i over k is equal to the order of $S_{n-1}/S_{n-1}^{(i+1)}$, the homomorphism $\overline{h}'$ is an isomorphism for all i. Therefore the homomorphism h' is an isomorphism. □

Let Θ be the set of all isomorphism from F_n to H_{n-1} over K. For $\Phi' \in \Theta$, we consider the following commutative diagram:

$$\begin{array}{ccc} F_n & = & F_n \\ \Phi \downarrow & & \downarrow \Phi' \\ H_{n-1} & \xrightarrow{h} & H_{n-1}. \end{array}$$

Since the isomorphism $h : H_{n-1} \to H_{n-1}$ is defined over $\mathbf{F}_{p^{n-1}}$, we see that Φ' is defined over L. Then the Galois group $\mathrm{Gal}(L/k)$ acts on Θ.

Corollary 4.7. *The action of* $\mathrm{Gal}(L/k)$ *on* Θ *is simply transitive.*

Proof. For $\Phi' \in \Theta$, we get $h \in S_{n-1}$ by the above commutative diagram. Let $\sigma \in \mathrm{Gal}(L/k)$ be the corresponding element under the isomorphism $h' : \mathrm{Gal}(L/k) \xrightarrow{\cong} S_{n-1}$. Then we have a commutative diagram:

$$\begin{array}{ccc} F_n & = & F_n \\ \Phi \downarrow & & \downarrow \Phi^{\sigma} \\ H_{n-1} & \xrightarrow{h} & H_{n-1}. \end{array}$$

By comparing two commutative diagrams, we see that $\Phi^{\sigma} = \Phi'$. This shows that the action is transitive.

If $\sigma \in \mathrm{Gal}(L/k)$ satisfies $\Phi^{\sigma} = \Phi$, then we have $\Phi_i^{\sigma} = \Phi_i$ for all i. Since L is generated by Φ_i for $i = 0, 1, \ldots$ over k, we see that $\sigma = id$. This completes the proof □

Let $\mathrm{Aut}(k)$ be the group consisting of the automorphisms of the topological field k. We consider $\mathrm{Aut}(k)$ acts k on the right. There is a homomorphism $\widetilde{G}_n(\mathbf{F}) \to \mathrm{Aut}(k)$ given by $g \mapsto \alpha(g)$. Since L is algebraic over k, there is a unique valuation on L extending the valuation on k. We regard L as a topological field by means of this valuation. Let $\mathrm{Aut}(L)$ be the automorphism group of L as a topological field. We denote by $A(L/k)$ the subgroup of $\mathrm{Aut}(L)$ consisting of the automorphisms which preserve the subfield k:

$$A(L/k) = \{\theta \in \mathrm{Aut}(L) |\ \theta(k) = k\}.$$

Then we have a restriction homomorphism

$$A(L/k) \to \mathrm{Aut}(k).$$

Let $\mathcal{G} = \widetilde{G}_n(\mathbf{F}) \times_{\mathrm{Aut}(k)} A(L/k)$ be the fibre product:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p} & \widetilde{G}_n(\mathbf{F}) \\ \Big\downarrow & & \Big\downarrow \\ A(L/k) & \longrightarrow & \mathrm{Aut}(k). \end{array}$$

By Lemma 4.3, the natural projection $p : \mathcal{G} \to \widetilde{G}_n(\mathbf{F})$ is surjective. It is clear that the kernel of p is the Galois group $\mathrm{Gal}(L/k)$. Hence we have an exact sequence:

$$1 \to \mathrm{Gal}(L/k) \longrightarrow \mathcal{G} \xrightarrow{p} \widetilde{G}_n(\mathbf{F}) \to 1.$$

Let $G_{n-1}(L)$ be the automorphism group of H_{n-1} over L in generalized sense. By the same way as Lemma 3.1, we have an isomorphism $G_{n-1}(L) \cong \mathrm{Aut}(L) \ltimes S_{n-1}$. Let $A(L/k) \ltimes S_{n-1}$ be the subgroup of $G_{n-1}(L)$. For $(g, \widetilde{g}) \in \mathcal{G}$, we consider the following commutative diagram:

$$\begin{array}{ccc} F_n & \xrightarrow{t(g)} & \alpha(g)^* F_n \\ \Phi \Big\downarrow & & \Big\downarrow \Phi^{\widetilde{g}} \\ H_{n-1} & \xrightarrow{h(g,\widetilde{g})} & H_{n-1}. \end{array}$$

This diagram defines a map $f : \mathcal{G} \to A(L/k) \ltimes S_{n-1}$ by $(g, \widetilde{g}) \mapsto (\widetilde{g}, h(g, \widetilde{g}))$.

Lemma 4.8. *The map $f : \mathcal{G} \to A(L/k) \ltimes S_{n-1}$ is a homomorphism.*

Proof. For $(g', \widetilde{g}') \in \mathcal{G}$, we have a commutative diagram:

$$\begin{array}{ccc}
\alpha(g)^*F_n & \xrightarrow{t(g')^{\alpha(g)}} & \alpha(g)^*\alpha(g')^*F_n \\
\Phi^{\widetilde{g}} \Big\downarrow & & \Big\downarrow \Phi^{\widetilde{g}'\widetilde{g}} \\
H_{n-1} & \xrightarrow{h(g',\widetilde{g}')^{\widetilde{g}}} & H_{n-1}.
\end{array}$$

Then we get a commutative diagram:

$$\begin{array}{ccccc}
F_n & \xrightarrow{t(g)} & \alpha(g)^*F_n & \xrightarrow{t(g')^{\alpha(g)}} & \alpha(g)^*\alpha(g')^*F_n \\
\Phi \Big\downarrow & & \Big\downarrow \Phi^{\widetilde{g}} & & \Big\downarrow \Phi^{\widetilde{g}'\widetilde{g}} \\
H_{n-1} & \xrightarrow{h(g,\widetilde{g})} & H_{n-1} & \xrightarrow{h(g',\widetilde{g}')^{\widetilde{g}}} & H_{n-1}.
\end{array}$$

This means that

$$f((g', \widetilde{g}') \cdot (g, \widetilde{g})) = (\widetilde{g}'\widetilde{g}, h(g', \widetilde{g}')^{\widetilde{g}} \cdot h(g, \widetilde{g})).$$

Hence f is a homomorphism. $\square$

There are homomorphisms $A(L/k) \to \mathrm{Aut}(k) \to \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ where the first homomorphism is restriction and the second homomorphism is obtained by considering the induced automorphism on the residue field. These homomorphisms are compatible with the action on the Morava stabilizer group S_{n-1}. Hence we get a homomorphism $f' : A(L/k) \ltimes S_{n-1} \to G_{n-1}(\mathbf{F})$.

There are homomorphisms

$$\mathcal{G} \xrightarrow{f' \circ f} G_{n-1}(\mathbf{F}) \longrightarrow \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p).$$

By Corollary 3.4, we have a natural isomorphism $\widetilde{G}_n(\mathbf{F}) \cong G_n(\mathbf{F})$. We identify $\widetilde{G}_n(\mathbf{F})$ and $G_n(\mathbf{F})$ by this isomorphism. Then we have homomorphisms

$$\mathcal{G} \xrightarrow{p} G_n(\mathbf{F}) \longrightarrow \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p).$$

We verify that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p} & G_n(\mathbf{F}) \\ {\scriptstyle f'\circ f}\Big\downarrow & & \Big\downarrow \\ G_{n-1}(\mathbf{F}) & \longrightarrow & \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p). \end{array}$$

Then we get a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 1 \to & \mathrm{Gal}(L/k) & \longrightarrow & \mathcal{G} & \xrightarrow{p} & G_n(\mathbf{F}) & \to 1 \\ & \Big\downarrow & & {\scriptstyle f'\circ f}\Big\downarrow & & \Big\downarrow & \\ 1 \to & S_{n-1} & \longrightarrow & G_{n-1}(\mathbf{F}) & \longrightarrow & \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) & \to 1. \end{array}$$

The left vertical arrow is an isomorphism from Theorem 4.6. Hence we get the following Theorem.

Theorem 4.9. *There are isomorphisms*

$$\mathcal{G} \cong G_n(\mathbf{F}) \times_{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)} G_{n-1}(\mathbf{F}) \cong \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes (S_n \times S_{n-1}).$$

5 Action of $\mathcal{G}$ on L

By Theorem 4.9, $\mathcal{G}$ is a profinite group. There is an action of $\mathcal{G}$ on L by using the projection $\mathcal{G} \to A(L/k)$. In this section we show that the profinite group $\mathcal{G}$ acts on L continuously.

Since L is algebraic over k, there is a unique valuation v on L extending the valuation on k. We regard L as a metric space by using v. We note that every $g \in \mathcal{G}$ preserves the valuation: $v(x^g) = v(x)$ for all $x \in L$. In particular $g \in \mathcal{G}$ induces a homeomorphism $g : L \to L$. For any $x \in L$ and $m > 0$, we define $U(x, m)$ as an open neighbourhood of x given by $U(x, m) = \{y \in L | \ v(y - x) > m\}$.

Lemma 5.1. *If $\mathcal{G}$ acts on L_i continuously for all i, then $\mathcal{G}$ acts on L continuously.*

Proof. We take any $x \in L$ and any $m > 0$. There exists i such that $x \in L_i$. Since $\mathcal{G}$ acts on L_i continuously, there exists an open neighbourhood V of the identity $\mathcal{G}$ such that $x \cdot V \subset U(x, m)$. Then for any $y \in U(x, m)$ and $h \in V$, we have

$$\begin{aligned} v(y^{hg} - x^g) &= v((y^{hg} - x^{hg}) + (x^{hg} - x^g)) \\ &\geq \min\{v(y^{hg} - x^{hg}), v(x^{hg} - x^g)\} \\ &= \min\{v(y - x), v(x^h - x)\} \\ &> m. \end{aligned}$$

This shows that $\mathcal{G}$ acts continuously on L. $\square$

Hence we consider the action of $\mathcal{G}$ on L_i for fixed i. We denote by v' the valuation of L_i such that $v'(\Phi_i) = 1$. Let $U_i(x, m)$ be the open nighbourhood of x given by $\{y \in L_i |\ v'(y - x) > m\}$.

Lemma 5.2. *For any $m > 0$, there exists an open neighbourood V of the identity of $\mathcal{G}$ such that $v'({\Phi_i}^g - \Phi_i) > m$ for all $g \in V$.*

Proof. The action of $(g, \widetilde{g}) \in \mathcal{G}$ is described by the commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow{t(g)} & \alpha(g)^* F_n \\ {\scriptstyle \Phi}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi^{\widetilde{g}}} \\ H_{n-1} & \xrightarrow{h(g,\widetilde{g})} & H_{n-1}. \end{array}$$

Hence we have a relation

$$\sum_{i,j\geq 0} {}^{H_{n-1}}{\Phi_j}^{\widetilde{g}_*} t_i(g)^{p^j} X^{p^{i+j}} = \sum_{i,j\geq 0} {}^{H_{n-1}} h_j \Phi_i^{p^j} X^{p^{i+j}}.$$

If $v'(t_0(g)-1) > m, v'(t_1(g)) > m, \ldots, v'(t_i(g)) > m$ and $h_0 = 1, h_1 = 0, \ldots, h_i = 0$, then $v'(\Phi_j^{\tilde{g}} - \Phi_j) > m$ for $j = 1, \ldots, i$. There exists an open subgroup $S_n^{(j)}$ of S_n such that $v'(t_0(g)-1) > m, v'(t_1(g)) > m, \ldots, v'(t_i(g)) > m$ for all $g \in S_n^{(j)}$. Then the open subgroup $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes (S_n^{(j)} \times S_{n-1}^{(m+1)})$ of $\mathcal{G}$ satisfies the condition. $\square$

Proposition 5.3. *The profinite group $\mathcal{G}$ acts on the metric space L continuously.*

Proof. By Lemma 5.1, it is sufficient to prove that $\mathcal{G}$ acts on L_i continuously for all i. Let $x \in L_i$ such that

$$x = \sum_{j=m'}^{\infty} x_j {\Phi_i}^j$$

where $x_j \in \mathbf{F}$ for $j \geq m'$. For any $m > 0$, by Lemma 5.2, there exists an open neighbourhood V of the identity of $\mathcal{G}$ such that $v'(\Phi_i^g - \Phi_i) > m - m' + 1$ for all $g \in V$. Let $W = V \cap S_n \times S_{n-1}$. Since $S_n \times S_{n-1}$ is an open subgroup, W is an open set. Then we have $x \cdot W \subset U_i(x, m)$. For any $y \in U_i(x, m)$ and any $h \in W$,

$$\begin{aligned} v'(y^{hg} - x^g) &= v'((y^{hg} - x^{hg}) + (x^{hg} - x^g)) \\ &\geq \min\{v'(y^{hg} - x^{hg}), v'(x^{hg} - x^g)\} \\ &= \min\{v'(y - x), v'(x^h - x)\} \\ &> m. \end{aligned}$$

This shows that $\mathcal{G}$ acts on L_i continuously. $\square$

6 Vanishing of some cohomology

Let G be a topological group and let M be a topological G-module. In this section we define a cohomology group of G with the coefficients in M parameterized by a topological space. Then we consider a vanishing condition of this cohomology group.

Let X be a topological space and let A be a subspace of X. We denote by (X, A) such a pair of topological spaces. We define a homogeneous n-cochain of G with the coefficients in M over a topological pair (X, A) to be a continuous map f from $X \times G^{n+1}$ to M such that

$$f(x; \sigma\sigma_0, \sigma\sigma_1, \ldots, \sigma\sigma_n) = \sigma \cdot f(x; \sigma_0, \sigma_1, \ldots, \sigma_n).$$

and

$$f(a; \sigma_1, \ldots, \sigma_n) = 0 \quad \text{if } a \in A.$$

We denote by $C^n_{(X,A)}(G; M)$ the abelian group of all homogeneous n-cochains for G in M over (X, A). As usual the coboundary map $d : C^n_{(X,A)}(G; M) \to C^{n+1}_{(X,A)}(G; M)$ is given by

$$df(x; \sigma_0, \ldots, \sigma_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x; \sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_{n+1}).$$

Let $H^*_{(X,A)}(G; M)$ be the cohomology group of the cochain complex $C^*_{(X,A)}(G; M)$. For $(X, A) =$pt (one point space), we see that $H^*_{\mathrm{pt}}(G; M) = H^*(G; M)$ is the continuous cohomology group of G with the coefficients in M.

In the following of this section we assume that G is a finite group and regard it as a discrete group. Let $C(X, A; M)$ be the abelian group of all continuous functions $f : X \to M$ such that $f(a) = 0$ for $a \in A$. Then a homogeneous n-cochain of G with the coefficients in M over (X, A) is naturally identified with a homogeneous n-cochain of G with the coefficients in $C(X, A; M)$. In particular, we have a natural isomorphism $H^*_{(X,A)}(G; M) \cong H^*(G; C(X, A; M))$.

We recall some vanishing condition of the cohomology group of G (cf. Chap. I §6 [5]). A mean on M is an additive function I which associates with each map $f : G \to M$ an element $I(f) \in M$ such that

1. if $f(\sigma) = m \in M$ for all $\sigma \in G$, then $I(f) = m$,
2. for all $\sigma \in G$, $I(\sigma \cdot f) = \sigma \cdot I(f)$ where $(\sigma \cdot f)(\tau) = \sigma \cdot f(\sigma^{-1}\tau)$.

Proposition 6.1. *If M is a G-module which admits a mean, then $H^n(G; M) = 0$ for all $n > 0$.*

Proof. Let f be a homogeneous n-cocycle for G in M. For fixed $\sigma_1, \dots, \sigma_n$, we consider the map

$$\phi(f; \sigma_1, \dots, \sigma_n) : \sigma \mapsto f(\sigma, \sigma_1, \dots, \sigma_n).$$

This map has a mean value $(I_n f)(\sigma_1, \dots, \sigma_n) \in M$. Since $\phi(f; \sigma\sigma_1, \dots, \sigma\sigma_n) = \sigma \cdot \phi(f; \sigma_1, \dots, \sigma_n)$, we get $(I_n f)(\sigma\sigma_1, \dots, \sigma\sigma_n) = \sigma \cdot (I_n f)(\sigma_1, \dots, \sigma_n)$. Hence $I_n f$ is a homogeneous $(n-1)$-cochain for G in M. We show that $d(I_n f) = f$. Since f is an n-cocycle,

$$\begin{aligned} 0 &= df(x, \sigma_0, \sigma_1, \dots, \sigma_n) \\ &= f(\sigma_0, \dots, \sigma_n) - \textstyle\sum_{i=0}^n (-1)^i f(x, \sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_n). \end{aligned}$$

So we have

$$\sum_{i=0}^n (-1)^i \phi(f; \sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_n) = f(\sigma_0, \dots, \sigma_n).$$

By taking mean values of both sides, we get

$$\sum_{i=0}^{n}(-1)^i(I_nf)(\sigma_0,\dots,\widehat{\sigma}_i,\dots,\sigma_n)=f(\sigma_0,\dots,\sigma_n).$$

The left hand side is equal to $d(I_nf)(\sigma_0,\dots,\sigma_n)$. This completes the proof.

□

Let R be a commutative ring. We denote by $R[G]$ the group ring of G over R. Let $\underline{g}$ $(g\in G)$ be a canonical base of $R[G]$.

Lemma 6.2. *The G-module $R[G]$ admits a mean.*

Proof. A map $f:G\to R[G]$ has a following form:

$$f(g)=\sum_{g'\in G}f_{g'}(g)\underline{g'}$$

where $f_{g'}:G\to R$. We define $I(f)$ as

$$I(f)=\sum_{g\in G}f_g(g)\underline{g}.$$

Then it is easy to verify that I is a mean. □

If R is a topological commutative ring, the group ring $R[G]$ is naturally topological G-module. Let $f:X\times G\to R[G]$ be a continuous function. Fixed $x\in X$, the function $f(x,-):G\to R[G],\ g\mapsto f(x,g)$ has a mean value $I(f)(x)\in R[G]$ by using the mean of the proof of Lemma 6.2.

Lemma 6.3. *The function $I(f):X\to R[G]$ is continuous.*

Proof. The function $f:X\times G\to R[G]$ is a following form:

$$f(x,g)=\sum_{g'\in G}f(x,g)_{g'}\underline{g'}.$$

Then f is continuous if and only if $f(-,g)_{g'} : X \to R$ is continuous for all $g, g' \in G$. On the other hand, the function $I(f) = \sum_{g \in G} I(f)_g \underline{g}$ is continuous if and only if $I(f)_g : X \to R$ is continuous for all $g \in G$. By the proof of Lemma 6.2, $I(f)_g(x) = I(f(x,-))_g = f(x,g)_g$. This complete the proof. $\square$

Corollary 6.4. *The G-module $C(X, A; R[G])$ admits a mean.*

Proof. Let $f : G \to C(X, A; R[G])$ be a map. The adjoint $\mathrm{ad}(f) : X \times G \to R[G]$ is a continuous map. By Lemma 6.3, we have $I(\mathrm{ad}(f)) \in C(X, A; M)$. Then the function $I(\mathrm{ad}(-))$ is a mean on $C(X, A; M)$. $\square$

Proposition 6.5. *Let R be a topological commutative ring. If $M \cong R[G]$ as topological G-modules, then $H^n_{(X,A)}(G; M) = 0$ for all $n > 0$.*

Proof. Since $H^*_{(X,A)}(G; M) \cong H^*(G; C(X, A; M))$, this follows from Proposition 6.1 and Corollary 6.4. $\square$

Remark 6.6. We can also define the cohomology $H^*_{(X,A)}(G; M)'$ of G in M over a topological pair (X, A) by using a (normalized) nonhomogeneous cochain complex of G in M. If G is finite, we also have a natural isomorphism

$$H^*_{(X,A)}(G; M)' \cong H^*(G; C(X, A; M)).$$

Hence, under the same condition as Proposition 6.5, we have $H^*_{(X,A)}(G; M)' = 0$ for all $n > 0$.

7 Normalized cochain complex

Let G be a topological group and let M be a topological G-module. A continuous n-cochain for G in M is a continuous function $f : G^n \to M$. We denote by $C^n = C^n(G; M)$ the abelian group of all continuous n-cochains for G in M. The coboundary map $d : C^n \to C^{n+1}$ is given by

$$\begin{aligned} df(\gamma_1, \dots, \gamma_{n+1}) &= \gamma_1 \cdot f(\gamma_2, \dots, \gamma_{n+1}) \\ &\quad + \textstyle\sum_{i=1}^{n} (-1)^i f(\gamma_1, \dots, \gamma_i\gamma_{i+1}, \cdots, \gamma_{n+1}) \\ &\quad + (-1)^{n+1} f(\gamma_1, \dots, \gamma_n). \end{aligned}$$

A "normalized" continuous n-cochain for G in M is a continuous function $f : G^n \to M$ such that $f(\gamma_1, \dots, \gamma_n) = 0$ if γ_i is equal to the identity e for some i $(1 \le i \le n)$. We denote by A^n the abelian group of all "normalized" continuous n-cochains for G in M. It is easy to verify that A^* is a sub-cochain complex of C^*. In this section we show that the natural cochain map $A^* \hookrightarrow C^*$ induces isomorphisms on cohomology groups.

We define a filtration of the cochain complex C^*. Let F^pC^n be a subgroup of C^n consisting of all continuous n-cochains for G in M such that $f(\gamma_1, \dots, \gamma_n) = 0$ if γ_i is equal to the identity e for some i $(1 \le i \le p)$. Then we have a filtration of the abelian group C^n:

$$C^n = F^0C^n \supset F^1C^n \supset \cdots \supset F^{n-1}C^n \supset F^nC^n = F^{n+1}C^n = \cdots .$$

It is easy to see that $d(F^pC^n) \subset F^pC^{n+1}$. Hence F^pC^* is a sub-cochain complex of C^*. We obtain a filtration of the cochain complex C^*:

$$C^* = F^0C^* \supset F^1C^* \supset \cdots \supset F^pC^* \supset \cdots .$$

Note that $\cap_p F^pC^* = A^*$. In the following we show that the inclusion $F^pC^* \hookrightarrow F^{p-1}C^*$ induces isomorphisms on cohomology groups.

For $1 \le p \le n$, we define an abelian group $B^{p,n}$ to be the set of all continuous functions $f : G^{n-1} \to M$ such that $f(\gamma_1, \dots, \gamma_{n-1}) = 0$ if γ_i is equal to the identity e for some i $(1 \leq i < p)$. For $p > n$, we set $B^{p,n} = 0$. There is an exact sequence for all $p \geq 1$ and n:

$$0 \to F^pC^n \longrightarrow F^{p-1}C^n \longrightarrow B^{p,n} \to 0$$

where the right hand map is given by the restriction of the source

$$G^{n-1} = G^{p-1} \times \{e\} \times G^{n-p} \hookrightarrow G^n.$$

There is a section of $F^{p-1}C^n \to B^{p,n}$ obtained by the projection forgetting the pth component of G:

$$G^n = G^{p-1} \times G \times G^{n-p} \longrightarrow G^{p-1} \times \{e\} \times G^{n-p} = G^{n-1}.$$

Since F^pC^* is a sub-cochain complex of $F^{p-1}C^*$, there is a map $d : B^{p,n} \to B^{p,n+1}$ which makes $B^{p,*}$ a cochain complex. The coboundary map $d : B^{p,n} \to B^{p,n+1}$ is given by the following form:

$$\begin{aligned}(df)(\gamma_1, \ldots, \gamma_n) &= \textstyle\sum_{i=p}^{n-1}(-1)^i f(\gamma_1, \ldots, \gamma_{p-1}, \gamma_p, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_n) \\ &\quad +(-1)^n f(\gamma_1, \ldots, \gamma_{p-1}, \gamma_p, \ldots, \gamma_{n-1}).\end{aligned}$$

Hence it is sufficient to show that all the cohomology groups of the cochain complex $B^{p,*}$ vanish so as to prove that $H^n(F^pC) \to H^n(F^{p-1}C)$ are isomorphisms for all n.

We define a map $s : B^{p,n} \to B^{p,n-1}$. For $f \in B^{p,n}$, the function $s(f) : G^{n-2} \to M$ is defined by

$$s(f)(\gamma_1, \ldots, \gamma_{n-2}) = f(\gamma_1, \ldots, \gamma_{p-1}, e, \gamma_p, \ldots, \gamma_{n-2}).$$

Clearly $s(f) \in B^{p,n-1}$. Note that if $f \in B^{p,p}$, then $s(f) = 0$. We compute

$d(s(f))$ and $s(d(f))$. First we have

$$\begin{aligned} d(s(f))(\gamma_1, \dots, \gamma_{n-1}) &= \textstyle\sum_{i=p}^{n-2} (-1)^i s(f)(\gamma_1, \dots, \gamma_{p-1}, \gamma_p, \dots, \\ &\quad \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_{n-1}) \\ &\quad +(-1)^{n-1} s(f)(\gamma_1, \dots, \gamma_{n-2}) \\ &= \textstyle\sum_{i=p}^{n-2} (-1)^i f(\gamma_1, \dots, \gamma_{p-1}, e, \gamma_p, \dots, \\ &\quad \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_{n-1}) \\ &\quad +(-1)^{n-1} f(\gamma_1, \dots, \gamma_{p-1}, e, \gamma_p, \dots, \gamma_{n-2}). \end{aligned}$$

Second we have

$$\begin{aligned} s(d(f))(\gamma_1, \dots, \gamma_{n-1}) &= d(f)(\gamma_1, \dots, \gamma_{p-1}, e, \gamma_p, \dots, \gamma_{n-1}) \\ &= (-1)^p f(\gamma_1, \dots, \gamma_{p-1}, \gamma_p, \dots, \gamma_{n-1}) \\ &\quad + \textstyle\sum_{i=p}^{n-2} (-1)^{i+1} f(\gamma_1, \dots, \gamma_{p-1}, e, \gamma_p, \dots, \\ &\quad \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_{n-1}) \\ &\quad +(-1)^n f(\gamma_1, \dots, \gamma_{p-1}, e, \gamma_p, \dots, \gamma_{n-2}). \end{aligned}$$

Hence we get

$$d(s(f)) + s(d(f)) = (-1)^p f.$$

This shows that $H^n(B^{p,*}) = 0$ for all n.

Theorem 7.1. *The cochain map $A^* \hookrightarrow C^*$ induces isomorphisms on cohomology groups*

$$H^*(A) \xrightarrow{\cong} H^*(C).$$

8 Inflation maps

Let G be a Hausdorff topological group and let K be a finite normal subgroup. We denote by H the quotient group G/K and $\pi : G \to H$ the quotient map.

In this section we assume that there is a continuous section $s : H \to G$ such that $s(e) = e$. Note that s is not necessarily group homomorphism. For example, if G is a profinite group, then there is such a section [15]. Let M be a topological G module. The fixed submodule M^K is naturally a topological H module. In this section we study the inflation map $H^*(H; M^K) \to H^*(G; M)$ under some conditions.

A normalized continuous n-cochain for G in M is a continuous map $f : G^n \to M$ such that $f(\gamma_1, \dots, \gamma_n) = 0$ if γ_i is equal to the identity e for some i $(1 \leq i \leq n)$. We denote by $A^n = A^n(G; M)$ the abelian group of all normalized continuous n-cochains for G in M. By definition, $A^0 = M$. The non-homogeneous coboundary map $d : A^n \to A^{n+1}$ is given by

$$\begin{aligned}(df)(\gamma_1, \dots, \gamma_{n+1}) &= \gamma_1 \cdot f(\gamma_2, \dots, \gamma_{n+1}) \\ &\quad + \textstyle\sum_{i=1}^{n} (-1)^i f(\gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_{n+1}) \\ &\quad + (-1)^{n+1} f(\gamma_1, \dots, \gamma_n).\end{aligned}$$

By Theorem 7.1, the cohomology of A^* is the continuous cohomology $H^*(G; M)$.

We define a filtration of the cochain complex A^*. For $j = 0$, we set $A^n_0 = A^n$. For $0 < j \leq n$, F^jA^n is defined as a subgroup of A^n consisting of $f \in A^n$ such that $f : G^n \to M$ factors through the continuous map $f' : G^{n-j} \times H^j \to M$. For $j > n$, we set $F^jA^n = 0$. Hence we get a filtration of A^n:

$$A^n = F^0A^n \supset F^1A^n \supset \cdots F^nA^n \supset F^{n+1}A^n = 0.$$

It is easy to verify that $d(F^jA^n) \subset F^jA^{n+1}$. Hence $(F^jA^*)_{j \geq 0}$ is a filtration of the cochain complex A^*:

$$A^* = F^0A^* \supset F^1A^* \supset \cdots \supset F^nA^* \supset \cdots.$$

Let N be the topological K module obtained from the topological G module M by the restriction of the action to K. In the following we assume that there is a topological commutative ring R such that the topological K module N is isomorphic to the group ring $R[K]$ as topological K modules.

Since K is discrete and finite, the normalized continuous n-cochain group $A^n(K;N)$ is naturally isomorphic to a direct product of finite many copies of N. We introduce a topology on $A^n(K;N)$ by using this isomorphism and the product topology. Let $A^j(G;A^i(K;N))$ be the abelian group of all normalized continuous j-cochains of G in $A^i(K;M)$. Note that a map from a topological space X to $A^i(K;M)$ is continuous if and only if the adjoint $X \times K^i \to M$ is continuous. We define a homomorphism $r_j : F^j A^{i+j} \to A^j(H; A^i(K;M))$ by

$$r_j(f)(\sigma_1, \dots, \sigma_j)(\tau_1, \dots, \tau_i) = f'(\tau_1, \dots, \tau_i, \sigma_1, \dots, \sigma_j)$$

where $f' : G^i \times H^j \to M$ is a continuous map such that

$$f(\gamma_1, \dots, \gamma_n) = f'(\gamma_1, \dots, \gamma_{n-j}, \pi\gamma_{n-j+1}, \dots, \pi\gamma_n).$$

It is easy to see that $r_j(f) = 0$ if $f \in F^{j+1}A^{i+j}$. Hence we get a homomorphism

$$\overline{r}_j : F^j A^{i+j}/F^{j+1}A^{i+j} \longrightarrow A^j(H; A^i(K;M)).$$

We note that $\overline{r}_j : F^j A^j/F^{j+1}A^j \to A^j(H;M)$ is an isomorphism. Let d be the coboundary operator of $F^jA^*/F^{j+1}A^*$. The coboundary operator of $A^*(K;M)$ induces a homomorphism

$$d_K : A^j(H; A^*(K;M)) \to A^j(H; A^{*+1}(K;M)).$$

Then we obtain that $d_K \circ \overline{r}_j = \overline{r}_j \circ d$.

Lemma 8.1. *$H(A^j(H;A^*(K;M)),d_K)=A^j(H;M^K)$ for all j.*

Proof. Let T_n be a subspace of G^n given by

$$T_n=\cup_{k=1}^n G^{k-1}\times\{e\}\times G^{n-k-1}\subset G^n.$$

By Remark 6.6, $H^*_{(G^n,T_n)}(K;M)'=M^K$. This shows that the lemma holds. $\square$

Lemma 8.2. *$\overline{r}_j:H^j(F^jA^*/F^{j+1}A^*)\xrightarrow{\cong}A^j(H;M^K)$.*

Proof. Let $f\in F^jA^j$ such that $df\in F^{j+1}A^{j+1}$. Then $d_K(r_j(f))=0$. By Lemma 8.1, $r_j(f)\in A^j(H;M^K)$. Conversely, let $\widetilde{f}\in A^j(H;M^K)\subset A^j(H;M)$. We define $f\in F^jA^j$ by

$$f(\gamma_1,\dots,\gamma_j)=\widetilde{f}(\pi\gamma_1,\dots,\pi\gamma_j).$$

Then for any $\tau\in K$, we easily see that

$$df(\gamma_1\tau,\gamma_2,\dots,\gamma_{j+1})=df(\gamma_1,\gamma_2,\dots,\gamma_{j+1}).$$

$\square$

Lemma 8.3. *$H^n(F^jA^*/F^{j+1}A^*)=0$ for all $n>j$.*

Proof. Put $i=n-j-1\geq 0$. Let $f\in F^jA^n$ such that $df\in F^{j+1}A^{n+1}$. Since $d_K\circ\overline{r}_j=\overline{r}_j\circ d$, we have $d_K(r_j(f))=0$. By Lemma 8.1, there is $u\in A^j(H;A^i(K;M))$ such that $d_Ku=r_j(f)$. We define a continuous function $g:K^i\times G^j\to M$ by

$$g(\sigma_1,\dots,\sigma_i,\gamma_1,\dots,\gamma_j)=u(\pi\gamma_1,\dots,\pi\gamma_j)(\sigma_1,\dots,\sigma_i).$$

Set $g_0 = g$. We define a sequence of continuous functions $g_1, \dots, g_i$ such that g_k is defined on $G^k \times K^{i-k} \times G^j$ with its values in M and g_k is an extension of g_{k-1} for all $1 \leq k \leq i$. We write $\rho_s^t = (\rho_s, \dots, \rho_t) \in G^{t-s+1}, \gamma_s^t = (\gamma_s, \dots, \gamma_t) \in G^{t-s+1}$ and $\sigma_s^t = (\sigma_s, \dots, \sigma^t) \in K^{t-s+1}$ for $1 \leq s \leq t$. Let

$$g_1(\rho, \sigma_2^i, \gamma_1^j) = s(\pi\rho) \cdot g(s(\pi\rho)^{-1}\rho, \sigma_2^i, \gamma_1^j) - f(s(\pi\rho), s(\pi\rho)^{-1}\rho, \sigma_2^i, \gamma_1^j).$$

For $k > 1$, we define the g_k's recursively by

$$\begin{aligned} g_k(\rho_1^k, \sigma_{k+1}^i, \gamma_1^j) &= g_{k-1}(\rho_1^{k-2}, \rho_{k-1}s(\pi\rho_k), s(\pi\rho_k)^{-1}\rho_k, \sigma_{k+1}^i, \gamma_1^j) \\ &\quad +(-1)^k f(\rho_1^{k-1}, s(\pi\rho_k), s(\pi\rho_k)^{-1}\rho_k, \sigma_{k+1}^i, \gamma_1^j). \end{aligned}$$

Then we can show that $f - dg_i \in F^{j+1}A^n$ as the proof of Theorem 2.2.1 of [5]. $\square$

Therefore we get an E_1-term of the spectral sequence associated with the filtration $(F^jA^*)_{j\geq 0}$ of the cochain complex A^*

$$E_1^{p,q} \cong \begin{cases} A^p(H; M^K) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

It is easy to verify that the differential d_1 is given by the coboundary map of the normalized continuous cochain complex $A^*(H; M^K)$ of H in M^K. Hence we get an E_2-term

$$E_2^{p,q} \cong \begin{cases} H^p(H; M^K) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

The spectral sequence collapses from E_2-term and converges to the cohomology groups $H^*(A^*) = H^*(G; M)$. It is easy to verify that the edge homomorphism $E_2^{p,0} \to H^p(A^*)$ is identified with the inflation map $H^p(H; M^K) \to H^p(G; M)$. Hence we get the following theorem.

Theorem 8.4. *Let G be a Hausdorff topological group, K a finite normal subgroup and $H = G/K$ the quotient group. We assume that there is a continuous section $s : H \to G$. Let M be a topological G module such that M is isomorphic as topological K-modules to a group ring $R[K]$ for some topological commutative ring R. Then the inflation map $H^*(H; M^K) \to H^*(G; M)$ is an isomorphism*

$$H^*(H; M^K) \xrightarrow{\cong} H^*(G; M).$$

9 Cohomology of $G_{n-1}(\mathbf{F})$ in $\mathbf{F}[w^{\pm 1}]$

Let $\mathbf{F}$ be a finite field containing the finite fields $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$. The profinite group $G_{n-1}(\mathbf{F})$ is a semi-direct product $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_{n-1}$. There is an action of $G_{n-1}(\mathbf{F})$ on the graded field $\mathbf{F}[w^{\pm 1}]$ where the degree of w is -2. In this section we study the cohomology of $G_{n-1}(\mathbf{F})$ in the coefficients $\mathbf{F}[w^{\pm}]$.

We recall that the group $G_{n-1}(\mathbf{F})$ is the automorphism group of the Honda formal group law H_{n-1} over the field $\mathbf{F}$ in the generalized sense. The Morava stabilizer group S_{n-1} which is the automorphism group of H_{n-1} in usual sense is a normal subgroup of $G_{n-1}(\mathbf{F})$ and its quotient group is the Galois group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$. In fact we have an isomorphism

$$G_{n-1}(\mathbf{F}) \cong \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_{n-1}.$$

The profinite group $G_{n-1}(\mathbf{F})$ acts on the graded field $\mathbf{F}[w^{\pm 1}]$ from the right as follows. For every $h \in S_{n-1}$, h has a following expression

$$h = h_0 + h_1 T + h_2 T^2 + \cdots, \qquad h_i \in W\mathbf{F}_{p^{n-1}},\ h_i^{p^{n-1}} = h_i,\ h_0 \neq 0$$

where $T^{n-1} = p$ and $Th_i = h_i^p T$ for all $i \geq 0$. The subgroup S_{n-1} of $G_{n-1}(\mathbf{F})$ acts on $\mathbf{F}[w^{\pm 1}]$ as $\mathbf{F}$-algebra automorphisms by

$$w^h = \pi(h_0)^{-1} w, \quad h \in S_{n-1}$$

where $\pi : W\mathbf{F}_{p^{n-1}} \to \mathbf{F}_{p^{n-1}}$ is the projection to the residue field. The subgroup $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ acts on $\mathbf{F}[w^{\pm 1}]$ by

$$(aw^n)^\sigma = a^\sigma w^n, \quad \sigma \in \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p), a \in \mathbf{F}, n \in \mathbf{Z}.$$

Then we obtain an action of $G_{n-1}(\mathbf{F})$ on $\mathbf{F}[w^{\pm 1}]$ compatible to the above actions of the subgroups S_{n-1} and $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$.

We consider the cohomology of $G_{n-1}(\mathbf{F})$ in $\mathbf{F}[w^{\pm 1}]$. Since we have an exact sequence

$$1 \to S_{n-1} \longrightarrow G_{n-1}(\mathbf{F}) \longrightarrow \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \to 1$$

and the quotient group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ is finite, we have a Lyndon-Hochschild-Serre spectral sequence

$$E_2^{*,*} = H^*(\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p); H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])) \Longrightarrow H^*(G_{n-1}(\mathbf{F}); \mathbf{F}[w^{\pm 1}]).$$

The cohomology $H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])$ is $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ module over $\mathbf{F}$ and the action of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ satisfies the relation $(cm)^\sigma = c^\sigma m^\sigma$ for all $c \in \mathbf{F}$ and $m \in H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])$. By Lemma 5.4 of [2], the E_2-term is given by

$$E_2^{p,q} \cong \begin{cases} H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)}, & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Hence the spectral sequence collapses from E_2-term and we obtain an isomorphism

$$H^*(G_{n-1}(\mathbf{F}); \mathbf{F}[w^{\pm 1}]) \cong H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)}.$$

We denote by $\Gamma(n-1)$ the Galois group of the extension $\mathbf{F}_{p^{n-1}}/\mathbf{F}_p$. The subfield $\mathbf{F}_{p^{n-1}}[w^{\pm 1}] \subset \mathbf{F}[w^{\pm 1}]$ is stable under the action of S_{n-1}. Hence we have

$$H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}]) \cong H^*(S_{n-1}; \mathbf{F}_{p^{n-1}}[w^{\pm 1}]) \otimes_{\mathbf{F}_{p^{n-1}}} \mathbf{F}.$$

This isomorphism implies an isomorphism

$$H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)} \cong H^*(S_{n-1}; \mathbf{F}_{p^{n-1}}[w^{\pm 1}])^{\Gamma(n-1)}.$$

Therefore we get the following proposition.

Proposition 9.1. *There is an isomorphism*

$$H^*(G_{n-1}(\mathbf{F}); \mathbf{F}[w^{\pm 1}]) \cong H^*(S_{n-1}; \mathbf{F}_{p^{n-1}}[w^{\pm 1}])^{\Gamma(n-1)}.$$

10 Cohomology of G_n in $\mathbf{F}((u_{n-1}))[u^{\pm 1}]$

The profinite group acts on the graded field $\mathbf{F}((u_{n-1}))[u^{\pm 1}]$ where the degree of u is -2. In this section we study the cohomology of G_n in the coefficients $\mathbf{F}((u_{n-1}))[u^{\pm 1}]$.

Let $k = \mathbf{F}((u_{n-1}))$. The profinite group $G_n = \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_n$ acts on the field k continuously from the right. Explicitly, the Morava stabilizer group S_n acts on k as F-algebra automorphisms by

$${u_{n-1}}^g = t_0(g)^{-(p^{n-1}-1)} u_{n-1}, \quad g \in S_n$$

where $t_0(g)$ is the leading coefficient of the homomorphism

$$t(g) : F_n \to \alpha(g)^* F_n, \quad t(g)(X) = \sum_{i \geq 0}{}^{\alpha(g)^* F_n} t_i(g) X^{p^i}.$$

The Galois group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ acts on $\mathbf{F}$ naturally and on u_{n-1} trivially.

We define an action of the profinite group G_n on the graded field $k[u^{\pm 1}]$ extending the action on the degree 0 part given above. For $g \in S_n$, we set

$$u^g = t_0(g)^{-1}u$$

and $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ acts on u trivially. Then we see that this defines a continuous action of G_n on $k[u^{\pm 1}]$ from the right.

We consider the cohomology of G_n in the coefficients $k[u^{\pm 1}]$. Since there is an exact sequence

$$1 \to S_n \longrightarrow G_n \longrightarrow \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \to 1$$

and the quotient group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$ is finite, we have a Lyndon-Hochschild-Serre spectral sequence

$$E_2^{*,*} = H^*(\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p); H^*(S_n; k[u^{\pm 1}])) \Longrightarrow H^*(G_n; k[u^{\pm 1}]).$$

By the same reason as the case of $H^*(S_{n-1}; \mathbf{F}[w^{\pm 1}])$, we see that the E_2-term is given by

$$E_2^{p,q} \cong \begin{cases} H^*(S_n; k[u^{\pm 1}])^{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)}, & \text{if } q = 0, \\ 0 & \text{if } q \neq 0 \end{cases}$$

and the spectral sequence collapses from E_2-term. Since the subfield $\mathbf{F}_{p^n}((u_{n-1}))[u^{\pm 1}] \subset k[u^{\pm 1}]$ is stable under the action of G_n, we obtain the following proposition.

Proposition 10.1. *There is an isomorphism*

$$H^*(G_n; k[u^{\pm 1}]) \cong H^*(S_n; \mathbf{F}_{p^n}((u_{n-1}))[u^{\pm 1}])^{\mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)}.$$

11 Cohomology of $\mathcal{G}(i)$

Let $\mathbf{F}$ be a finite filed containing the finite fields $\mathbf{F}_{p^n}$ and $\mathbf{F}_{p^{n-1}}$. Let $k = \mathbf{F}((u_{n-1}))$. We denote by L_i the totally ramified Galois extension $k(\Phi_0, \ldots, \Phi_i)$ for $i \geq 0$ where Φ_i are coefficients of the isomorphism

$$\Phi : F_n \longrightarrow H_{n-1}, \quad \Phi(X) = \sum_{i \geq 0}{}^{H_{n-1}} \Phi_i X^{p^i}.$$

Let $L_{-1} = k$ and $L = \cup_{i \geq 0} L_i$. We consider a graded field $L[u^{\pm 1}]$ where the degree of u is -2. In this section we define quotient groups $\mathcal{G}(i)$ of the profinite group $\mathcal{G}$ which act on the graded field $L_i[u^{\pm 1}]$ for $i \geq -1$. Then we study the cohomologies of $\mathcal{G}(i)$ in the coefficients $L_i[u^{\pm 1}]$.

We recall that $\mathcal{G}$ is a fibre product $\widetilde{G}_n(\mathbf{F}) \times_{\mathrm{Aut(k)}} A(L/k)$ where $\widetilde{G}_n(\mathbf{F})$ is the automorphism group of the deformation F_n in $\mathbf{F}[\![u_{n-1}]\!]$, $Aut(k)$ is the automorphism group of the local field k, and $A(L/k)$ is the subgroup of the automorphism group of the valuation field L, whose elements preserve the subfield k. By Theorem 4.9, there is an isomorphism $\mathcal{G} \cong \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes (S_n \times S_{n-1})$. Hence $\mathcal{G}$ is a profinite group. There is an action of $\mathcal{G}$ on L by using the projection $\mathcal{G} \to A(L/k)$. By Proposition 5.3, the action is continuous with respect to the profinite topology on $\mathcal{G}$ and the valuation topology on L.

Let $L[u^{\pm 1}]$ be a graded field where the degree of u is -2. We define an action of $\mathcal{G}$ on $L[u^{\pm 1}]$ as automorphisms of graded field, which is an extension of the action of $\mathcal{G}$ on the degree 0 part L. An element g of $\widetilde{G}_n(\mathbf{F})$ is a pair $(\alpha(g), t(g))$ where $\alpha(g)$ is a continuous automorphism of $\mathbf{F}[\![u_{n-1}]\!]$ and $t(g)$ is an isomorphism $t(g) : F_n \to \alpha(g)^* F_n$ over $\mathbf{F}[\![u_{n-1}]\!]$. The automorphism $t(g)$

has a form

$$t(g)(X) = \sum_{i\geq 0}{}^{\alpha(g)^*F_n} t_i(g)X^{p^i}.$$

For $(g, \widetilde{g}) \in \mathcal{G} = \widetilde{G}_n(\mathbf{F}) \times_{\mathrm{Aut}(k)} A(L/k)$, we set

$$u^{(g,\widetilde{g})} = t_0(g)^{-1}u.$$

This defines a continuous action of $\mathcal{G}$ on $L[u^{\pm 1}]$ as automorphism of a graded field. We note that under the isomorphism $\mathcal{G} \cong \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes (S_n \times S_{n-1})$, the subgroup $G_{n-1} = \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_{n-1}$ acts on u trivially and on L as a Galois group $\mathrm{Gal}(L/\mathbf{F}_p((u_{n-1})))$.

We recall that a quotient group $S_{n-1}(i)$ of S_{n-1}. An element h of the Morava stabilizer group S_{n-1} has a form

$$h = h_0 + h_1T + h_2T^2 + \cdots, \quad h_i \in W\mathbf{F}_{p^{n-1}}, h_i^{p^{n-1}-1} = h_i, h_0 \neq 0$$

where $T^{n-1} = p$ and $Th_i = h_i^pT$. For $i \geq 0$, there is an open normal subgroup $S_{n-1}^{(i)}$ given by

$$S_{n-1}^{(i)} = \{h \in S_{n-1} |\ h_0 = 1, h_1 = \cdots h_{i-1} = 0\}.$$

We denote by $S_{n-1}(i)$ the quotient group $S_{n-1}/S_{n-1}^{(i+1)}$. Since $S_{n-1}^{(i+1)}$ is open, $S_{n-1}(i)$ is a finite group and its order is equal to $(p^{n-1}-1)p^{(n-1)i}$.

For $i \geq -1$, we define a quotient group $\mathcal{G}(i)$ of $\mathcal{G}$. Under the isomorphism $\mathcal{G} \cong \mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes (S_n \times S_{n-1})$, we see that the subgroup $S_{n-1}^{(i+1)}$ is normal. We denote by $\mathcal{G}(i)$ the quotient group $\mathcal{G}/S_{n-1}^{(i+1)}$. In particular, $\mathcal{G}(-1) = G_n$. Hence there is an exact sequence of profinite groups

$$1 \to S_{n-1}^{(i+1)} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}(i) \to 1.$$

By Lemma 4.3, the action of $\mathcal{G}$ on $L[u^{\pm 1}]$ induces an action of $\mathcal{G}$ on the subfield $L_i[u^{\pm 1}]$ for all $i \geq -1$. Then it is easy to verify that the action of $\mathcal{G}$ on $L_i[u^{\pm 1}]$ factors through the quotient group $\mathcal{G}(i)$.

There is an exact sequence

$$1 \to S_{n-1}^{(i)}/S_{n-1}^{(i+1)} \longrightarrow \mathcal{G}(i) \longrightarrow \mathcal{G}i - 1 \to 1.$$

By Theorem 4.6 and its proof, the kernel $S_{n-1}^{(i)}/S_{n-1}^{(i+1)}$ is identified with the Galois group of the extension L_i/L_{i-1}. Hence the invariant sub ring of the action of $S_{n-1}^{(i)}/S_{n-1}^{(i+1)}$ on $L_i[u^{\pm 1}]$ is equal to $L_{i-1}[u^{\pm 1}]$. We consider the inflation map

$$H^*(\mathcal{G}(i-1); L_{i-1}[u^{\pm 1}]) \longrightarrow H^*(\mathcal{G}(i); L_i[u^{\pm 1}])$$

for $i \geq 0$. For a finite Galois extension L_i/L_{i-1}, the existence of a normal basis implies that the $\mathrm{Gal}(L_i/L_{i-1})$ module L_i is a regular representation over the discrete valuation field L_{i-1}. By Theorem 8.4, we obtain the following theorem.

Proposition 11.1. *The inflation map*

$$H^*(\mathcal{G}(i); L_i[u^{\pm 1}]) \longrightarrow H^*(\mathcal{G}(i+1); L_{i+1}[u^{\pm 1}])$$

is an isomorphism for all $i \geq -1$.

12 Ring homomorphism

In this section we construct a ring homomorphism from the cohomology of G_{n-1} in the coefficients $\mathbf{F}[w^{\pm 1}]$ to the cohomology of G_n in the coefficients $\mathbf{F}((u_{n-1}))[u^{\pm 1}]$.

Let G_{n-1} be the profinite group $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_{n-1}$. We denote by $G_{n-1}(i)$ the quotient group of G_{n-1} given by $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes S_{n-1}(i)$ for $i \geq 0$. The action of G_{n-1} on $\mathbf{F}[w^{\pm 1}]$ factors through $G_{n-1}(i)$. The following lemma is well-known on the cohomology of the profinite group.

Lemma 12.1 (cf. [15]). $H^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) \cong \varinjlim_i H^*(G_{n-1}(i); \mathbf{F}[w^{\pm 1}])$.

Let $k = \mathbf{F}((u_{n-1}))$, $L_i = k(\Phi_0, \Phi_1, \ldots, \Phi_i)$ for $i \geq 0$ and $L = \cup_{i \geq 0} L_i$. The action of $\mathcal{G}$ on the graded field $L[u^{\pm 1}]$ induces the action of the quotient group $\mathcal{G}(i)$, which is isomorphic to $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p) \ltimes (S_n \times S_{n-1}(i))$, on the subfield $L_i[u^{\pm 1}]$. We identify the graded field $\mathbf{F}[w^{\pm 1}]$ as the subfield of $L[u^{\pm 1}]$ by using the relation

$$w = \Phi_0^{-1} u.$$

Lemma 12.2. $\mathbf{F}[w^{\pm 1}]$ *is stable under the action of* $\mathcal{G}$. *The subgroup* S_n *of* $\mathcal{G}$ *acts trivially on* $\mathbf{F}[w^{\pm 1}]$. *The action of the subgroup* G_{n-1} *of* $\mathcal{G}$ *coincides with the action defined in § 9.*

Proof. For $g \in S_n$, we have

$$\Phi_0^g = t_0(g)^{-1} \Phi_0, \quad u^g = t_0(g)^{-1}.$$

Hence S_n acts on w trivially. For $h \in S_{n-1}$, we have

$$\Phi_0^h = \pi(h_0) \Phi_0, \quad u^h = u.$$

Hence we obtain

$$w^h = \pi(h_0)^{-1} w.$$

This shows that $\mathbf{F}[w^{\pm 1}]$ is stable under $\mathcal{G}$ and the action of G_{n-1} is the same as defined in § 9. $\square$

By Lemma 12.2, we see that the inclusion $\mathbf{F}[w^{\pm 1}] \hookrightarrow L_i[u^{\pm 1}]$ is compatible with the projection map $\mathcal{G}(i) \to G_{n-1}(i)$ for all $i \geq 0$. Hence we get an inflation map

$$H^*(G_{n-1}(i); \mathbf{F}[w^{\pm 1}]) \longrightarrow H^*(\mathcal{G}(i); L_i[u^{\pm 1}]).$$

We consider the homomorphism of systems

$$\begin{array}{ccccc}
H^*(G_{n-1}(0); \mathbf{F}[w^{\pm 1}]) & \to & H^*(G_{n-1}(1); \mathbf{F}[w^{\pm 1}]) & \to \cdots \to \\
\Big\downarrow & & \Big\downarrow & \\
H^*(\mathcal{G}(0); \mathbf{F}[w^{\pm 1}]) & \to & H^*(\mathcal{G}(1); \mathbf{F}[w^{\pm 1}]) & \to \cdots \to
\end{array}$$

By Theorem 11.1, the homomorphisms in the bottom sequence are all isomorphisms and we have an isomorphism

$$H^*(G_n; k[u^{\pm 1}]) \xrightarrow{\cong} H^*(\mathcal{G}(i); L_i[u^{\pm 1}])$$

for all $i \geq 0$. By passing to the direct limits of the systems we obtain that

$$\begin{array}{rcl}
H^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) & \cong & \varinjlim_i H^*(G_{n-1}(i); \mathbf{F}[w^{\pm 1}]) \\
\longrightarrow \varinjlim_i H^*(\mathcal{G}(i); L_i[u^{\pm 1}]) & \cong & H^*(G_n; k[u^{\pm 1}]).
\end{array}$$

Hence we obtain the following theorem.

Theorem 12.3. *There is a ring homomorphism*

$$\varphi : H^*(G_{n-1}; \mathbf{F}[w^{\pm 1}]) \longrightarrow H^*(G_n; k[u^{\pm 1}]).$$

Remark 12.4. We recall that $H^*(S_1; \mathbf{F}_p[w^{\pm 1}])$ is an exterior algebra generated by ζ_1 for $p > 2$. Then we can show that $\varphi(\zeta) = t_1$ which is nontrivial by Shimomura.

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