HOLE PROBABILITY AND LARGE DEVIATIONS IN THE DISTRIBUTION OF THE ZEROS OF GAUSSIAN RANDOM HOLOMORPHIC FUNCTIONS.

by

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DEDICATION

I would like to dedicate this work to my late grandfather Francis Kelly who made the world and my world a better place.

1. Introduction

Random polynomials and random holomorphic functions are studied as a way to gain insight into difficult problems in physics, geometry and analytic number theory, and because of these applications, random holomorphic functions have been studied in a flurry of recent research. Gaussian random holomorphic functions are defined as

$$\psi_\alpha(z) = \sum \alpha_j \psi_j(z), \tag{1}$$

where $\{\alpha_j\}$ are independent identically distributed standard centered complex Gaussian random variables and for all j, $\psi_j \in \mathcal{O}(M^m)$, M an m manifold. A particularly interesting case of random holomorphic functions is when the random functions can be defined so that they obey certain invariance laws with respect to the natural isometries of the manifold M^m. The following choices for $\{\psi_j\}$ are particularly well studied for this reason, ([BSZ2], [ST1]):

Name	Related space	Isometries for $(m=1)$	Random function	Expected zero set
entire flat	$\mathbb{C}^m$	$z \mapsto e^{i\theta} z + \zeta$ $A \in U(m)$	$\sum_{j \in \mathbb{N}^m} \alpha_j \frac{z^j}{\sqrt{j!}}$	$\frac{i}{2\pi} \sum dz_i \wedge d\overline{z}_i$
SU$(m+1)$ polynomials	$\mathbb{C}P^m$	$z \mapsto \frac{az+b}{-\overline{b}z+\overline{a}}$, $\lvert a\rvert^2 + \lvert b\rvert^2 = 1$	$\sum_{\lvert j\rvert=0}^{N} \alpha_j \binom{N}{j}^{\frac{1}{2}} z^j$	$\frac{Ni}{2\pi} \frac{\sum dz_i \wedge d\overline{z}_i}{(1+\lvert z\rvert^2)^2}$
SU$(1,1)$, analytic on D	$D \subset \mathbb{C}^1$	$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$, $\lvert a\rvert^2 - \lvert b\rvert^2 = 1$	$\sum \alpha_j z^k$ $z \in \mathbb{C}^1$	$\frac{1}{2\pi i} \frac{dz \wedge d\overline{z}}{(1-\lvert z\rvert^2)^2}$
random holomorphic sections	(M^m, ω) $L > 0$ M cmpt Kähler	None in general	$\sum_j \alpha_j S_j(z)$ $\{S_j\}$ any o.n. basis of $H^0(M, L^N)$	Asymptotic to $N \cdot \omega$

While the topic of random holomorphic functions, and the zeros of said functions, is of recent interest there is also an old history complete with many results from the first half of the twentieth century, which is now experiencing a renaissance. Historically, many mathematicians have studied various properties of random functions including Polya, Erdos-Renyi, Salem-Zygmund, Paley-Wiener. Additionally, the study of the zeros of random holomorphic functions was born with the work of Kac and Rice, who determined a formula for the expected distribution of zeros of real polynomials in a certain cases, ([Kac], [Ric]), work which has been subsequently generalized throughout the years, [EK]. An excellent reference for other results regarding the general properties of random functions is Kahane's text, [Kah]. More recently there was a series of papers of Offord which is particularly relevant to questions involving the hole probability of random holomorphic functions and the distribution of values of random holomorphic functions, [Off1], [Off2].

These classical results have been built upon with recent research which statistically studies the behavior of zeros of random holomorphic functions. In particular first and second moments for the number of zeros of a random holomorphic function have been computed, as well as a central limit law ([ST1], [SZ]). Additionally, recent work suggests that the zero sets of random polynomials and holomorphic functions are much more natural objects than they may initially appear. For example, Bleher, Shiffman and Zelditch have shown that for any positive line bundle over a compact complex manifold, the random holomorphic sections to L^N (defined intrinsically) have universal high N correlation functions, [BSZ2].

In addition to a plethora of results describing the typical behavior, there have also been several results in 1 (real or complex) dimension for Gaussian random holomorphic functions where there is a large deviation in the number of zeros present, including the event where there is a hole. A hole is an event where there are no (complex) zeros present in a domain where many are expected. Let $Hole_r$ be the event: $\{f$, in a class of holomorphic functions, such that $\forall z \in B(0,r),\ f(z) \neq 0\}$, where $B(0,r) = \{z : |z| < r\}$. For the complex zeros in one complex dimension, there is a general upper bound for the hole probability of a Gaussian random holomorphic function: $Prob(Hole_r) \leq e^{-c\rho(B(0,r))}$, $\rho(\Omega) = \int_\Omega E[Z_{\psi_\alpha}]$ as in theorem 2.12, [Sod]. In one case this estimate was shown by Peres and Virág to be sharp: $Prob(Hole_r) = e^{-\frac{\rho(B(0,r))}{24} + o(\rho(B(0,r)))}$, [PV]. These last two results on hole probability might suggest that when the expected zero set of a Gaussian random holomorphic functions is invariant with respect to the local isometries then the rate of decay of the hole probability would be the same as that which would be arrived at if the zeros where distributed according to an appropriate Poisson process. However, as the zeros repel in 1 dimension [BSZ1], one might expect there to be a quicker decay for hole probability of a random holomorphic function. This is the case for Gaussian entire flat functions in 1 complex variable, [ST2] :

$$Prob(Hole_r) \leq e^{-c_1 r^4} = e^{-c\rho(B(0,r))^2}, \; and \; Prob(Hole_r) \geq e^{-c_2 r^4} = e^{-c\rho(B(0,r))^2}$$

Additionally questions related to hole probability have been answered for a specific class of real Gaussian polynomials, [DPSZ], [LS]. However, there had never been any non trivial results concerning the order of the decay of the hole probability in more than one complex variable.

In this thesis work we will examine just such problems by studying large deviations of the zero set of Gaussian entire flat functions, in n variables, and random SU$(m+1)$ polynomials. This will be done in three sections. In the first section we shall state, and where necessary prove results which are typical of those found in the literature, even though in certain cases these typical results will need to be adopted to our m variable cases. We will also discuss various technical details concerning Gaussian random holomorphic functions. The second section will concern a new and interesting result concerning the decay of the hole probability of Gaussian entire flat functions. In the third section we will, using several lemmas from section 2, derive the order of the decay of the hole probability for Gaussian random SU$(m+1)$ polynomials. This last result is also a new and interesting result for the one variable case, or when $m=1$, as well as for the higher dimensional case.

More specifically in section 2, we will prove two main results concerning Gaussian entire flat functions and the volume of the zero sets of these functions. To describe the volume of the zero set we will be using the unitegrated counting function which is defined as:

$$n_{\psi_{\alpha,N}(r)} = \frac{1}{(m-1)!r^{2m-2}\pi^{m-1}} \cdot \text{Volume of the zero set of } \psi_{\alpha,N} \bigcap B(0,r),$$

where $B(0,r) = \{z \in \mathbb{C}^m : |z| < r\}$.

The first two of these main results generalize Sodin and Tsirelison's result for Gaussian entire flat functions in one variable, [ST2]. The first of these main results is:

Theorem 1.1. *If $\psi_\alpha(z)$ is a Gaussian entire flat function, i.e.*

$$\psi_\alpha(z_1, z_2 \ldots, z_m) = \sum_{j \in \mathbb{N}^m} \alpha_j \frac{z_1^{j_1} z_2^{j_2} \ldots z_m^{j_m}}{\sqrt{j_1! \cdot j_m!}},$$

where α_j are independent identically distributed centered complex Gaussian random variables, then for all $\delta > 0$, there exists $c_{\delta,m} > 0$ and $R_{m,\delta}$ such that for all $r > R_{m,\delta}$

$$Prob\left(\left\{\left|n_{\psi_\alpha}(r) - \frac{1}{2}r^2\right| \geq \delta r^2\right\}\right) \leq e^{-c_\delta r^{2m+2}}$$

where $n_{\psi_\alpha}(r)$ is the unintegrated counting function for ψ_α.

We will often use the standard multi index notation to simplify our formulas. Specifically, if $z \in \mathbb{C}^m$ and $j \in \mathbb{N}^m$ then

$$z^j = z_1^{j_1} z_2^{j_2} \cdot \ldots \cdot z_m^{j_m}$$

$$j! = j_1! \cdot j_2! \cdot \ldots \cdot j_m!$$
$$\binom{N}{j} = \frac{N!}{j! \cdot (N - |j|)!}$$

As an example, a Gaussian entire flat function may be written as $\psi_\alpha(z) = \sum \alpha_j \frac{z^j}{\sqrt{j!}}$.

Theorem 1.2. *If $\psi_\alpha(z)$ is a Gaussian entire flat function and if*

$$Hole_r = \{\forall z \in B(0,r),\ \psi_\alpha(z) \neq 0\},$$

then there exists R_m, c_m, and $C_m > 0$ such that for all $r > R_m$

$$e^{-C_m r^{2m+2}} \leq Prob(Hole_r) \leq e^{-c_m r^{2m+2}}$$

The proof of theorems 1.1 and 1.2, will use value distribution theory, which relates the expected number of zeroes with the rate of growth of the maximum of a function, basic probability theory, and standard techniques of complex analysis. Additionally, an invariance rule for Gaussian entire flat functions with respect to translation will be used. These results, using the mainly same techniques, were already proven in the case where m=1 by Sodin and Tsirelson, [ST2]. Additionally, the results of section 2 will be appearing in the *Michigan Mathematical Journal*, [Zre1].

In section 3 of this thesis we will consider the class of random polynomials whose zeros are distributed on $\mathbb{C}P^m$ according to the Fubini-Study measure. The random functions which will be studied here are called Gaussian random SU$(m+1)$ polynomials and can be written as:

$$\psi_{\alpha,N}(z) = \sum_{|j|=0}^{N} \alpha_j \sqrt{\binom{N}{j}} z^j \tag{1}$$

where $\forall j,\ \alpha_j$, are independent identically distributed standard centered complex Gaussian random variables (mean 0 and variance 1). For these Gaussian random SU$(m+1)$ polynomials we will prove two results regarding the distribution of the zeros, by using similar techniques to those used to solve the m variable entire flat case:

Theorem 1.3. *If $\psi_{\alpha,N}$ is a degree N Gaussian random SU$(m+1)$ polynomial,*

$$\psi_{\alpha,N}(z) = \sum_{|j|=0}^{N} \alpha_j \sqrt{\binom{N}{j}} z^j,$$

where α_j are independent identically distributed centered complex Gaussian random variables, and if $n_{\psi_{\alpha,N}}(r)$ is the unintegrated counting function then for all $\Delta > 0$, and $r > 0$ there exists $A_{\Delta,r,m}$ and $N_{\Delta,r,m}$ such that for all $N > N_{\Delta,r,m}$

$$Prob\left(\left\{\left|n_{\psi_{\alpha,N}}(r) - \frac{Nr^2}{1+r^2}\right| \geq \Delta N\right\}\right) < e^{-A_{\Delta,r,m}N^{m+1}}.$$

Typically, $n_{\psi_{\alpha,N}}(r)$ should be about $\frac{Nr^2}{1+r^2}$. Theorem 1.3 gives an upper bound on the rate of decay of the hole probability, and we will be able to prove a lower bound for the decay rate of the same order:

Theorem 1.4. *Let $\psi_{\alpha,N}$ be as in theorem 1.3, and let*

$$Hole_{N,r} = \{\alpha : \forall z \in B(0,r),\ \psi_{\alpha,N}(z) \neq 0\},$$

then there exists $c_{1,r,m}$, $c_{2,r,m} > 0$ and N_r such that for all $N > N_{r,m}$

$$e^{-c_{2,r}N^{m+1}} \leq Prob(Hole_{N,r}) \leq e^{-c_{1,r}N^{m+1}}$$

As an immediate consequence of this result, the order of the probability specified in the Theorem 1.3 is the correct order of decay.

Random SU$(m+1)$ polynomials of the form studied here are the simplest examples of a class of natural random holomorphic sections of large N powers of a positive line bundle on a compact Kähler manifold. Most of the results stated in this paper may be restated in terms of Szegö kernels, whose leading order scaling asymptotics exhibit universal behavior in the large N limit, when appropriately scaled. Hopefully, this paper may provide insight into proving a similar decay rate for this more general setting. This has already been done for other properties of random holomorphic sections, e.g. correlation functions, [BSZ2].

2. Common results

2.1. **Defining Gaussian random holomorphic functions.** There are two general, and equivalent perspectives of how to define a Gaussian random holomorphic function: either one establishes a Gaussian probability distribution on a Hilbert space of functions or alternatively, or one uses line (1) of the introduction. These two perspectives are equivalent, but we shall always explicitly use the second, while occasionally implicitly using the first. By doing this we may avoid the technical, abstract definitions of Gaussian Hilbert spaces and Gaussian fields.

Our definition of a Gaussian random holomorphic function makes use of the standard definition of Gaussian random variables:

Definition 2.1. *A Gaussian random variable is a complex-valued random variable with the following probability distribution function:*

$$d\nu(z) = \frac{1}{\pi\sigma} e^{\frac{-|z|^2}{\sigma^2}} dm(z),$$

This is a stronger definition than most in that it assumes that the mean is zero, and is often called a centered complex Gaussian random variable. In this paper we will only use (centered complex) Gaussian random variables, and thus we will not bother including the two words centered and complex when discussing Gaussian random variables. Additionally, we will call a Gaussian random variable whose variance is 1, a standard Gaussian random variable. As is always the case, a (complex) Gaussian random variable can be written as the sum of two independent jointly normal variables for $\mathbb{R}^2$, where the variance of the real part is the same as the variance of the imaginary part (these random variables are independent).

A straightforward computation shows that Gaussian random variables have the following moments:

$E[z] := \int_{\mathbb{C}} z d\nu(z) = 0$

$E[1] := \int_{\mathbb{C}} d\nu(z) = 1$

$E[|z|^2] := \sigma^2 = Var(z)$

$\forall p > 0,\ E[|z|^p] = \sigma(\Gamma(\frac{p+1}{2}))^{\frac{1}{p}}$

To finish the definition of a Gaussian random holomorphic function we will now choose $\{\psi_j\}$ to be a specific orthonormal basis of one of two Hilbert spaces of holomorphic functions. The first of these two Hilbert spaces relates to Gaussian entire flat functions and is:

$$H^2(\mathbb{C}^n, e^{-|z|^2}dm(z)) = \left\{ f \in \mathcal{O}(\mathbb{C}^m) : \int_{z\in\mathbb{C}^m} |f(z)|^2 e^{-|z|^2} dm(z) < \infty \right\},$$

where dm is Lebesque measure. This Hilbert space has the following two orthonormal bases: $\left\{ \frac{z_1^{j_1} \cdot z_2^{j_2} \cdot \ldots z_n^{j_n}}{\sqrt{j_1! \cdot \ldots \cdot j_m!}} \right\}_{j\in\mathbb{N}^m} = \{\psi_j(z)\}_{\mathbb{N}^m}$ and $\left\{ e^{-\frac{1}{2}|\zeta|^2 - z\bar{\zeta}} \frac{(z+\zeta)^j}{\sqrt{j!}} \right\}_{j\in\mathbb{N}^m}$.

We will now prove that the previously mentioned collections of functions are in fact orthonormal bases for $H^2(\mathbb{C}^m, e^{-|z|^2}dm(z))$:

Proof. It suffices to show that for all $\zeta \in \mathbb{C}^m$ the second collection of functions $\left\{ e^{-\frac{|\zeta|^2}{2} - z\bar{\zeta}} \frac{(z+\zeta)^j}{\sqrt{j!}} \right\}_{j\in\mathbb{N}^m}$ is an orthonormal basis:

$$\begin{aligned}
\left\| e^{-\frac{1}{2}|\zeta|^2 - z\bar{\zeta}} \frac{(z+\zeta)^j}{\sqrt{j!}} \right\| &= \frac{1}{(j!\pi^m)} \int_{\mathbb{C}^m} |z+\zeta|^{2j} e^{-|\zeta|^2 - z\bar{\zeta} - \zeta\bar{z}} e^{-|z|^2} dm(z) \\
&= \frac{1}{(j!\pi^m)} \int_{\mathbb{C}^m} |z|^{2j} e^{-|z|^2} dm(z) \\
&= \frac{1}{(j!)} \int_{\mathbb{R}^{+,m}} u^j e^{-u}, \text{ by setting } u = r^2 \\
&= 1e^{-u}|_{u=0}^{u=\infty} = 1, \text{ by integrating by parts}
\end{aligned}$$

By symmetry this basis, is orthogonal, and as any holomorphic function can be written as a (convergent) power series, this is a basis.

□

Our second set of basis functions comes from a Hilbert space of polynomials and is the set of polynomials in one variable whose degree is less than or equal to N, $Poly_N$ along with the SU$(m+1)$ invariant norm, [BSZ2]:

$$\|f\|_N^2 := \frac{N+m!}{N!\pi^m} \int_{z\in\mathbb{C}^m} |f(z)|^2 \frac{dm(z)}{(1+|z|^2)^{N+m+1}},$$

where dm is just the usual Lebesque measure. For this norm $\left\{\sqrt{\binom{N}{j}} z^j\right\}$ is an orthonormal basis, as is $\left\{\sqrt{\binom{N}{j}} \prod_{k=0}^{m} (\sum_{l=1}^{m} a_{k,l} z_l + a_{k,0})^{j_k}\right\}$, where $A = (a_{k,l})$ and $A \cdot \overline{A^T} = I$, and $j_0 = N - |j|$. Specifically, one alternate orthonormal basis is, for any $\zeta \in \mathbb{C}^1$, $\left\{\sqrt{\binom{N}{j}} \left(\frac{z_1-\zeta}{\sqrt{1+|\zeta|^2}}\right)^{j_1} \left(\frac{1+\bar{\zeta}z_1}{\sqrt{1+|\zeta|^2}}\right)^{N-|j|} \prod_{k=2}^{m} z_k^{j_k}\right\}_{|j|\le N}$.

Clearly, by line (1), a Gaussian random SU$(m+1)$ polynomial is defined as, $\psi_{\alpha,N}(z) = \sum_{|j|=0}^{|j|=N} \alpha_j \psi_j(z)$, where α_j are i.i.d. standard complex Gaussian random variables, and $\{\psi_j\}$ is the first orthonormal basis. Any basis for $Poly_N$ could have been used and the Gaussian random SU$(m+1)$ polynomials would be probabilistically identical, as for $\{\alpha_j\}$ a sequence of i.i.d. Gaussian random variables there exists another sequence of i.i.d. Gaussian random variables, $\{\alpha'_j\}$, such that

$$\sum \alpha_j \sqrt{\binom{N}{j}} z^j = \sum_{|j|=0}^{N} \alpha'_j \sqrt{\binom{N}{j}} \left(\frac{z_1-\zeta}{\sqrt{1+|\zeta|^2}}\right)^{j_1} \left(\frac{1+\bar{\zeta}z_1}{\sqrt{1+|\zeta|^2}}\right)^{N-|j|} \prod_{k=2}^{N} z_k^{j_k}.$$

We will now prove that these collections of functions are in fact orthonormal bases for $Poly_N$:

Proof. By making a specific SU$(m+1)$ transformation we may get from the first basis to the second:

$$\left\langle \sqrt{\binom{N}{j}} z^j, \sqrt{\binom{N}{k}} z^k \right\rangle = \frac{(N+m)!}{N!\pi^m} \int_{\mathbb{C}^m} \binom{N}{j} \frac{z_1^{j_1} \overline{z}_1^{k_1} \tilde{z}^{\tilde{j}} \overline{\tilde{z}}^{\tilde{k}}}{(1+|z_1|^2+|\tilde{z}|^2)^{N+2}} dm(z),$$

where $\tilde{z} = (z_2, z_3, \ldots, z_m)$ and $\tilde{j} = (j_2, j_3, \ldots, j_m)$. We now make the following SU$(m+1)$ transformation:

Let $a = \dfrac{1}{\sqrt{1+|\zeta|^2}}$ and $b = \dfrac{-\zeta}{\sqrt{1+|\zeta|^2}}$, which guarantees that $|a|^2 + |b|^2 = 1$.

$$\begin{aligned}
\text{Let } z_1 &= \tfrac{aw_1+b}{-\overline{b}w_1+\overline{a}}. \\
\text{Let } z_k &= \tfrac{w_k}{-\overline{b}w_1+\overline{a}}, \text{ for all } k \neq 1. \\
\text{Hence, } dz_1 &= \tfrac{dw_1}{(-\overline{b}w_1+\overline{a})^2}, \\
\text{and } dz_k &= \tfrac{dw_k}{(-\overline{b}w_1+\overline{a})} + \tfrac{-\overline{b}w_k}{(-\overline{b}w_1+\overline{a})^2} dw_1.
\end{aligned}$$

Making this transformation:

$$\frac{N!\pi^m}{(N+m)!\sqrt{\binom{N}{k}}\sqrt{\binom{N}{j}}} \left\langle z^j,\ z^k \right\rangle$$

$$\begin{aligned}
&= \int_{\mathbb{C}^m} \frac{\left(\frac{aw+b}{-\overline{b}w+\overline{a}}\right)^{j_1} \overline{\left(\frac{aw+b}{-\overline{b}w+\overline{a}}\right)}^{k_1} \tilde{z}^{\tilde{j}} \overline{\tilde{z}}^{\tilde{k}}}{\left(1+\left|\frac{aw+b}{-\overline{b}w+\overline{a}}\right|^2+|\tilde{z}|^2\right)^{N+m+1}} \frac{dm(w)}{|-\overline{b}w+\overline{a}|^{2m+2}}, \\
&= \int_{\mathbb{C}^m} \frac{(aw+b)^{j_1}(\overline{aw}+\overline{b})^{k_1}(-\overline{b}w+\overline{a})^{(N-|j|)}(-b\overline{w}+a)^{(N-|k|)}\tilde{w}^{2\tilde{j}} dm(w)}{\left(|-\overline{b}w+\overline{a}|^2+|aw+b|^2+|\tilde{w}|^2\right)^{N+m+1}}, \\
&= \int_{\mathbb{C}^m} \frac{(aw+b)^{j_1}(\overline{aw}+\overline{b})^{k_1}(-\overline{b}w+\overline{a})^{(N-|j|)}(-b\overline{w}+a)^{(N-|k|)}\tilde{w}^{2\tilde{j}} dm(w)}{(1+|w|^2)^{N+m+1}},
\end{aligned}$$

as $|a|^2 + |b|^2 = 1$.

$$= \left\langle \frac{\sqrt{\binom{N}{j}}(z_1-\zeta)^{j_1}(1+\overline{\zeta}z)^{N-|j|}\tilde{z}^{\tilde{j}}}{(1+|\zeta|^2)^{\frac{N}{2}}}, \frac{\sqrt{\binom{N}{k}}(z_1-\zeta)^{k_1}(1+\overline{\zeta}z)^{N-|k|}\tilde{z}^{\tilde{k}}}{(1+|\zeta|^2)^{\frac{N}{2}}} \right\rangle,$$

We now show that both these bases are normalized by demonstrating that the first one is. Explicitly, this is accomplished by switching to polar coordinates and integrating by parts:

$$
\begin{aligned}
\left\| \sqrt{\binom{N}{j}} z^j \right\|^2 &= \frac{1}{\pi^m} \int_{\mathbb{C}^m} \frac{(N+m)!}{j!(N-|j|)!} \frac{|z_1|^{2j_1} |\tilde{z}|^{2\tilde{j}} dm(z)}{(1+|z_1|^2+|\tilde{z}|^2)^{N+m+1}}, \\
&= \frac{1}{\pi^{m-1}} \int_{\mathbb{C}^{m-1}} \int_{r=0}^{\infty} \frac{2(N+m)!}{j!(N-|j|)!} \frac{r^{2j_1+1} |\tilde{z}|^{2\tilde{j}} dr dm(\tilde{z}}{(1+|r|^2+|\tilde{z}|^2)^{N+m+1}}), \\
&= \frac{1}{\pi^{m-1}} \int_{\mathbb{C}^{n-1}} \int_{u=0}^{\infty} \frac{(N+m)!}{j!(N-|j|)!} \frac{u^{j_1} |\tilde{z}|^{2\tilde{j}} du dm(\tilde{z})}{(1+u+|\tilde{z}|^2)^{N+m+1}}, \\
&= \frac{1}{\pi^{m-1}} \int_{\mathbb{C}^{m-1}} \int_{u=0}^{\infty} \frac{(N+m-1)!}{(j_1-1)!\tilde{j}!(N-|j|)!} \frac{u^{j_1-1} |\tilde{z}|^{2\tilde{j}} du dm(\tilde{z})}{(1+u+|\tilde{z}|^2)^{N+m}}, \\
&\vdots \\
&= \frac{1}{\pi^{m-1}} \int_{\mathbb{C}^{m-1}} \int_{u=0}^{\infty} \frac{(N+m-j_1)!}{\tilde{j}!(N-|j|)!} \frac{|\tilde{z}|^{2\tilde{j}} dm(\tilde{z})}{(1+u+|\tilde{z}|^2)^{N+m-j_1}}, \\
&= \frac{1}{\pi^{m-1}} \int_{\mathbb{C}^{m-1}} \frac{(N+m-j_1-1)!}{\tilde{j}!(N-|j|)!} \frac{|\tilde{z}|^{2\tilde{j}} dm(\tilde{z})}{(1+|\tilde{z}|^2)^{N+m-j_1-1}}, \\
&\vdots \\
&= (N-|j|-1)! \int_{u=0}^{\infty} \frac{du}{(N-|j|)!(1+u)^{N-|j|}}, \\
&= 1,
\end{aligned}
$$

□

Now let us recall our definition of a Gaussian random function, line (1):

$$\psi_\alpha(z) = \sum \alpha_j \psi_j(z).$$

We define Gaussian entire flat functions to be Gaussian random functions with ψ_j being the first basis we chose for $H^2(\mathbb{C}^m, e^{-|z|^2}dm)$:

$$\psi_\alpha(z) = \sum \alpha_j \frac{z^j}{\sqrt{j!}}, \tag{2}$$

where $\{\alpha_j\}$ are i.i.d. Gaussian random variables and we are using the standard multi-index notation. Prior to proving Theorem 2.3 this is an abstract definition, but after we prove said theorem we will know that a Gaussian entire flat function a.s. defines an entire function. Additionally, while this definition appears to depend on the basis chosen, this is not the case as may be seen by skipping ahead to Theorem 2.5, and the following corollary.

Likewise to define Gaussian random SU$(m+1)$ polynomials we choose ψ_j to be a basis for $Poly_N$, which gives

$$\psi_{\alpha,N}(z) = \sum_{|j|=0}^{N} \alpha_j \sqrt{\binom{N}{j}} z^j, \tag{3}$$

where $\{\alpha_j\}$ are i.i.d. Gaussian random variables. As this sum is finite, we see immediately that Gaussian random SU$(m+1)$ polynomials are almost surely degree N polynomials.

2.2. **The radius of convergence of a random power series.** In this section we will show that a Gaussian entire flat function, is in fact, a.s., an entire function.

Proposition 2.2. *Let α be a standard complex Gaussian Random Variable, then: a-i) $Prob(\{|\alpha| \geq \lambda\}) = e^{-\lambda^2}$*

a-ii) $Prob(\{|\alpha| \leq \lambda\}) = 1 - e^{-\lambda^2} \in [\frac{\lambda^2}{2}, \lambda^2], if \lambda \leq 1$

a-iii) if $\lambda \geq 1$ then $Prob(\{|\alpha| \leq \lambda\}) \geq \frac{1}{2}$.

b) If $\{\alpha_j\}_{\mathbb{N}^m}$ is a sequence of i.i.d. standard random variables then $\limsup |\alpha_j|^{\frac{1}{|j|}} = 1$, almost surely.

Proof. of a)

$$\begin{aligned} Prob(\{|\alpha| \geq \lambda\}) &= \frac{1}{\pi}\int_{|\alpha|>\lambda} e^{-|\alpha|^2} dm(\alpha) \\ &= \int_{r>\lambda} 2re^{-r^2} dr \\ &= e^{-\lambda^2} \end{aligned}$$

□

Proof. of b) We will first prove that for all $\varepsilon \in (0, 1]$, $\limsup |\alpha_j|^{\frac{1}{|j|}} > 1 - \varepsilon$, almost surely.

$Prob(|\alpha_j| > (1-\varepsilon)^{|j|}) = e^{(1-\varepsilon)^{2|j|}} \geq e^{-(1-\varepsilon)^2}$. Thus the probability of this event is greater than or equal to the event where in an infinite sequence of coin flips involving a coin, which takes the value heads with probability $\frac{1}{e^{2-2\varepsilon}}$, has at least one "heads" appear after the R^{th} flip, for any fixed R. This event has probability one by the zero-one law.

We now prove that for all $\varepsilon > 0$, $\limsup |\alpha_j|^{\frac{1}{|j|}} < 1 + 2\varepsilon$, almost surely.

$Prob(|\alpha_j| > (1+\varepsilon)^{|j|}) = e^{-(1+\varepsilon)^{2|j|}}$. Therefore,

$$\sum_{j\in\mathbb{N}^n} Prob(|\alpha_j| > (1+\varepsilon)^{|j|}) = \sum_{j\in\mathbb{N}^m} e^{-(1+\varepsilon)^{2|j|}} < \infty.$$

Hence,

$$Prob(\limsup |\alpha_j|^{\frac{1}{|j|}} > 1+\varepsilon) = 0.$$

□

Theorem 2.3. *If for all* $j \in \mathbb{N}$, $\psi_j(z) \in \mathcal{O}(\Omega)$, *such that for all compact* $K \subset \Omega$, *there exists an* $\varepsilon_K > 0$ *such that* $\sum (1+\varepsilon_K)^{|j|} \max_{z\in K} |\psi_j(z)| < \infty$, *and if* $\{\alpha_j\}$ *is a sequence of i.i.d. Gaussian random variables then for a.a.-*α, $\sum \alpha_j \psi_j(z)$, *converges locally uniformly on* $B(0,r)$, *and hence defines a holomorphic function on* Ω.

Proof. Let K be compact, and $w \in K$.

$\left|\sum_{j\in\mathbb{N}} \alpha_j \psi_j(w)\right| \leq \sum_{j\in\mathbb{N}} |\alpha_j||\psi_j(w)| \leq \sum_{j\in\mathbb{N}} |\alpha_j| \max_{z\in K} |\psi_j(z)|$. The result now follows by Proposition 2.2 and the ratio test.

□

Corollary 2.4. *If* $\psi_\alpha(z)$ *is a Gaussian entire flat function,* $\psi_\alpha(z) = \sum \alpha_j \frac{z^j}{\sqrt{j!}}$ *then* $\psi_\alpha(z)$ *is a.s. an entire holomorphic function.*

Proof. For all r, $\sum_{j\in\mathbb{N}^m} \frac{r^{2|j|}}{j!} = e^{mr^2} < \infty$, and hence the ψ_α is a.s. a holomorphic function on any polydisk by the previous theorem. □

2.3. **Invariance laws.** We will now show that defining random holomorphic functions as $\sum \psi_j \alpha_j$, where $\{\psi_j\}$ is any basis is a good definition, well defined independent of basis.

Theorem 2.5. *If* $\{\psi_j\}_{j\in\Lambda}$ *and* $\{\phi_j\}_{j\in\Lambda}$ *are orthonormal bases for* H *a Hilbert space of holomorphic functions, and if there exists* $\varepsilon > 0$ *such that*

$\sum(1+\varepsilon)^j|\phi_j(z)|^2$ and $\sum(1+\varepsilon)^j|\psi_j(z)|^2$, converge locally uniformly in a domain R, and if $\{\alpha_j\}_{j\in\Lambda}$ are i.i.d. standard Gaussian random variables then $\exists\{\beta_j\}_{j\in\Lambda}$ i.i.d. standard Gaussian random variables s.t.

$$\forall z \in R,\ \psi_\alpha(z) = \phi_\beta(z),\ a.s.$$

Proof. An evaluation map is a linear map from H to $\mathbb{C}$ (since $|\psi_j|^2$ converges locally uniformly), and hence there exists $\{a_j\}$ such that $\sum a_j\psi_j$ is a representative for this fixed evaluation map:

$$\forall f \in H,\ eva_{z_0}(f) = \langle f, \sum a_j\psi_j\rangle_H = f(z_0)$$

In particular,

$$eva_{z_0}(\psi_j) = \overline{a}_j = \psi_j(z_0)$$

As both $\{\psi_j\}$ and $\{\phi_j\}$ are orthonormal bases we have the following relations:

$$\begin{aligned}
\langle \phi_j, \psi_k\rangle &= u_{j,k},\ where\ \forall j,\ \sum_k |u_{j,k}|^2 = 1,\ and\ \forall k,\ \sum_j |u_{j,k}|^2 = 1\\
\langle \psi_k, \phi_j\rangle &= \overline{u}_{j,k}\\
(\overline{u}_{j,k}) &= u^{k,j}\\
\sum_k u^{l,k}u_{k,j} &= \delta_{j,l}
\end{aligned}$$

For all j we define a new Gaussian random variable by $\beta_j = \langle \psi_\alpha, \phi_j\rangle_H$, and using these new Gaussian random variables we define a new Gaussian random holomorphic function:

$$\begin{aligned}
\phi_\beta &= \sum \beta_j \phi_j = \sum \langle \psi_\alpha, \phi_j \rangle \phi_j. \\
\psi_\alpha(z_0) &= \langle \psi_\alpha(z), \sum a_j \psi_j \rangle \\
&= \psi_\alpha(z_0)
\end{aligned}$$

$$\begin{aligned}
\phi_\beta(z_0) &= \langle \sum_k \langle \psi_\alpha, \phi_k \rangle \phi_k, \sum a_j \psi_j \rangle \\
&= \sum_j \overline{a}_j \langle \sum_k \langle \psi_\alpha, \phi_k \rangle \phi_k, \psi_j \rangle \\
&= \sum_{j,k} \overline{a}_j \langle \psi_\alpha, \phi_k \rangle \cdot \langle \phi_k, \psi_j \rangle \\
&= \sum_{j,k,l} \overline{a}_j \overline{u}_{k,l} \alpha_l \cdot u_{k,j} \\
&= \sum_{j,k,l} \overline{a}_j u^{l,k} \cdot u_{k,j} \alpha_l \\
&= \sum_j \overline{a}_j \alpha_j \\
&= \psi_\alpha(z_0)
\end{aligned}$$

□

Corollary 2.6. *(Invariance laws)*

1) If $\psi_{\alpha,N}$ is a Gaussian entire flat function then there exists $\{\beta_j\}$ a sequence of i.i.d. standard Gaussian random variables such that

$$\sum_{j \in \mathbb{N}^m} \alpha_j \frac{z^j}{\sqrt{j!}} = e^{-\frac{1}{2}|\zeta|^2 - z\overline{\zeta}} \left(\sum_{j \in \mathbb{N}^m} \beta_j \frac{(z+\zeta)^j}{\sqrt{j!}} \right)$$

2) If $\psi_{\alpha,N}$ is a degree N Gaussian random SU($m+1$) polynomial then for all $\zeta \in \mathbb{C}^1$ there exists $\{\beta_j\}$ a sequence of i.i.d. standard Gaussian random *variables such that*

$$\sum_{|j|=0}^{N} \alpha_j \sqrt{\binom{N}{j}} z^j = \sum_{|j|=0}^{N} \beta_j \frac{\sqrt{\binom{N}{j}}(z_1-\zeta)^{j_1}(1+\overline{\zeta}z_1)^{N-|j|}\prod z_k^{j_k}}{(1+|\zeta|^2)^{\frac{N}{2}}}$$

2.4. **Zero sets of m variable holomorphic functions and currents.** In this section, we will examine the basic properties of zero sets. As a simple example, consider the function $f(z) = \sin(z)$. As a set

$$Z_f = f^{-1}(\{0\}) = \{(2n+1)\pi,\ n \in \mathbb{N}\}.$$

To this set we may associate a distribution:

$$Z_f = \sum_{n\in\mathbb{N}} \delta_{n\pi}(z),$$

where δ_a is a Dirac delta function centered at a. This measure defines a linear function on $C_c^\infty(\mathbb{C}^1)$: if $\phi \in C_c^\infty(\mathbb{C}^1)$ then

$$\langle Z_f, \phi\rangle := \sum_{n\in\mathbb{Z}} \phi(n\pi),$$

and as $supp(\phi)$ is compact this sum is finite.

We now generalize such a construction to m variables by defining currents and forms. These results and definitions can be found in ([Sch],[GH]). Let $K \subset M$, be compact. Let $\mathcal{D}_K^{(j,p)}$ be the (j,m) forms whose support is in K. Let $\mathcal{D}_U^{(j,p)} = dir\ lim_{K\subset U}\mathcal{D}_K^{(j,p)}$.

example 2.7. *For* $M = \mathbb{C}^m$,

$$\mathcal{D}^{(1,1)}_{\mathbb{C}^m} = \{\sum_{i,j} \phi_{i,j}(z) dz_i \wedge d\overline{z}_j, \phi_{i,j} \in C^\infty_c\}$$

As long as $supp(\phi) = K \subset U$, U a single coordinate chart, any (j,k) form can be written in the above form.

Forms have an associated exterior derivative $d : \mathcal{D}^{(j,p)}_M \to \mathcal{D}^{(j+1,p)}_M \oplus \mathcal{D}^{(j,p+1)}_M$. This derivative can be "split" into holomorphic and anti-holomorphic parts: $d = \partial + \overline{\partial}$. Explicitly, locally on generators of $\mathcal{D}^{(j,p)}_M$, $\partial f dz^I \wedge dz^J = \sum \frac{\partial f}{\partial z_i} dz_i \wedge dz^I \wedge d\overline{z}^J$. This gives the following identities:

1) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge (d\beta)$

2) $dd = 0 = \partial\partial = \overline{\partial\partial}$, $d\overline{\partial} = \partial\overline{\partial}$, $\partial\overline{\partial} = (-1)\overline{\partial}\partial$

3) $\alpha \in \mathcal{D}^{(m-1,m)}_{M^m} \oplus \mathcal{D}^{(m,m-1)}_{M^m} \Rightarrow \displaystyle\int_M d\alpha = \int_{\partial M} \alpha = 0$, $(Stoke's\ theorem)$

Just as in 1 variable Z_f was dual to C^∞_c, in m dimensions the zero set is an analytic variety and this variety will be dual to $(m-1, m-1)$ forms (by taking restrictions). This brings up the definition of currents:

Definition 2.8. *A current of bi-degree* (j,p) *is a linear map,* $T : \mathcal{D}^{(m-j,m-p)}(M) \to \mathbb{C}$ *that is continuous in the direct limit topology.*

The set of all currents of degree (j,p) *is thus Dual to* $\mathcal{D}^{(m-j,m-p)}_M$*, and will be denoted as:* $\mathcal{D}'^{(j,p)} := \left(\mathcal{D}^{(m-j,m-p)}_M\right)'$

A (j,p) current is closely related to a (j,p) form as for a particular $\rho \in \mathcal{D}_M^{(j,p)}$, $\int_M \rho \wedge \cdot \in \mathcal{D}'^{(j,p)}$ is a linear function on $\mathcal{D}_M^{(m-j,m-p)}$.

The zero sets of holomorphic functions are often given the added structure of divisors, which can be written as (1,1) currents. For zero sets, this added structure is unique up to multiplication by nonzero holomorphic functions. In one dimension this becomes: for $f \in \mathcal{O}(M^1)$,

$$div_0(f) = \sum_{z \in Z_f} n_j [z_j],$$

where n_j is the order of f at z_j, or the highest integer n_j st $m \leq n_j \Rightarrow f^{(m)}(z_j) = 0$. For $M^1, \mathcal{D}^{(0,0)}(M) = \mathbb{C}_c^\infty(M)$, and as a current,

$$\langle div_0(f), \phi \rangle = \sum_{z \in Z_f} n_j \phi(z_j)$$

where the compactness of the support of ϕ guarantees that the former sum is finite.

example 2.9. *As a simple example in m variables consider:* $f(z) = z_1,\ z \in \mathbb{C}^m,\ Z_f = 1 \cdot [z_1 = 0] = \delta_{z_1=0} dz_1 \wedge d\overline{z}_1$, *the Dirac delta function,* $\phi \in \mathcal{D}^{(m-1,m-1)}(\mathbb{C})$ *where* $\phi(z) = \sum f_{i,j}(z) dz_1 \wedge \ldots \wedge \hat{dz_i} \wedge \ldots \wedge \hat{d\overline{z}_j} \wedge \ldots \wedge d\overline{z}_n$, *and*

$$\begin{aligned}\langle Z_f, \phi \rangle &= \int_{supp(\phi)} (\delta_{z_1=0}(z) dz_1 \wedge d\overline{z}_1) \wedge \phi(z) \\ &= \int_{supp(\phi) \cap \{z_1=0\}} f_{1,1}(0, z_2, \ldots, z_n) dz_2 \wedge d\overline{z}_2 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n\end{aligned}$$

The above example is highly related to Dirac delta functions. Further, $\log(|z|^2)$, is a fundamental solution to the Laplace equation:

$$\delta_0 = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2$$

or in other words $Z_{f(z)=z} = \delta_0 = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2 \in \mathcal{D}'^{(1,1)}$. In one dimension it is very easy to prove a generalization of this:

$$\frac{i}{2\pi}\partial\overline{\partial}\log|f(z)|^2 = div_0(f)$$

For M^m an m dimensional manifold, the situation is very similar, but there are more details to verify. For $f \in \mathcal{O}(M)$, $f : M^m \to \mathbb{C}$, $f^{-1}(\{0\})$ is still a divisor, and is an $m-1$ analytic variety. Hence the regular points of Z_f are a manifold, and it makes sense (taking restriction), to identify forms in $D_M^{(m-1,m-1)}$ with ones in $D_{Z_{f,reg}}^{(m-1,m-1)}$. As $Z_{f,reg}$ is an n-1 complex manifold, $\int_{Z_{f,reg}}$ is a (1,1) current on M, which we will denote $Z_{f,reg}$ (abusing notation).

Theorem 2.10. *(Poincare-Lelong formula)*
If $f \in \mathcal{O}(M^m)$, M an m complex manifold,

then $Z_f = \frac{i}{2\pi}\partial\overline{\partial}\log|f|^2$, as $(1,1)$ currents on M.

This theorem can be proved by using a theorem of Federer dealing with integration of forms near singularities [Shi]. Computation rules for using currents resemble those for differential forms, and the weak forms of the standard properties of differential forms invariably hold:

Proposition 2.11. *(Some basic properties of currents)*

1) $T \in \mathcal{D}'^{(p,q)},\ \phi \in \mathcal{D}'^{(m-p-1,q)},\ \langle \partial T, \phi\rangle = (-1)^{p+q+1}\langle T, \partial\phi\rangle$

2) $d^2T = 0$

3) $d(T \wedge \beta) = d(T) \wedge \beta + (-1)^{deg(T)} T \wedge d\beta$

2.5. **The expected zero set of a Gaussian random holomorphic function.** We will now prove a standard result wherein the expected zero set of Gaussian random holomorphic functions is determined. Let us first consider the one variable case. If one chooses a random polynomial of degree 1, the zero set of this polynomial will consist of point charges in the complex plane, counted with multiplicity. This polynomial occurs with zero probability, and when we compute the average zero set these singular delta measures will be smeared out into one which is absolutely continuous with respect to Lebesque measure. The theorem below is a more precise and further reaching statement of this heuristic reasoning.

Theorem 2.12. *If* $E[|\psi_\alpha(z)|^2] = \Pi(z,z)$ *converges locally uniformly in* Ω *then*

$$E[Z_\alpha \bigcap \Omega] = (\frac{i}{2\pi}\partial\overline{\partial}\log||\psi(z)||^2_{\ell^2})\big|_\Omega = (\frac{i}{2\pi}\partial\overline{\partial}\log(\Pi(z,z)))\big|_\Omega.$$

Many various forms of the following theorem have been proven, [EK], [Kac] and [Sod]. For my purposes it is important that the proof is valid in n dimensions, and for infinite sums. Many of the proofs resemble this one. After a conversation with Steve Zelditch, I was able to simplify a previously complicated argument into the current form. This simplification is already known to other researchers including Mikhail Sodin.

As a matter of notation we define $\Pi(z,z)$ to be the Szegö Kernel, which for any orthonormal basis has the form

$$\Pi(z,w) = \sum_{j\in\mathbb{N}} \psi_j(z)\overline{\psi_j(w)}.$$

Proof. Let $\alpha \in D^{m-1,m-1}(\Omega)$

To simplify the notation, let $\alpha = \phi\ dz_2 \wedge d\overline{z}_2 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n$.

$$\begin{aligned}
\langle Z_{\psi_\alpha}, \alpha\rangle &= \langle \tfrac{i}{2\pi}\partial\overline{\partial}\log(|\psi_\alpha(z)|^2), \alpha\rangle \\
&= \langle \tfrac{i}{2\pi}\log\left(|\psi_\alpha(z)|^2\right), \partial\overline{\partial}\alpha\rangle \\
&= \langle \tfrac{i}{2\pi}\left(\log(\Pi(z,z)) + \log\left(\tfrac{|\psi_\alpha(z)|^2}{\Pi(z,z)}\right)\right), \partial\overline{\partial}\alpha\rangle
\end{aligned}$$

Taking the expectation of both sides we compute:

$$\begin{aligned}
E[\langle Z_{\psi_\alpha}, \alpha\rangle] = \tfrac{1}{2\pi} & \int_\alpha \int_{z\in\Omega} \log(\Pi(z,z))\tfrac{\partial^2\phi}{\partial z_1\partial\overline{z}_1}\ dm(z)\ d\nu(\alpha) \\
& + \tfrac{1}{2\pi}\int_\alpha \int_{z\in\Omega} \log\left(\tfrac{|\psi_\alpha(z)|^2}{\Pi(z,z)}\right)\tfrac{\partial^2\phi}{\partial z_1\partial\overline{z}_1}\ dm(z)\ d\nu(\alpha)
\end{aligned}$$

The first term is the desired result (which by assumption is integrable and finite), while the second term will turn out to be zero. We first must establish that it is in fact integrable:

$$\begin{aligned}
\int_{z\in\Omega}\int_\alpha |\log\left(\tfrac{|\psi_\alpha(z)|^2}{\Pi(z,z)}\right) & \tfrac{\partial^2\phi}{\partial z_1\partial\overline{z}_1} d\nu(\alpha)|\ dm(z) \\
& \leq c\int_{z\in K}\int_\alpha \left|\log\left(\tfrac{|\psi_\alpha(z)|^2}{\Pi(z,z)}\right)\ d\nu(\alpha)\right|\ dm(z) \\
& = c\int_{z\in K}\int_\alpha |\log(|\alpha'|^2)|\ d\nu(\alpha')\ dm(z)
\end{aligned}$$

where α' is a standard centered Gaussian ($\forall z$). Integrability follows as:

$$\int_{z\in\Omega}\int_\alpha \left|\log\left(\frac{|\psi_\alpha(z)|^2}{\Pi(z,z)}\right)\right| \leq C\int_{|x|<1} |\log(x)|dm(x) + c\int_{|x|>1} |xe^{-x^2}|dm(x) \leq c$$

Finally,

$$\int_\Omega \alpha \wedge E[Z_{\psi_\alpha}] = \frac{i}{2\pi}\int_\Omega \partial\overline{\partial}\alpha \log(\Pi(z,z)) \ + \ \int_\Omega C\frac{\partial^2\phi}{\partial z_i \partial \overline{z}_j} dm(z)$$

$$= \frac{i}{2\pi}\int_\Omega \partial\overline{\partial}\alpha \wedge \log(\Pi(z,z))$$

□

Corollary 2.13. *Expected Zero set of specific gaussian random holomorphic functions*

1) For a Gaussian random entire flat function ψ_α*,*

$$E[Z_{\psi_\alpha}] = \frac{i}{2\pi}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + \ldots + dz_m \wedge d\overline{z}_m)$$

2) For a Gaussian random SU($m+1$*) polynomial* $\psi_{\alpha,N}$*,*

$$E[Z_{\psi_{\alpha,N}}] = \frac{Ni}{2\pi}\frac{(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + \ldots + dz_m \wedge d\overline{z}_m)}{(1+|z|^2)^2}$$

In theorem 2.12, we proved that the expected zero set is determined by the variance of a of the random function when evaluated at a point. More can be said:

Theorem 2.14. *Let* $\{\psi_j\}$ *and* $\{\phi_j\}$ *be two collections of holomorphic functions defined on a domain* Ω *such that for all compact* $K \subset \Omega$*, there exists* $\varepsilon_K > 0$ *such that* $\sum(1+\varepsilon)^j \max_K |\psi_j(z)^2| < \infty$, $\sum(1+\varepsilon)^j \max_K |\phi_j(z)^2| < \infty$ *and let* $\{\alpha_j\}$ *be a sequence of i.i.d. Gaussian random variables.*

If $E[Z_{\psi_\alpha}] = E[Z_{\phi_\alpha}]$ then there exists $h \in \mathcal{O}(\Omega)^x$ and a sequence of i.i.d. standard random variables, $\{\beta_j\}$, such that for all $z \in \Omega$

$$\psi_\alpha(z) = h(z)\phi_\beta(z)$$

This theorem is proven in one dimension by Sodin [Sod], and the same proof works in m-dimensions.

Proof. Fix a domain Ω, and a collection of holomorphic functions, $\{\psi_j\}_{j\in\Lambda}$, such that $\sum |\psi_j(z)|^2 < \infty$. Let $\Pi_1(z,w) = \sum_{j\in\Lambda} \psi_j(z)\psi_j(w)$.

By design, $E[\psi_\alpha(z)\overline{\psi_\alpha(w)}] = \Pi_1(z,w)$ and by theorem 2.12 this determines the expected zero set.

Suppose $E[\phi_\beta(z)\overline{\phi_\beta(z)}] = \Pi_2(z,z)$ has the same zero set. Hence, by theorem 2.12:

$$\partial\overline{\partial}\log\Pi_1(z,z) = \partial\overline{\partial}\log\Pi_2(z,z)$$

Or, in other words

$$\log\frac{\Pi_1(z,z)}{\Pi_2(z,z)} = H(z),\ \ \partial\overline{\partial}H(z) = 0$$

As $\partial\overline{\partial}H(z) = 0$, $H(z) = 2Re(g(z))$, and $\Pi_1(z,z) = e^{g(z)}e^{\overline{g(z)}}\Pi_2(z,z)$. While a priori, this is only true on the diagonal, the values on the diagonal uniquely determine $\Pi_1(z,w)$ in a neighborhood, hence

$$\Pi_1(z,w) = e^{g(z)}e^{\overline{g(w)}}\Pi_2(z,w)$$

As the Szegő kernel is a trace class, and evaluation maps are linear functionals we get that

$$\phi_\beta(z) = e^{g(z)}\psi_\alpha(z).$$

□

3. New results concerning large deviations in the distribution of zeros of a Gaussian entire flat function.

3.1. **An estimate for the growth rate of Gaussian entire flat functions.** In this section we begin working towards our main results. The first step in this process will be to prove Lemma 3.2 which is interesting in and of itself as it proves that Gaussian entire flat functions are of finite order 2, a.s.

We will be need the following elementary estimate:

Proposition 3.1. *If $j \in \mathbb{N}^{+,m}$ then $\frac{|j|^{|j|}}{j^j} \leq m^{|j|}$*

Proof. Let $u_k = \frac{j_k}{|j|} \geq 0$, hence $\sum_{k=1}^n u_k = 1$.
As $\sum u_k \delta_{u_k^{-1}}$ is a probability measure:

$$\begin{aligned}\sum_{j=1}^m u_k \log\left(\frac{1}{u_k}\right) &\leq \log \sum \frac{1}{u_k} u_k, \text{ by Jensen's inequality.} \\ &= \log(m)\end{aligned}$$

$$\begin{aligned}\text{Hence, } m^{|j|} &\geq \prod_k (u_k)^{-|j|u_k} \\ &= \tfrac{|j|^{|j|}}{j^j}\end{aligned}$$

□

Let $M_{r,\alpha} = \max_{z \in B(0,r)} |\psi_\alpha(z)|$

Lemma 3.2. *For all $\delta > 0$ there exists $R_{m,\delta}$ and $c_{\delta,m} > 0$ such that for all $r > R$,*

$$\left| \frac{\log(M_{r,\alpha})}{r^2} - \frac{1}{2} \right| \leq \delta,$$

except for an event whose probability is less than $e^{-c_{\delta,m} r^{2m+2}}$.

Proof. We will first prove that: $Prob(\{\frac{\log(M_{r,\alpha})}{r^2} \geq \frac{1}{2} + \delta\} \leq e^{-c_{\delta,1} r^{2m+2}}$ and we will prove this by specifying an event Ω_r of measure almost 1 where $M_{r,\alpha} \leq e^{(\frac{1}{2}+\delta)r^2}$.

Let Ω_r be the event where: $i)\ |\alpha_j| \leq e^{\frac{\delta r^2}{4}}, \quad |j| \leq 2e \cdot m \cdot r^2$

$ii) |\alpha_j| \leq 2^{\frac{|j|}{2}}, \quad |j| > 2e \cdot m \cdot r^2$

$$\begin{aligned}
Prob(\Omega_r^c) &\leq \sum_{|j| \leq 2e \cdot mr^2} Prob(\{|\alpha_j| > e^{\frac{\delta r^2}{4}}\}) + \sum_{|j| > 2e \cdot mr^2} Prob(\{|\alpha_j| > 2^{\frac{|j|}{2}}\}) \\
&\leq c_m r^{2m} e^{\left(-e^{\frac{\delta r^2}{2}}\right)} + \sum_{|j|_> 2e \cdot mr^2} e^{-2^{|j|}} \\
&\leq e^{-e^{cr^2}} + ce^{-2^{cr^2}}, \ \forall r > R_0 \\
&\leq e^{-e^{cr^2}}
\end{aligned}$$

We now have that $Prob(\Omega_r^c) < e^{-e^{cr^2}} < e^{-cr^{2m+2}}$. It now remains for us to show that $\forall \alpha \in \Omega_r$, $\frac{\log |M_{r,\alpha}|}{r^2} \leq \frac{1}{2} + \frac{1}{2}\delta$. For all $\alpha \in \Omega_r$, and for all $z \in B(0,r)$,

$$\begin{aligned}
M_{r,\alpha} &\leq \max_{z \in B(0,r)} \left(\sum_{|j|=0}^{|j| \leq 4e \cdot m(\frac{1}{2}r^2)} |\alpha_j| \frac{|z|^j}{\sqrt{j!}} + \sum_{|j| > 4e \cdot m(\frac{1}{2}r^2)} |\alpha_j| \frac{|z|^j}{\sqrt{j!}} \right) \\
&= \max_{z \in B(0,r)} \sum^{1} + \max_{z \in B(0,r)} \sum^{2}
\end{aligned}$$

Using the Cauchy-Schwartz inequality:

$$\max_{z\in B(0,r)} \sum^{1} \leq (e^{\frac{1}{4}\delta r^2})\sqrt{c(r^2)^m} \max_{z\in B(0,r)} \left(\sum_j \frac{|z^{2j}|}{j!}\right)^{\frac{1}{2}}$$
$$\leq c_m e^{\frac{\delta r^2}{4}} r^m e^{\frac{1}{2}r^2}$$
$$\leq e^{(r^2)(\frac{1}{2}+\frac{1}{3}\delta)},\ \forall r > R_m.$$

Using Stirling's Formula:

$$\max_{z\in B(0,r)} \sum^{2} \leq \max_{z\in B(0,r)} \sum_{|j|>4e\cdot mr^2} (2)^{\frac{|j|}{2}} \frac{|z^j|}{\sqrt{j!}}$$
$$\leq \sum_{|j|>4e\cdot mr^2} (2)^{\frac{|j|}{2}} \left(\frac{|j|}{4em}\right)^{\frac{|j|}{2}} \prod_k \left(\frac{e}{j_k}\right)^{\frac{j_k}{2}}$$
$$\leq C, \text{ by Proposition 3.1.}$$

Hence, for all $\alpha \in \Omega_r$, $\log(M_{r,\alpha}) \leq (\frac{1}{2}+\frac{1}{2}\delta)r^2$, proving half of the result.

Let $M'_{r,\alpha} = \max_{z\in P(0,r)} |\psi_\alpha(z)|$, where $P(0,r) := \{z \in \mathbb{C}^m :\ \forall i,\ |z_i| < r\}$

It now remains for me to show that for all $\delta > 0$, and $\forall r > R_{m,\delta}$:

$$Prob\left(\left\{\frac{\log(M_{r,\alpha})}{r^2} \leq \frac{1}{2} - \delta\right\}\right) \leq e^{-c_{\delta,2}r^{2m+2}}.$$

It suffices to prove this result only for small δ. However, we will actually prove a stronger claim: that for all δ and for all $r > R_{\delta,m}$ there exists $c_{\delta,m}$ such that

$$Prob\left(M'_{r,\alpha} \leq \frac{m}{2}r^2 - \delta r^2\right) < e^{-c_{\delta,m}r^{2m+2}}$$

This is a stronger claim as: $M_{r,\alpha} \geq M'_{\frac{1}{\sqrt{m}}r,\alpha} \Rightarrow \left\{M_{r,\alpha} < \left(\frac{1}{2}-\delta\right)r^2\right\} \subset \left\{M'_{r,\alpha} < \left(\frac{1}{2}-\delta\right)r^2\right\}$ and the desired probability estimate therefore will hold by monotonicity.

Let us assume that

$$\log(M'_{r,\alpha}) \leq \left(\frac{m}{2} - \delta\right) r^2.$$

By Cauchy's Integral Formula: $\left|\frac{\partial^j \psi_\alpha}{\partial z^j}\right|(0) \leq j! M'_{r,\alpha} r^{-|j|}$. Further, by direct computation using line (1):

$$\left|\frac{\partial^j \psi_\alpha}{\partial z^j}\right|(0) = |\alpha_j|\sqrt{j!}$$

Therefore: $|\alpha_j| \leq cM'_{r,\alpha}\sqrt{j!}r^{-|j|}$, and using Sterling's formula, $(j! \leq e^{\frac{1}{12}}\sqrt{2\pi}\sqrt{j}j^j e^{-j})$, we get that for all k, $j_k \neq 0$:

$|\alpha_j| \leq (2\pi e^{\frac{1}{12}})^{\frac{m}{2}}(\prod_k j_k^{\frac{1}{4}})e^{(\frac{m}{2}-\delta)r^2+\sum \frac{j_k}{2}\log(j_k)-(|j|)\log r-\frac{|j|}{2}}$.

The $(2\pi e^{\frac{1}{12}})^{\frac{m}{2}} j^{\frac{1}{4}}$ term is will not matter in the end so we will focus instead on the exponent, which we will now call A.

$$\begin{aligned} A &= \left(\frac{m}{2} - \delta\right) r^2 - \frac{|j|}{2} + \sum_k \left(\frac{j_k}{2}\log(j_k)\right) - (|j|)\log(r) \\ &= \sum_{k=1}^{k=m} \left(\frac{j_k}{2}\right)\left(\left(1 - \frac{2\delta}{m}\right)\frac{r^2}{j_k} - 1 + \log(j_k) - 2\log(r)\right) \end{aligned}$$

If for all k, $j_k = r^2$ then $A = -\delta r^2$. We now will generalize this observation for $j_k \approx r^2$ by letting $j_k = \gamma_k r^2$.

$$\begin{aligned} A &= \sum_{k=1}^{k=m} \left(\frac{\gamma_k r^2}{2}\right)\left(\left(1 - \frac{2\delta}{m}\right)\frac{1}{\gamma_k} - 1 + \log(\gamma_k)\right) \\ &= -\delta r^2 + m f(\gamma_k)\frac{r^2}{2}, \text{ where } f(\gamma_k) = 1 - \gamma_k + \gamma_k \log(\gamma_k) \end{aligned}$$

$f(\gamma_k) = (1-\gamma_k)^2 - (1-\gamma_k)^3 + o((1-\gamma)^4)$ near 1.

Hence there exists Δ such that $\forall\delta \le \Delta$ if $\gamma_k \in \left[1 - \sqrt{\frac{\delta}{m}}, 1 + \sqrt{\frac{\delta}{m}}\right]$ then $A \le \frac{-\delta r^2}{2}$

Therefore for j as above $|\alpha_j| \le (2\pi e^{\frac{1}{12}})^{\frac{m}{2}} (\prod_k j_k^{\frac{1}{4}}) e^{-\frac{\delta r^2}{2}} \le c_m r^{\frac{m}{2}} e^{-\frac{\delta r^2}{2}}$. This holds true for all α_j, j in terms of r. Specializing our work for large r, we have that $\forall\varepsilon > 0$, there exists $R_{m,\delta}$, such that for all $r > R_{m,\delta}$, $|\alpha_j| \le e^{-\frac{1}{2}(\delta-\varepsilon)r^2}$. Here the factor of ε is used to compensate for the $e^{\frac{1}{12}}\sqrt{2\pi} j_k^{\frac{1}{4}}$ terms, which had been ignored.

We now prove the desired probability decay rate:

$Prob(\{\log M'_{r,\alpha} \le (\frac{1}{2} - \delta)r^2\})$

$\le Prob(\{|\alpha_j| \le e^{-\frac{1}{2}(\delta-\varepsilon)r^2},\ and\ j_k\ \in [(1 - \sqrt{\frac{\delta}{m}})r^2,\ (1 + \sqrt{\frac{\delta}{m}})r^2]\})$

$\le (e^{-(\delta-\varepsilon)r^2})^{(2\sqrt{\frac{\delta}{m}}r^2)^m} = e^{-2^m(1+o(\delta))\delta^{\frac{m+2}{2}} r^{2m+2}} = e^{-c_{1,\delta} r^{2m+2}}$, using the independence of α_j and Proposition 2.2.

□

Corollary 3.3. *Let* $z_0 \in \overline{B(0,r)} \backslash B(0, \frac{1}{2}r)$.
For all $\delta > 0$ *there exists* $R_{\delta,m}$ *and* $c_{\delta,m} > 0$ *such that for all* $r > R_{\delta,m}$, *there exists* $\zeta \in B(z_0, \delta r)$ *such that*

$$\log|\psi_\alpha(\zeta)| > \left(\frac{1}{2} - 3\delta\right)|z_0|^2,$$

except for an event whose probability is $\le e^{-c_{\delta,m} r^{2m+2}}$

Proof. Without loss of generality assume that $\delta < \frac{1}{4}$. By Lemma 3.2:

$$Prob(\{\max_{z\in\partial B(0,r)} \log|\psi_\alpha(z)| - \frac{1}{2}|z|^2 \le -\delta r^2\}) \le e^{-c_\delta r^{2m+2}}.$$

By Corollary 2.6, we have that for $z_0 \in B(0,r)\backslash B(0,\frac{1}{2}r)$, $z \in B(z_0, \delta r)$:

$$Prob(\{\max_{z\in\partial B(0,\delta r)} \log|\psi_\alpha(z-z_0)| - \frac{1}{2}|z-z_0|^2 \leq -\delta(\delta r)^2\}) \leq e^{-c_\delta r^{2m+2}}.$$

Hence, there exists $c_{\delta,m}, R_{\delta,m}$ and $z \in B(z_0,\delta r)$ such that $\log|\psi_\alpha(z-z_0)| - \frac{1}{2}|z-z_0|^2 \geq -\delta(\delta r)^2$, except for an event whose probability for $r > R_{\delta,m}$ is less than $e^{-cr^{2m+2}}$.

By hypothesis, $|z_0| \in [\frac{1}{2}r, r)$, hence $|z-z_0| \leq \delta r \leq \frac{1}{4}r = \frac{r}{2}\frac{1}{2} \leq \frac{1}{2}|z_0|$.

Hence, $|z_0 - z|^2 \geq |z_0|^2 - \delta r^2 \geq |z_0|^2(1-2\delta)$.

$$\begin{aligned} \log|\psi_\alpha(z-z_0)| &\geq \tfrac{1}{2}|z-z_0|^2 - \delta^3 r^2 \geq |z_0|^2\tfrac{1}{2}(1-2\delta)^2 - 4\delta^3|z_0|^2 \\ &\geq \tfrac{1}{2}|z_0|^2 - 2\delta|z_0|^2 - \tfrac{1}{4}\delta|z_0|^2 \\ &\geq \tfrac{1}{2}|z_0|^2 - 3\delta|z_0|^2 \end{aligned}$$

After setting $\zeta = z - z_0$ we have proven what we set out to prove. □

Using that $\log \max_{B(0,r)} |\psi_\alpha|$ is an increasing function in terms of r, we may prove the following corollary:

Corollary 3.4. *For all* $\delta > 0$

a) $Prob\left(\left\{\lim_{r\to\infty} \frac{(\log\max_{z\in B(0,r)}|\psi_\alpha(z)|) - \frac{1}{2}r^2}{r^2} \notin [-\delta,\delta]\right\}\right) = 0$

b) $Prob\left(\left\{\lim_{r\to\infty} \frac{(\log\max_{z\in B(0,r)}|\psi_\alpha(z)|) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0$

This corollary has been proven by more direct methods, [SZ].

Proof. Part b follows immediately from part a, which we now prove.

Let $E_{\delta,R} = \left\{ \frac{\log \max_{B(0,R)} |\psi_\alpha(z)| - \frac{1}{2}R^2}{R^2} \notin [-\delta, \delta] \right\}$, and let $R_t = r + \delta(t+1)r,\ r > 0$. Let $s_t \in [R_{t-1}, R_t]$.

Claim: $\forall t > T_\delta,\ \forall s_t,\ E_{\delta,s_t} \subset E_{\frac{1}{3}\delta,R_t} \bigcup E_{\frac{1}{3}\delta,R_{t-1}}$

Let $T_\delta = \max\{T_{1,\delta},\ T_{2,\delta}\}$, which may be specifically determined.

Case i: for $\alpha \in E_{\delta,s_t}$, $\log \max\limits_{B(0,s_t)} |\psi_\alpha| \geq \dfrac{1}{2}s_t^2 + \delta s_t^2$

$$\begin{aligned} \log \max_{B(0,R_t)} |\psi_\alpha| &\geq \tfrac{1}{2}s_t^2 + \delta s_t^2, \\ &\geq \tfrac{1}{2}(1+t\delta)^2 r^2 + \delta(1+t\delta)^2 r^2 \\ &> \tfrac{1}{2}R_t^2 + \tfrac{1}{3}\delta R_t^2,\ \forall t > M_{1,\delta} \end{aligned}$$

Therefore, $\alpha \in E_{\frac{\delta}{3},R_t}$

Case ii: for $\alpha \in E_{\delta,s_t}$, $\log \max\limits_{B(0,s_t)} \psi_\alpha \leq \dfrac{1}{2}s_t^2 - \delta s_t^2$

$$\begin{aligned} \log \max_{B(0,R_{t-1})} |\psi_\alpha| &\leq \tfrac{1}{2}s_t^2 - \delta s_t^2 \\ &\leq \tfrac{1}{2}(1+(t-1)\delta)^2 r^2 - \delta(1+t\delta)^2 r^2 \\ &\leq \tfrac{1}{2}R_{t-1}^2 - \tfrac{1}{3}\delta R_{t-1}^2,\ \forall t > M_{2,\delta} \end{aligned}$$

Therefore, $\alpha \in E_{\frac{\delta}{3},R_{t-1}}$. Hence, $\forall t > T_\delta$ and $\forall s \in [R_{t-1}, R_t]$, $E_{\delta,s} \subset E_{\frac{1}{3}\delta,R_{t-1}} \cup E_{\frac{1}{3}\delta,R_t}$. Hence, $Prob\left(\bigcup_{s\in[R_{t-1},R_t]} E_{\delta,s}\right) \leq 2e^{-c_\delta r^{2m+2} t^{2m+2}}$, and

$$\sum_{t\in\mathbb{N}} Prob\left(\bigcup_{s\in[R_{t-1},R_t]} E_{\delta,s}\right) = \sum_{t\in\mathbb{N}} e^{-c_\delta t^{2m+2}} < \infty.$$

The result follows. ☐

3.2. **Large deviations in the value of the Nevanlinna characteristic function of a Gaussian entire flat function.** In order to prove Theorem

1.1 we will need to estimate $\int_{S_r} \log|\psi_\alpha| d\mu_r$, where $d\mu_r$ is the rotationally invariant Haar probability measure on the sphere $S_r = \partial B(0,r)$. This will be proved by approximating a surface integral using Riemann integration.

In order to establish notation I state the Poisson integral formula: for $\zeta \in B(0,r)$, and h a harmonic function,

$$h(\zeta) = \int_{S_r} P_r(\zeta, z) h(z) d\mu_r(z),$$

where P_r is the Poisson kernel for $B(0,r)$. For this notation, the Poisson kernel is given by the equation:

$$P_r(\zeta, z) = r^{2m-2} \frac{(r^2 - |\zeta|^2)}{|\zeta - z|^{2m}}$$

Lemma 3.5. *There exists R_m and $c_m > 0$ such that for all $r > R_m$,*

$$\int_{S_r} |\log(|\psi_\alpha|)| d\mu_r(z) < (3^{2m} + 1) r^2,$$

except for an event whose probability is less than or equal to $e^{-c_m r^{2m+2}}$.

Proof. By Lemma 3.2, with the exception of an event whose probability is less than $e^{-c_m r^{2m+2}}$, there exists $\zeta_0 \in \partial B(0, \frac{1}{2}r)$ such that $\log(|\psi_\alpha(\zeta_0)|) > 0$. Hence,

$$\int_{\partial B(0,r)} P_r(\zeta_0, z) \log(|\psi_\alpha(z)|) d\mu_r(z) \geq \log(|\psi(\zeta_0)|) \geq 0.$$

Alternatively,

$$\int_{\partial(B(0,r))} P_r(\zeta_0, z) \log^-(|\psi_\alpha(z)|) \leq \int_{\partial(B(0,r))} P_r(\zeta_0, z) \log^+(|\psi_\alpha(z)|)$$

Since $\zeta \in \partial B(0, \frac{1}{2}r)$ and $z \in \partial B(0, r)$, we have: $\frac{r^2}{4} \leq |z - \zeta|^2 \leq \frac{9}{4}r^2$. Estimating the values of the Poisson Kernel for these values of z and ζ yields:

$$\frac{1}{3}\left(\frac{2}{3}\right)^{2m-2} \leq P_r(\zeta, z) \leq (2)^{2m-2}3$$

Therefore,

$$\int_{S_r} \log^+(|\psi_\alpha(z)|) d\mu_r(z) \leq \log M_r \leq \left(\frac{1}{2} + \delta\right) r^2 \leq r^2,$$

except for an event whose probability is less than $e^{-c_m r^{2m+2}}$, by Lemma 3.2.

It remains to compute $\int \log^- |\psi_\alpha|$:

$$\begin{aligned}
\int_{\partial(B(0,r))} P(\zeta_0, z) \log^+(|\psi_\alpha(z)|) &\leq \mu_r(S_r) \log(M_r) 3(2)^{2m-2} \\
&\leq 3(2)^{2m-2} r^2 \\
\int_{\partial(B(0,r))} \log^-(|\psi_\alpha(z)|) d\mu_r(z) &\leq \frac{1}{\min_z P(\zeta_0, z)} \int_{\partial(B(0,r))} P(\zeta_0, z) \log^+(|\psi_\alpha(z)|) \\
&\leq 3\left(\tfrac{3}{2}\right)^{2m-2} \int_{\partial(B(0,r))} P(\zeta_0, z) \log^+(|\psi_\alpha(z)|) \\
&\leq 9\left(\tfrac{3}{2}\right)^{2m-2} (2)^{2m-2} r^2 \\
&\leq 3^{2m} r^2
\end{aligned}$$

The result now follows immediately.

□

In order to use Riemann integration to prove Lemma 3.9 we will need to be able to choose "evenly" spaced points on the sphere. This choice will be made according to the next lemma:

Lemma 3.6. *(A partition of a Sphere)*

For all $t \in \mathbb{N}^+$, let $Q = (2m)t^{2m-1}$, then $S_r^{2m} \subseteq \mathbb{R}^{2m}$ can be partitioned into

Q disjoint measurable sets $\{I_1^r, I_2^r, \ldots, I_Q^r\}$ such that

$$diam(I_j^r) \leq \frac{\sqrt{2m-1}}{t} r = \frac{c_m}{Q^{\frac{1}{2m-1}}} r.$$

Proof. Surround S_r with $2m$ pieces of planes: $P_{+,1},\ P_{+,2},\ \ldots,\ P_{+,m},\ P_{-,1},\ \ldots,\ P_{-,m}$, where

$$P_{+,j} = \{x \in \mathbb{R}^{2m} :\ ||x||_{L^\infty} = r,\ x_j = r\}$$

$$P_{-,j} = \{x \in \mathbb{R}^{2m} :\ ||x||_{L^\infty} = r,\ x_j = -r\}$$

Subdivide each piece into t^{2m-1} identical closed $2m-1$ cubes, in the usual way, and enumerate these sets $R'_1, \ldots,\ R'_Q$. These sets may intersect one another on the boundary, which is a problem that may be fixed by letting $R_j = R'_j \backslash \bigcup_{k<j} R'_k$.

Let $I_j^r = \{x \in S_r :\ \lambda x \in R_j,\ \lambda > 0\}$. Clearly, $x \in S_r \Rightarrow \exists\ !\ j$ such that $x \in I_j^r$.

Also, by design $\lambda \geq 1$ and $x,\ y \in I_j^r$ therefore,

$$d(x, y) \leq d(\lambda_1 x, \lambda_2 y) \leq \frac{2}{t} r = diam(R_j).$$

$\square$

For the following result note that the integration is with respect to w, which is not the same variable of integration that is usually used in the Poisson integral formula.

Lemma 3.7. *For $\kappa < 1$, $z \in \partial B(0, r)$*

$$\int_{w \in S^m_{\kappa r}} P_r(w, z) d\mu_{\kappa r}(w) = 1$$

Proof. If $w \in S^{2m}_{\kappa r} \subseteq \mathbb{R}^{2m}$, then the Poisson Kernel can be rewritten as a function of $|z - w|$, and as such $\forall \Upsilon \in U_n(\mathbb{R}^n)$, $P_r(\Upsilon w, \Upsilon z) = P_r(w, z)$

Let $f(z) = \int_{w \in S^n_{\kappa r}} P_r(w, z) d\mu_{\kappa r}(w)$

$f(z) \quad = \int_{w \in S^m_{\kappa r}} P_r(w, z) d\mu_{\kappa r}(w)$
$= \int_{w \in S^m_{\kappa r}} P_r(\Upsilon w, \Upsilon z) d\mu_{\kappa r}(w)$, by the above work.
$= \int_{w \in S^m_{\kappa r}} P_r(\Upsilon w, \Upsilon z) d\mu_{\kappa r}(\Upsilon w)$, as $d\mu_{\kappa r}$ is invariant under rotations.
$= \int_{w \in S^n_{\kappa r}} P_r(w, \Upsilon z) d\mu_{\kappa r}(w)$, by a change of coordinates.
$= f(\Upsilon z)$

Hence $f(z) = c$, $\forall z \in S^m_r$

By switching the order of integration we compute that:

$$1 = \int_{w \in S^m_{\kappa r}} \int_{z \in S^m_r} P_r(w, z) d\mu_r(z) d\mu_{\kappa r}(w) = c$$

□

Lemma 3.8. *If $r > 0$ and $\delta \in (0, \min\{\frac{1}{r}, 1\})$, $\kappa = 1 - \delta^{\frac{1}{2(2m+2)(2m-1)}}$, and $\{I^{\kappa r}_j\}^Q_{j=1}$ forms a partition of the sphere of radius κr, and is chosen as per lemma 3.6, and if for each j a point ζ_j is chosen such that $d(\zeta_j, I^{\kappa r}_j) < \delta r$ then*

$$\max_{z \in \partial(B(0,r))} \left| \sum_{j=1}^{Q} \mu_{\kappa r}(I^{\kappa r}_j)(P_r(\zeta_j, z) - 1) \right| \leq C_m \delta^{\frac{1}{2(2m-1)}}$$

Proof. Let $\mu_j = \mu_{\kappa r}(I^{\kappa r}_j)$

$\forall z \in \partial B(0,r)$, $\int_{\zeta\in\partial B(0,\kappa r)} P_r(\zeta,z)d\mu_{\kappa r}(\zeta) = 1$, by Lemma 3.7. Hence,

$$1 = \sum_{j=1}^{j=N} \mu_j P_r(\zeta_j, z) + \sum_{j=1}^{j=N} \int_{\zeta\in I_j^{\kappa r}} (P_r(\zeta,z) - P_r(\zeta_j,z))d\mu_{\kappa r}(\zeta).$$

$$|\sum_{j=1}^{j=N} \mu_j(P_r(\zeta_j,z)-1)| = |\sum_{j=1}^{j=N} \int_{\zeta\in I_j^{\kappa r}}(P_r(\zeta,z) - P_r(\zeta_j,z))d\mu_{\kappa r}(\zeta)|$$
$$\leq \max_{j,\ \zeta\in I_j^{\kappa r}} |\zeta-\zeta_j| \cdot \max_{w\in B(0,(\kappa+\delta)r)\setminus B(0,(\kappa-\delta)r)} \left|\frac{\partial P_r(w,z)}{\partial w}\right|$$

We now approximate the LHS:

$$\frac{\partial P_r(w,z)}{\partial w} = -r^{2m-2}\frac{\overline{w}|z-w|^2 + (r^2-|w|^2)m(\overline{z}-\overline{w})}{|z-w|^{2m+2}}$$

As $|z| = r$, and $|w| = (1-\varepsilon)r \in [(\kappa-\delta)r, (\kappa+\delta)r]$ this last line may be approximated as:

$$\left|\frac{\partial P_r(w,z)}{\partial w}\right| \leq \frac{2+4\varepsilon m}{r\varepsilon^{2m+2}} \leq \frac{c_m}{r\varepsilon^{2m+2}} = \frac{c_m}{r}\delta^{-\frac{1}{2(2m-1)}}$$

Using that $\max_\zeta |\zeta-\zeta_j| \leq diam(I_j) + \delta r \leq c\delta^{\frac{1}{2m-1}} r + \delta r \leq c' r \delta^{\frac{1}{2m-1}}$, we may finish the proof of the claim:

$$\left|\sum_{j=1}^{j=N} \mu_j(P_r(\zeta_j,z)-1)\right| \leq C\delta^{\frac{1}{2m-1}} \cdot \delta^{-\frac{1}{2(2m-1)}} = C\delta^{\frac{1}{2(2m-1)}}$$

.

□

Now we are able to prove our final lemma.

Lemma 3.9. *For all* $\Delta > 0$, *there exists* $R_{m,\Delta} > 0$ *and* $c_{\Delta,m} > 0$ *such that for all* $r > R_{m,\Delta}$

$$\frac{1}{r^2}\int_{z\in\partial B(0,r)} \log|\psi_\alpha| d\mu_r(z) \geq \frac{1}{2} - \Delta,$$

except for an event whose probability is less than $e^{-c_{m,\Delta}r^{2m+2}}$.

Proof. It suffices to prove the result for small Δ. Let $a_m = \frac{1}{2(2m+2)(2m-1)}$. Set $\delta = \left(\frac{1}{\lambda}\Delta\right)^{(\frac{1}{a_m})} < \frac{1}{6}$, $\lambda > 0$ to be determined later. Choose $m \in \mathbb{N}$ such that $\frac{1}{(2m)m^{2m-1}} \leq \delta$, and for notational purposes let $N = (2m)m^{2m-1}$. Also, let $\kappa = 1 - \delta^{a_m}$.

Form a disjoint partition, $\{I_j^{\kappa r}\}$, of $S_{\kappa r}$ as in Lemma 3.6. In particular,

$$diam(I_j^{\kappa r}) \leq c\delta^{\frac{1}{2m-1}} r.$$

Let $\mu_j = \mu_{\kappa r}(I_j^{\kappa r})$, which does not depend on r, and for all j fix a point $x_j \in I_j^{\kappa r}$.

By Corollary 3.3, for each j there exists a $\zeta_j \in B(x_j, \delta r)$ such that

$$\log(|\psi_\alpha(\zeta_j)|) > \left(\frac{1}{2} - 3\delta\right)|x_j|^2 = \left(\frac{1}{2} - 3\delta\right)\kappa^2 r^2$$

except for N different events each of which has probability less than $e^{-c' r^{2m+2}}$, and thus the union of all of these also has probability less than $e^{-cr^{2m+2}}$.

As we have the same estimate for each j for $|\psi_\alpha(\zeta_j)|$, and $\sum \mu_j = 1$ we have:

$$\begin{aligned}\left(\frac{1}{2}-3\delta\right) \quad & (1-\delta^{a_m})^2 r^2 \leq \sum_{j=1}^{N} \mu_j \log(|\psi_\alpha(\zeta_j)|) \\ & \leq \int_{\partial B(0,r)} \left(\sum_j \mu_j P_r(\zeta_j, z) \log(|\psi_\alpha(z)|) d\mu_r(z) \right) \\ & = \int_{\partial(B(0,r))} \left(\sum_j \mu_j (P_r(\zeta_j, z) - 1) \right) \log(|\psi_\alpha(z)|) d\mu_r(z) \\ & \quad + \int_{\partial(B(0,r))} \log(|\psi_\alpha(z)|) d\mu_r(z)\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\partial B(0,r)} \quad & \log(|\psi_\alpha|) d\mu_r \\ & \geq (\tfrac{1}{2}-3\delta)(1-\delta^{a_m})^2 r^2 - \int |\log|\psi_\alpha|| d\mu_r \cdot \max_z |\textstyle\sum_j \mu_j (P_r(\zeta_j, z) - 1)| \\ & \geq (\tfrac{1}{2}-3\delta)(1-\delta^{a_m}) r^2 - (3^{2m}+1) r^2 \cdot C_m \delta^{\frac{1}{2(2m-1)}} \geq \tfrac{1}{2} r^2 - \lambda \delta^{a_m} r^2\end{aligned}$$

by Lemma 3.5 and lemma 3.8.

□

This lemma gives an alternate proof for the growth rate of the characteristic function. Let $T(f,r) = \int_{S_r} \log^+ |f(z)| d\mu_r(z)$, the Nevanlinna characteristic function.

Corollary 3.10. *For all* $\delta \in (0, \frac{1}{3}]$

$$a) \quad Prob\left(\left\{ \lim_{r\to\infty} \frac{(\int_{S_r} \log|\psi_\alpha| d\mu_r) - \frac{1}{2}r^2}{r^2} \notin [-\delta, \delta] \right\}\right) = 0$$

$$b) \quad Prob\left(\left\{ \lim_{r\to\infty} \frac{(\int_{S_r} \log|\psi_\alpha| d\mu_r) - \frac{1}{2}r^2}{r^2} \neq 0 \right\}\right) = 0$$

$$c) \quad Prob\left(\left\{ \lim_{r\to\infty} \frac{T(\psi_\alpha, r) - \frac{1}{2}r^2}{r^2} \neq 0 \right\}\right) = 0$$

As $(\int_{S_r} \log|\psi_\alpha| d\mu_r)$ is increasing the proof of Corollary 3.4 can be used in conjunction with Lemma 3.9 to prove that $\psi_\alpha(z)$ is a.s. finite order 2. A more elementary proof of this is already available [SZ].

3.3. **Large deviations in the volume of the zero sets of Gaussian entire flat functions, and a derivation of the order of the decay of the hole probability.** We will now be able to put the pieces together to estimate the value of the unintegrated counting function of a Gaussian entire flat function $\psi_\alpha(z)$.

Definition 3.11. *For $f \in \mathcal{O}(B(0,r))$, $B(0,r) \subset \mathbb{C}^m$, the unintegrated counting function,*
$n_f(r) := \int_{B(0,t)\bigcap Z_f} (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{m-1} = \int_{B(0,t)} (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{m-1} \wedge \frac{i}{2\pi}\partial\overline{\partial}\log|f|$

The equivalence of these two definitions follows by the Poincare-Lelong formula. The above form $((\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{m-1})$ gives a projective volume, with which it is more convenient to measure the zero set of a random function. The Euclidean volume may be recovered as

$$\int_{B(0,t)\bigcap Z_f} (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{m-1} = \int_{B(0,t)\bigcap Z_f} (\frac{i}{2\pi t^2}\partial\overline{\partial}|z|^2)^{m-1}.$$

Lemma 3.12. *If $f \in \mathcal{O}(\overline{B}_r)$, and $f(0) \neq 0$ then*

$$\int_{t=r\neq 0}^{t=R} \frac{dt}{t} \int_{B_t} \frac{i}{2\pi}\partial\overline{\partial}\log(|f|^2) \wedge (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{m-1}$$
$$= \frac{1}{2}\int_{S_R} \log|f|^2 d\mu_R - \frac{1}{2}\int_{S_r} \log|f|^2 d\mu_r$$

This result is proven on pages 390-391 of Griffiths and Harris, [GH].

We may now prove one of our main theorems, Theorem 1.1:

Proof. (of theorem 1.1).

It suffices to prove the result for small δ.

As $n_{\psi_\alpha}(r)$ is increasing,

$$n_{\psi_\alpha}(r)\log(\kappa) \leq \int_{t=r}^{t=\kappa r} n_{\psi_\alpha}(t)\frac{dt}{t} \leq n_{\psi_\alpha}(\kappa r)\log(\kappa)$$

Let $\kappa = 1+\sqrt{\delta}$. Except for an event whose probability is less than $e^{-cr^{2m+2}}$, we have:

$$\begin{aligned}
n_{\psi_\alpha}(r)\log(\kappa) &\leq \int_{t=r}^{t=\kappa r} n_{\psi_\alpha}(t)\frac{dt}{t} \\
&= \int_{t=r}^{t=\kappa r}\int_{B(0,t)} \frac{i}{2\pi}\partial\overline{\partial}\log|\psi_\alpha(z)| \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2\right)^{m-1}\frac{dt}{t} \\
&= \frac{1}{2}\int_{S_{\kappa r}} \log|\psi_\alpha(z)|d\mu_{\kappa r} - \frac{1}{2}\int_{S_r}\log|\psi_\alpha(z)|d\mu_r, \text{ by Lemma 3.12.} \\
&\leq \frac{1}{2}\left(\left(\frac{1}{2}+\delta\right)\kappa^2 r^2 - \int_{S_r}\log|\psi_\alpha(z)|d\mu_r\right), \text{ by Lemma 3.2.} \\
&\leq \frac{1}{2}\left(\left(\frac{1}{2}+\delta\right)r^2\kappa^2 - \left(\frac{1}{2}-\delta\right)r^2\right), \text{ by Lemma 3.9.} \\
2\frac{n_{\psi_\alpha}(r)}{r^2} &\leq \frac{1}{\log(\kappa)}\left(\kappa^2\left(\frac{1}{2}+\delta\right) - \left(\frac{1}{2}-\delta\right)\right) \\
&= \frac{\kappa^2-1}{2\log(\kappa)} + \delta\frac{\kappa^2+1}{\log(\kappa)} \leq 1+c\sqrt{\delta}.
\end{aligned}$$

We have just shown that:

$$Prob\left(\left\{\frac{n_{\psi_\alpha}(r)}{r^2} \geq \frac{1}{2}+\delta\right\}\right) \leq e^{-c_\delta r^{2m+2}}$$

and now only half of the result remains to be proven. We now complete the proof by noting that, except for an event whose probability is less than $e^{-cr^{2m+2}}$, we have that: $n_{\psi_\alpha}(r)\log(\kappa) \ \geq \int_{t=\kappa^{-1}r}^{t=r} n_{\psi_\alpha}(t)\frac{dt}{t}$, by line (2).

$$
\begin{aligned}
&= \int_{t=\kappa^{-1}r}^{t=r}\int_{B(0,t)} \frac{i}{2\pi}\partial\overline{\partial}\log|\psi_\alpha(z)| \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2\right)^{m-1}\frac{dt}{t} \\
&= \frac{1}{2}\int_{S_r}\log|\psi_\alpha(z)|d\mu_r - \frac{1}{2}\int_{S_{\kappa^{-1}r}}\log|\psi_\alpha(z)|d\mu_{\kappa^{-1}r}, \text{ by Lemma 3.12.} \\
&\geq \frac{1}{2}\left[\left(\frac{1}{2}-\delta\right)r^2 - \int_{S_{\kappa^{-1}r}}\log|\psi_\alpha(z)|d\mu_{\kappa^{-1}r}\right], \text{ by Lemma 3.9.} \\
&\geq \frac{1}{2}\left[\left(\frac{1}{2}-\delta\right)r^2 - \left(\frac{1}{2}+\delta\right)r^2\kappa^{-2}\right], \text{ by Lemma 3.2.}
\end{aligned}
$$

$$
\begin{aligned}
2\frac{n_{\psi_\alpha}(r)}{r^2} &\geq \frac{1}{\log(\kappa)}\left(\left(\frac{1}{2}-\delta\right) - (\frac{1}{2}+\delta)\kappa^{-2}\right) \\
&= \frac{1-\kappa^{-2}}{2\log(\kappa)} - \delta\frac{1+\kappa^{-2}}{\log(\kappa)} \geq 1-2\sqrt{\delta}
\end{aligned}
$$

We have now finished the proof by showing that:

$$Prob\left(\left\{\frac{n_{\psi_\alpha}(r)}{r^2} \leq \frac{1}{2}-\delta\right\}\right) \leq e^{-c_\delta r^{2m+2}}$$

□

Using this estimate for the typical measure of the zero set of a random function we get an upper bound for the hole probability, and a lower bound with the same order of decay is easy to prove:

Proof. (of theorem 1.2).

The upper estimate follows by the previous theorem, as if there is a hole in a ball of radius r then $n_{\psi_\alpha}(r) = 0$, and this can only occur for an event with probability less than $e^{-cr^{2m+2}}$. Therefore it suffices to show that the event where there is a hole in the ball of radius r contains an event whose probability is larger than $e^{-cr^{2m+2}}$. We now design such a set. Let Ω_r be the event where:

i) $\quad |\alpha_0| \geq E_m + 1,$

ii) $\quad |\alpha_j| \leq e^{-(1+\frac{m}{2})r^2}, \ \forall j : 1 \leq |j| \leq \lceil 24mr^2 \rceil = \lceil (m \cdot 2 \cdot 12) r^2 \rceil$

iii) $\quad |\alpha_j| \leq 2^{\frac{|j|}{2}}, \ |j| > \lceil 24mr^2 \rceil \geq 24mr^2$

$Prob(\{|\alpha_j| \leq e^{-(1+\frac{m}{2})r^2}\}) \geq \frac{1}{2}(e^{-(1+\frac{m}{2})r^2})^2 = \frac{1}{2}e^{-(2+m)r^2}$, by Proposition 2.2

$$\#\{j \in \mathbb{N}^m | 1 \leq |j| \leq \lceil 24mr^2 \rceil\} = (\binom{\lceil 24mr^2 \rceil + m}{m}) \approx cr^{2m}$$

Hence, $Prob(\Omega_r) \geq C(e^{-c_m r^{2m+2}}) \geq e^{-cr^{2m+2}}$, by independence and Proposition 2.2. It now suffices to show that for all $\alpha \in \Omega_r$, and for all $z \in B(0, r)$, $\psi_\alpha(z) \neq 0$.

$$|\psi_\alpha(z)| \geq |\alpha_0| - \sum_{|j|=1}^{|j| \leq \lceil 24mr^2 \rceil} |\alpha_j| \frac{r^{|j|}}{\sqrt{j!}} - \sum_{|j| > \lceil 24mr^2 \rceil} |\alpha_j| \frac{r^{|j|}}{\sqrt{j!}} = |\alpha_0| - \sum^{1} - \sum^{2}$$

$$\begin{aligned} \sum^{1} \ &\leq e^{-(1+\frac{m}{2})r^2} \sum_{|j|=1}^{|j| \leq \lceil 24mr^2 \rceil} \frac{r^{|j|}}{\sqrt{j!}} \\ &\leq e^{-(1+\frac{m}{2})r^2} \sqrt{(24mr^2+1)^m} \sqrt{(e^{r^m})}, \text{ by Cauchy-Schwarz inequality.} \\ &\leq C_m r^m e^{-r^2} \leq ce^{-0.9r^2} < \tfrac{1}{2} \text{ for } r > R_m \end{aligned}$$

$$\sum^{2} \leq \sum_{|j|>24mr^2} 2^{\frac{|j|}{2}} \left(\frac{|j|}{24m}\right)^{\frac{|j|}{2}} \frac{1}{\sqrt{j!}}, \text{ as } r < \sqrt{\frac{|j|}{24m}}$$

$$\leq c \sum_{|j|>24mr^2} 2^{\frac{|j|}{2}} \left(\frac{|j|}{24m}\right)^{\frac{|j|}{2}} \prod_{k=1}^{k=m} \left(\frac{e}{j_k}\right)^{\frac{j_k}{2}}, \text{ by Stirling's formula}$$

$$= c \sum_{|j|>24mr^2} \frac{(|j|)^{\frac{|j|}{2}}}{\left(\prod_{k=1}^{k=m} j_k^{\frac{j_k}{2}}\right) m^{\frac{|j|}{2}}} \left(\frac{e}{12}\right)^{\frac{|j|}{2}}$$

$$\leq c \sum_{|j|>1} \left(\frac{1}{4}\right)^{\frac{|j|}{2}}, \text{ by Proposition 2.2.}$$

$$\leq c \sum_{l>1} \left(\frac{1}{2}\right)^{l} l^m \leq E_m$$

Hence, $|\psi_\alpha(z)| \geq E_m + 1 - \sum^{1} - \sum^{2} \geq \frac{1}{2}$ □

4. New results concerning large deviations of the distribution of zeros of a Gaussian random SU$(m+1)$ polynomial

4.1. **Large deviations of the maximum value of a random SU$(m+1)$ polynomial on a ball of radius r.** Our goal will now be to estimate $\max_{B(0,r)} \log|\psi_{\alpha,N}|$. This next lemma is key as it states that the maximum of the norm of a random SU$(m+1)$ polynomial on the ball of radius r tends to not be too far from its expected value.

Lemma 4.1. *For all $\delta \in (0,1)$, and for all $r > 0$ there exists $a_{r,\delta,m} > 0$ and $N_{\delta,m}$ such that for all $N > N_{\delta,m}$*

$$\max_{B(0,r)} |\psi_{\alpha,N}(z)| \in \left[(1+r^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}, (1+r^2)^{\frac{N}{2}}(1+\delta)^{\frac{N}{2}}\right],$$

except for an event whose probability is less than $e^{-a_{r,\delta}N^{m+1}}$.

Proof. We will first prove a sharper decay estimate for the probability of the event where a random SU$(m+1)$ polynomial takes on large values in the ball of radius r:

$$Prob\left(\{\max_{B(0,r)} |\psi_{\alpha,N}(z)| > (1+r^2)^{\frac{N}{2}}(1+\delta)^{\frac{N}{2}}\}\right) < e^{-c_m N^{2m}}.$$

To do this we consider the event $\Omega_N := \{\forall j,\ |\alpha_j| \leq N^m\}$, the complement of which has probability $\leq (N+1)^m e^{-N^{2m}}$, by Proposition 2.2. For $\alpha \in \Omega_N$,

$$\begin{aligned}
\max_{z\in B(0,r)} |\psi_{\alpha,N}(z)| &= \max_{z\in B(0,r)} \left|\sum \alpha_j \binom{N}{j}^{\frac{1}{2}} (z)^j\right| \\
&\leq \max_{z\in B(0,r)} \sum |\alpha_j| \binom{N}{j}^{\frac{1}{2}} |z|^j \\
&\leq \max_{z\in B(0,r)} N^m (N+1)^{\frac{m}{2}} (1+\sum |z_i|^2)^{\frac{N}{2}}, \\
&\quad \text{by the Schwartz inequality.} \\
&= N^m (N+1)^{\frac{m}{2}} (1+r^2)^{\frac{N}{2}}
\end{aligned}$$

For all $\delta > 0$, $\lim_{N\to\infty} c_m N^m \frac{(N+1)^{\frac{m}{2}}}{(1+\delta)^{\frac{N}{2}}} = 0$, therefore there exists $N'_{\delta,m,1} > 0$ such that if $N > N'_{\delta,m,1}$ then $c_m N^m (N+1)^{\frac{m}{2}} < (1+\delta)^{\frac{N}{2}}$. Hence, for $N > N'_{\delta,m,1}$

$$\max_{z\in B(0,r)} |\psi_{\alpha,N}(z)| < (1+\delta)^{\frac{N}{2}}(1+r^2)^{\frac{N}{2}}.$$

In other words, if $N > N'_{\delta,m,1}$ then

$$\left\{\max_{B(0,r)} |\psi_{\alpha,N}(z)| > (1+r^2)^{\frac{N}{2}}(1+\delta)^{\frac{N}{2}}\right\} \subset \Omega_N^c$$

and thus there exists $N'_{\delta,m} > N'_{\delta,m,1}$ such that this event has probability less than or equal to $(N+1)^m e^{-N^{2m}} < e^{-\frac{1}{2}N^{2m}}$, for all $N > N'_{\delta,m}$. This decay rate is independent of δ and r, and the estimate for the order of the decay of this probability could be improved upon.

We complete the proof by showing that:

$$Prob\left(\{\max_{B(0,r)} |\psi_{\alpha,N}(z)| < (1+r^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}\}\right) < e^{-a_{r,\delta}N^{m+1}}.$$

This will be done when we prove the following claim concerning a polydisk, $P(0, \frac{1}{\sqrt{m}}r) := \{z \in \mathbb{C}^m : |z_1| < \frac{1}{\sqrt{m}}r,\ |z_2| < \frac{1}{\sqrt{m}}r, \ldots,\ |z_m| < \frac{1}{\sqrt{m}}r\}$:

$$Prob\left(\{\max_{P(0,r)} |\psi_{\alpha,N}(z)| < (1+mr^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}\}\right) < e^{-a_{r,\delta}N^{m+1}}.$$

This second claim is stronger as $\max_{P(0,\frac{1}{\sqrt{m}}r)} |\psi_{\alpha,N}(z)| \leq \max_{B(0,r)} |\psi_{\alpha,N}(z)|$.

Consider the event where

$$M = \max_{P(0,r)} |\psi_{\alpha,N}(z)| < (1+mr^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}.$$

We will show that this event can only occur if certain Gaussian random variables, α_j, obey the inequality $|\alpha_j| < e^{-c_{\delta,m}N}$, where $c_{\delta,m} > 0$. Further we will show that this occurs whenever j is in a certain cube. This will give us the desired decay rate for the probability.

The Cauchy estimates for a holomorphic function state that:

$$|\psi^{(j)}_{\alpha,N}(0)| \leq j!\frac{M}{r^{|j|}}.$$

By differentiating a random SU$(m+1)$ polynomial of degree N and evaluating at zero we compute that

$$\psi^{(j)}_{\alpha,N}(0) = \sqrt{\binom{N}{j}} j!\alpha_j.$$

Combining this with Stirling's formula:

$$\sqrt{2\pi j_1} j_1^{j_1} e^{-j_1} < j_1! < \sqrt{2\pi j_1} j_1^{j_1} e^{-j_1} e^{\frac{1}{12}}$$

we get that:

$$\begin{aligned}
|\alpha_j| &\leq \frac{(1+mr^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}}{r^{|j|}\sqrt{\binom{N}{j}}} \\
&\leq e^{\frac{m}{12}+\frac{m}{2}\log(2\pi)} \left(\frac{(1+mr^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}(N-|j|)^{\frac{1}{2}(N-|j|+\frac{1}{2})} \prod (j_k)^{\frac{j_k+\frac{1}{2}}{2}}}{r^{|j|} N^{\frac{N+\frac{1}{2}}{2}}} \right) \\
&\leq e^{\frac{m}{12}+\frac{m}{2}\log(2\pi)} N^{\frac{m}{4}} \cdot \left(\frac{(1+mr^2)^{\frac{N}{2}}(1-\delta)^{\frac{N}{2}}(N-|j|)^{\frac{1}{2}(N-|j|)} \prod (j_k)^{\frac{j_k}{2}}}{r^{|j|} N^{\frac{N}{2}}} \right)
\end{aligned}$$

For the time being we focus on the term in parenthesis in the previous line which we call A. Writing j as $j = (j_k) = (x_k N),\ x_k \in (0,1)$, we now have:

$$A = (1-\delta)^{\frac{N}{2}} \left(\frac{(1+mr^2)}{r^{2|x|}} (1-|x|)^{(1-|x|)} \prod_{k=1}^{m} (x_k)^{x_k} \right)^{\frac{N}{2}}$$

If for all $k \in \{1,2,\dots,m\}$, $x_k = \frac{r^2}{1+mr^2}$ then $A = (1-\delta)^{\frac{N}{2}}$, which inspires the following claim:

Claim: Let $s_{r,m} = \frac{1}{2m(1+mr^2)} \min\{r^2, 1\}$.

If for each $k \in \{1,2,\dots,m\},\ x_k \in \left[\frac{r^2}{1+mr^2} - s_{r,m}\delta, \frac{r^2}{1+mr^2}\right] \subset (0,1)$, and thus $|x| < 1$ then

$$(1+mr^2)\left(\frac{x}{r^2}\right)^x(1-|x|)^{(1-|x|)} < (1-\delta)^{-\frac{1}{2}}.$$

Proof: We begin by setting $x_k = (1-\Delta_k)\frac{r^2}{1+mr^2}$ and $\Delta = \sum \Delta_k$. Therefore

$$\Delta_k \in \left[0, \frac{1}{2m}\min\left\{1, \frac{1}{r^2}\right\}\delta\right] \text{ and } \Delta \in \left[0, \frac{1}{2}\min\left\{1, \frac{1}{r^2}\right\}\delta\right].$$

Thus, $1-|x| = \frac{1+\Delta r^2}{1+mr^2}$, and from this we compute that:

$$\begin{aligned}
\frac{(1+mr^2)}{r^{2|x|}}(1-|x|)^{(1-|x|)}(x_k)^{x_k} &= \frac{1+mr^2}{r^{2|x|}}\left(\frac{1+\Delta r^2}{1+mr^2}\right)^{1-|x|} \\
&\quad \cdot \left(\frac{r^2}{1+mr^2}\right)^{|x|}\prod(1-\Delta_k)^{x_k} \\
&= \left(1+\Delta r^2\right)^{1-|x|}\prod(1-\Delta_k)^{x_k} \\
&\leq \left(1+\Delta r^2\right) \\
&\leq 1+\frac{1}{2}\delta \\
&\leq \left(\frac{1}{1-\delta}\right)^{\frac{1}{2}}.
\end{aligned}$$

Proving the claim.

Therefore if for each k, $x_k \in \left[\frac{r^2}{1+mr^2} - s_{r,m}\delta, \frac{r^2}{1+mr^2}\right]$, then $A < (1-\delta)^{\frac{N}{4}}$. This in turn guarantees that $|\alpha_j| < e^{\frac{m}{12}+\frac{m}{2}\log(2\pi)}N^{\frac{m}{4}}(1-\delta)^{\frac{N}{4}}$. The probability this occurs for a single α_j is less than or equal to

$$\left(e^{\frac{n}{12}+\frac{n}{2}\log(2\pi)}N^{\frac{m}{2}}(1-\delta)^{\frac{N}{4}}\right)^2.$$

Thus the probability this occurs for all α_j, $j_k \in \left[(\frac{r^2}{1+mr^2} - s_{r,m}\delta)N, (\frac{r^2}{1+mr^2})N\right]$, is less than or equal to

$$\left(e^{\frac{m}{12}+\frac{m}{2}\log(2\pi)}N^{\frac{m}{4}}(1-\delta)^{\frac{N}{4}}\right)^{2(\lfloor Ns_{r,m}\delta\rfloor)^m}.$$

Hence, there exists $a_{r,\delta,m} > 0$ and $N''_{\delta,m}$ such that for all $N > N''_{\delta,m}$,

$$\left(e^{\frac{m}{12}+\frac{m}{2}\log(2\pi)} N^{\frac{m}{4}} (1-\delta)^{\frac{N}{4}}\right)^{2(\lfloor N s_{r,m}\delta \rfloor)^m} < e^{-a_{r,\delta,m}N^{m+1}}$$

The result follows after setting $N_{\delta,m} = \max\{N'_{\delta,m}, N''_{\delta,m}\}$

□

A nice application of this lemma is the following:

Lemma 4.2. *For all $\Delta \in (0,1)$ and $a \in \mathbb{C}^n \backslash \{0\}$ there exists $N_{\Delta,|a|,m}$ and $c_{\Delta,|a|,m} > 0$, such that if $N > N_{\Delta,|a|,m}$ then*

$$\max_{z \in B(0,\Delta)} |\psi_{\alpha,N}(z-a)| > (1+|a|^2)^{\frac{N}{2}}(1-\Delta)^{\frac{N}{2}},$$

except for an event whose probability is less than $e^{-c_{\Delta,|a|,m}N^{m+1}}$.

Proof. As Gaussian random SU$(m+1)$ polynomials are rotationally invariant, as a random process, with out loss of generality we assume that a is of the form: $a = (\zeta_1, 0, \ldots, 0)$.

Let $\delta = \frac{\Delta}{2+2|\zeta|+2|\zeta|^2)}$.

By Lemma 4.1 and line (2), there exists $c_{\Delta,|a|,m} > 0$ and $N_{\Delta,|a|,m}$ such that if $N > N_{\Delta,|a|,m}$ then, except for an event whose probability is less than $e^{-c_{\Delta,|a|}N^{m+1}}$,

$$
\begin{aligned}
(1-\delta)^{\frac{N}{2}} &\leq \frac{\max_{B(0,\delta)}|\psi_{\alpha,N}(z)|}{(1+\delta)^{\frac{N}{2}}} \\
&= \max_{\partial B(0,\delta)} \frac{\left|\sum \alpha'_j \sqrt{\binom{N}{j}} (z_1-\zeta_1)^{j_1}(1+\overline{\zeta_1} z_1)^{N-|j|} \prod (z_k)^{j_k}\right|}{(1+|\zeta_1|^2)^{\frac{N}{2}}(1+\delta^2)^{\frac{N}{2}}}.
\end{aligned}
$$

In order to simplify this previous line, let $\phi(z) = (\frac{z_1-\zeta_1}{1+\overline{\zeta_1}z_1}, \frac{z_2}{1+\overline{\zeta_1}z_1}, \ldots, \frac{z_m}{1+\overline{\zeta_1}z_1})$, so that we may rewrite the previous equation as:

$$
\begin{aligned}
(1-\delta)^{\frac{N}{2}} &\leq \left(\max_{\partial B(0,\delta)} \frac{|1+\overline{\zeta_1}z_1|^N}{(1+|\zeta_1|^2)^{\frac{N}{2}}(1+\delta^2)^{\frac{N}{2}}}\right)\left(\max_{B(0,\delta)} |\psi_{\alpha',N}(\phi(z))|\right) \\
&\leq \left(\frac{(1+|\zeta_1|\delta)^N}{((1+|\zeta_1|)^2(1+\delta^2))^{\frac{N}{2}}}\right)\left(\max_{B(-\zeta_1,(2+2|\zeta_1|^2)\delta)} |\psi_{\alpha',N}(z)|\right),
\end{aligned}
$$

as the image of $\phi\mid_{B(0,\delta)} \subset B(-\zeta_1,(2+2|\zeta_1|^2)\delta)$, since:

$$
\begin{aligned}
\max_{z\in\partial B(0,\delta)} \left|\frac{z_1-\zeta}{1+\overline{\zeta}z_1}+\zeta\right| &= \max_{z\in\partial B(0,\delta)} \left|\frac{z_1-\zeta+\zeta+z_1|\zeta|^2}{1+\overline{\zeta}z}\right| \\
&= \delta \max_{z\in\partial B(0,\delta)} \left|\frac{(-1-\zeta^2)}{1+\overline{\zeta}z_1}\right| \\
&\leq 2\delta(|\zeta|^2+1)
\end{aligned}
$$

Rearranging the previous sets of equations we get the result:

$$
\begin{aligned}
\max_{B(0,\Delta)} |\psi_{\alpha',N}(z-\zeta_1)| &\geq \frac{(1+|\zeta_1|^2)^{\frac{N}{2}}(1+\delta^2)^{\frac{N}{2}}}{(1+|\zeta_1|\delta)^N} \cdot (1-\delta)^{\frac{N}{2}} \\
&\geq (1+|\zeta_1|^2)^{\frac{N}{2}}(1-(2+2|\zeta_1|)\delta)^{\frac{N}{2}} \\
&\geq (1-\Delta)^{\frac{N}{2}}(1+|\zeta_1|^2)^{\frac{N}{2}}
\end{aligned}
$$

□

4.2. **Large deviations in the value of the Nevanlinna characteristic function of a Gaussian SU($m+1$) polynomial of degree N.** The goal of this section will be to estimate

$$
\int_{S_r} \log|\psi_{\alpha,N}(z)| d\mu_r(z),
$$

where $d\mu_r(z)$ is the normalized rotationally invariant volume measure of the sphere of radius r, $S_r = \partial B(0,r)$, which we will accomplish when we prove lemma 4.4, using the same techniques as in [ST2], and as were used in section 2.

Lemma 4.3. *For all $r > 0$ there exists $c_{m,r}$, and N_m such that for all $N > N_m$,*

$$\int_{S_r} |\log(|\psi_{\alpha,N}(z)|)| d\mu_r(z) \leq \left(\frac{3^{2m}}{2} + \frac{1}{2}\right) N \log\left((2)(1+r^2)\right)$$

except for an event whose probability is $< e^{-c_{m,r}N^{m+1}}$.

Proof. By Lemma 4.1, there exists $c_{m,r}$, and $N_{m,r}$ such that if $N > N_{m,r}$ then, with the exception of an event whose probability is less than $e^{-c_{m,r}N^{m+1}}$, there exists $\zeta_0 \in \partial B(0, \frac{1}{2}r)$ such that $\log(|\psi_{\alpha,N}(\zeta_0)|) > 0$. This also implies that:

$$\int_{z\in S_r} P_r(\zeta_0, z) \log(|\psi_{\alpha,N}(z)|) d\mu_r(z) \geq \log(|\psi(\zeta_0)|) \geq 0,$$

Where P_r is the Poisson kernel for the sphere of radius r (whose area is normalized to 1): $P_r(\zeta, z) = r^{2m-2}\frac{r^2-|\zeta^2|}{|z-\zeta|^{2m}}$. Hence,

$$\int_{z\in S_r} P_r(\zeta_0, z) \log^-(|\psi_{\alpha,N}(z)|) d\mu_r(z) \leq \int_{z\in S_r} P(\zeta_0, z) \log^+(|\psi_{\alpha,N}(z)|) d\mu_r(z)$$

Now given the event where

$$\log \max_{B(0,r)} |\psi_\alpha(z)| < \frac{N}{2} \log\left((2)(1+r^2)\right),$$

(whose complement for $N > N_{m,r}$ has probability less than $e^{-c_{m,r}N^{m+1}}$), we may estimate that

$$\int_{z\in S_r} \log^+(|\psi_{\alpha,N}(z)|)d\mu_r(z) \leq \frac{N}{2}\log\big((2)(1+r^2)\big).$$

Since $\zeta_0 \in \partial B(0,\frac{1}{2}r)$ and $z = re^{i\theta}$, we have: $\frac{r^2}{4} \leq |z-\zeta_0|^2 \leq \frac{9}{4}r^2$. Hence, by using the formula for the Poisson Kernel,

$$\frac{2^{2m-2}}{3^{2m-1}} \leq P(\zeta_0, z) \leq 3\cdot 2^{2m-2}.$$

Putting the pieces together proves the result:

$$\int_{z\in S_r} P_r(\zeta_0,z)\log^+(|\psi_{\alpha,N}(z)|)d\mu_r \leq 3\cdot 2^{2m-3}N\log\big(2(1+r^2)\big)$$

$$\begin{aligned}\int_{z\in S_r} \log^-(|\psi_{\alpha,N}(z)|)d\mu_r(z) \;&\leq\; \frac{1}{\min P(\zeta_0,z)}\int_{z\in S_r} P_r(\zeta_0,z)\log^+(|\psi_{\alpha,N}|)d\mu_r(z)\\ &\leq\; \frac{3^{2m}}{2}N\log\big(2(1+r^2)\big)\end{aligned}$$

□

We now arrive at the main result of this section:

Lemma 4.4. *For all $r > 0$ and for all $\Delta \in (0,1)$ there exists $c_{\Delta,r,m} > 0$ and $N_{\Delta,r,m}$ such that for all $N > N_{\Delta,r,m}$,*

$$\int_{z\in S_r} \log(|\psi_{\alpha,N}(z)|)d\mu_r(z) > \frac{N}{2}\log\big((1+r^2)(1-\Delta)\big),$$

except for an event whose probability is less than $e^{-c_{\Delta,r,m}N^{m+1}}$.

Proof. It suffices to prove this result for small Δ. Set $\delta = 3^{-4m}\Delta^{4m}$. Let $s = \lceil\frac{1}{\delta}\rceil$, let $Q = (2m)s^{2m-1}$, and let $\kappa = 1-\delta^{\frac{1}{4m}}$.

In lemma 3.6 it was shown that the sphere of radius κr may be partitioned into Q measurable disjoint sets $\{I_1^{\kappa r}, I_2^{\kappa r}, \ldots, I_Q^{\kappa r}\}$ such that

$$diam(I^{\kappa r_j}) \leq \frac{\sqrt{2m-1}}{s}\kappa r = \frac{c_m}{Q^{\frac{1}{2m-1}}}\kappa r.$$

We choose such a partition and then we choose a ζ_j within $\delta r < 1$ of $I_j^{\kappa r}$ such that

$$\log(|\psi_{\alpha,N}(\zeta_j)|) > \frac{N}{2}\log\left((1+\kappa^2 r^2)(1-\delta r)\right),$$

for which, by Lemma 4.2, there exists $c'_{\Delta,r}$ and $N'_{\Delta,r}$ such that if $N > N_{\Delta,r}$ then the probability that this does not occur is less than $e^{-c'_{\delta,r}N^{m+1}}$. Therefore there exists $c_{\Delta,r} > 0$ and $N_{\Delta,r}$ such that if $N > N_\Delta$ the union of these m events has probability less than or equal to

$$\left(2m\left\lceil\frac{1}{\delta}\right\rceil^{2m-1}\right)e^{-c'_{\delta,r}N^{m+1}} < e^{-c_{\delta,r}N^{m+1}}.$$

Let $\mu_k = \mu_{\kappa r}(I_k^{\kappa r})$. As $\{I_1^{\kappa r}, I_2^{\kappa r}, \ldots, I_Q^{\kappa r}\}$ form a partition of $S_{\kappa r}$, $\sum_k \mu_k = 1$.

We now turn to investigating the average of $\log|\psi_{\alpha,N}(z)|$ on the sphere of radius r by using Riemann integration and line (3):

$$\begin{aligned}
\frac{N}{2}\log((1+\frac{1}{2}\kappa^2 r^2) \quad (1-\delta)) &\leq \sum_{k=1}^{k=Q}\mu_k \log|\psi_{\alpha,N}(\zeta_k)| \\
&\leq \int_{z\in S_r}\left(\sum_k \mu_k P_r(\zeta_k,z)\log(|\psi_{\alpha,N}(z)|)d\mu_r(z)\right) \\
&= \int_{z\in S_r}\left(\sum_k \mu_k(P_r(\zeta_k,z)-1)\right)\log(|\psi_{\alpha,N}(z)|)d\mu_r(z) \\
&\quad + \int_{z\in S_r}\log(|\psi_{\alpha,N}(z)|)d\mu_r(z)
\end{aligned}$$

This will simplify to:

$$
\begin{aligned}
\int \log(|\psi_{\alpha,N}(z)|)d\mu_r(z) \geq\ & \tfrac{N}{2}\log\left((1+\kappa^2 r^2)(1-\delta r)\right) \\
& -(\textstyle\int |\log|\psi_{\alpha,N}(z)||d\mu_r(z)) \\
& \quad \cdot \max_{z\in S_r}|\sum_k \mu_k(P_r(\zeta_k,z)-1)|
\end{aligned}
$$

In Lemma 3.8, it was computed that in exactly this situation that:

$$
\max_{|z|=r}\left|\sum_k \mu_k(P_r(\zeta_k,z)-1)\right| \leq C_m \delta^{\frac{1}{2(2m-1)}}
$$

Hence by Lemma 4.3 and line (4), there exists $c_{\delta,r,m}>0$ and $N_{\delta,r,m}$ such that if $N > N_{\delta,r,m}$, except for an event of probability $< e^{-c_{\delta,r,m}N^{m+1}}$:

$$
\begin{aligned}
\int \log(|\psi_{\alpha,N}|)d\mu(z) \ \geq\ & \frac{N}{2}\log\big((1+\kappa^2r^2)(1-\delta r)\big) \\
& -C_m N\log\big(2(1+r^2)\big)\,\delta^{\frac{1}{2(2m-1)}}, \\
=\ & \frac{N}{2}\log\Big((1+r^2-2\delta^{\frac{1}{4m}}r^2+\delta^{\frac{1}{2m}}r^2)(1-\delta r)\Big) \\
& -C_m N\log\big(2(1+r^2)\big)\,\delta^{\frac{1}{2(2m-1)}}, \\
=\ & \frac{N}{2}\log[(1+r^2)-2\delta^{\frac{1}{4m}}r^2 \\
& \quad +\delta^{\frac{1}{2m}}O(r^2+\delta^{\frac{4m-1}{4m}}r^3+\delta^{\frac{2m-2}{2m}}r)] \\
& -C_m N\log\big(2(1+r^2)\big)\,\delta^{\frac{1}{2(2m-1)}}, \\
\geq\ & \frac{N}{2}\log((1+r^2)(1-3\delta^{\frac{1}{4m}})), \text{ for sufficiently small } \delta,
\end{aligned}
$$

which is independent of N. The proof is thus completed by choosing sufficiently small Δ so that the previous line holds, (and $\delta r < 1$) .

□

4.3. **Main results concerning random SU$(m+1)$ polynomials.** We will now be able to estimate the value of the unitegrated counting function for a random SU$(m+1)$ polynomial, $\psi_{\alpha,N}$, and by doing so we will prove one of our two main theorems, Theorem 1.3:

Proof. (of theorem 1.3). It suffices to prove the result for small Δ. Let $\delta = \frac{\Delta^2}{4} < 1$. Let $\kappa = 1 + \sqrt{\delta} = 1 + \frac{\Delta}{2}$. As $n_{\psi_{\alpha,N}}(r)$ is increasing,

$$n_{\psi_{\alpha,N}}(r)\log(\kappa) \leq \int_{t=r}^{t=\kappa r} n_{\psi_{\alpha,N}}(t)\frac{dt}{t} \leq n_{\psi_{\alpha,N}}(\kappa r)\log(\kappa)$$

There exists $c_{\delta,r,m} > 0$ and $N_{\delta,r,m}$ such that for all $N > N_{\delta,r,m}$, except for an event of probability $\leq e^{-c_{\delta,r,m}N^{m+1}}$, we get that:

$$\begin{aligned} n_{\psi_{\alpha,N}}(r)\log(\kappa) &\leq \int_{S_{\kappa r}} \log|\psi_{\alpha,N}(z)|d\mu_{\kappa r}(z) - \int_{S_r} \log|\psi_{\alpha,N}(z)|d\mu_r(z) \\ &\leq \frac{N}{2}\left(\log\left((1+\kappa^2r^2)(1+\delta)\right) - \int_{S_r}\log|\psi_\alpha(re^{i\theta})|d\mu_r\right), \end{aligned}$$

by Lemma 4.1.

$$\leq \frac{N}{2}\left(\log\left((1+\kappa^2r^2)(1+\delta)\right) - \log\left((1+r^2)(1-\delta)\right)\right),$$

by Lemma 4.4.

$$\leq \frac{N}{2}\left(\frac{2\sqrt{\delta}r^2 + \delta r^2 + 2\delta + 2\delta r^2}{(1+r^2)} - \frac{2\delta r^4}{(1+r^2)^2} + O(\delta^{\frac{3}{2}})\right),$$

Therefore,

$$\begin{aligned} n_{\psi_{\alpha,N}}(r) &\leq N\left(\frac{r^2 + \frac{1}{2}\sqrt{\delta}r^2 + \sqrt{\delta} + \sqrt{\delta}r^2}{(1+r^2)} - \frac{\sqrt{\delta}r^4}{(1+r^2)^2} + O(\delta)\right) \\ &\qquad \cdot \left(1 + \frac{\sqrt{\delta}}{2} + O(\delta)\right) \\ &\leq \frac{Nr^2}{1+r^2} + 3N\sqrt{\delta} + O(\delta) \end{aligned}$$

This proves the probability estimate when the value of the unintegrated counting function $n_{\psi_{\alpha,N}}(r)$ is significantly above its typical value. We now modify the above the argument to finish the proof. There exists $c_{\delta,r,m}$ and $N_{\delta,r,m}$ such that if $N > N_{\delta,r,m}$, then except for an event whose probability is less than $e^{-c_{\delta,r,m}N^{m+1}}$, the following inequalities hold:

$$
\begin{aligned}
n_{\psi_{\alpha,N}}(r)\log(\kappa) &\geq \int_{S_r}\log|\psi_{\alpha,N}(z)|d\mu_r(z) - \int_{S_{\kappa^{-1}r}}\log|\psi_{\alpha,N}(z)|d\mu_{\kappa^{-1}r}(z)\\
&\geq \frac{N}{2}\left(\log\big((1+r^2)(1-\delta)\big) - \int_{S_{\kappa^{-1}r}}\log|\psi_\alpha(re^{i\theta})|d\mu\right),\\
&\text{by Lemma 4.4.}\\
&\geq \frac{N}{2}\big(\log\big((1+r^2)(1-\delta)\big) - \log\big((1+\kappa^{-2}r^2)(1+\delta)\big)\big),\\
&\text{by Lemma 4.1.}\\
&\geq \frac{N}{2}\log(1-\delta)\\
&\quad -\frac{N}{2}\log\left(1-\frac{2\sqrt{\delta}r^2}{1+r^2}+\frac{\delta r^2}{1+r^2}+\delta+O(\delta^{\frac{3}{2}})\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
n_{\psi_{\alpha,N}}(r) &\geq \frac{N}{2}\left(-\sqrt{\delta}+\frac{2r^2}{1+r^2}-\frac{\sqrt{\delta}r^2}{1+r^2}-\sqrt{\delta}+\frac{2\sqrt{\delta}r^4}{(1+r^2)^2}+O(\delta)\right)\\
&\qquad\cdot\left(1+\frac{\sqrt{\delta}}{2}+O(\delta)\right)\\
&\geq \frac{Nr^2}{1+r^2}-4N\sqrt{\delta}+O(\delta).
\end{aligned}
$$

□

We have just implicitly proven an upper bound on the order of the decay of the hole probability. We will now compute the lower bound to finish the proof theorem 1.4.

Proof. (of theorem 1.4) The desired upper bound for the order of the decay of the hole probability is a consequence of the previous theorem.

We must still prove the lower bound for the order of the decay of the hole probability, and we start this by considering the event, Ω which consists of α_j where:

$$|\alpha_0| \geq 1$$
$$|\alpha_j| < \binom{N}{j}^{\frac{-1}{2}} N^{-m} r^{-|j|}.$$

If $\alpha \in \Omega$, then $|\alpha_0| > \sum_{|j|>0}^{N} |\alpha_j| \binom{N}{j}^{\frac{1}{2}} r^j$. Hence for all $z \in B(0,r)$, $\psi_{\alpha,N}(z) \neq 0 \Rightarrow \Omega \subset Hole_{N,r}$. A lower bound for the probability of Ω will thus give a lower bound for the probability of $Hole_{N,r}$. First we restrict ourselves to considering the Gaussian random variables, α_j, whose indices, j, satisfy the equation: $\binom{N}{j}^{-1} N^{-2m} r^{-2|j|} \leq 1$. For these j

$$\begin{aligned}
Prob\left(\left\{ |\alpha_j| < \binom{N}{j}^{\frac{-1}{2}} N^{-m} r^{-|j|} \right\}\right) &\geq \frac{1}{2} \frac{1}{N^{2m} \binom{N}{j} r^{2|j|}}, \\
&\text{by Proposition 2.2.} \\
&= \frac{1}{2} \frac{(N-|j|)!j!}{N^{2m} N!} r^{-2|j|} \\
&\geq \frac{(2\pi)^{\frac{m}{2}}}{N^{2m} 2(m+1)^{N+\frac{m}{2}}} r^{-2|j|} e^{\frac{1}{12}} \\
&\geq e^{-(N+\frac{m}{2})\log(m+1)+c_m} \\
&\quad \cdot e^{-|j|\log(r)-2m\log(N)} \\
&\geq e^{-(N+\frac{m}{2})\log(m+1)+c_m} \\
&\quad \cdot e^{-|j|\log(\max\{r,1\})-2m\log(N)} \\
&\geq e^{-c_{m,r}N}
\end{aligned}$$

Whereas if for the index j, $\binom{N}{j}^{-1} N^{-2m} r^{-2|j|} > 1$ then

$$\begin{aligned}
Prob\left(\left\{ |\alpha_j| < \binom{N}{j}^{\frac{-1}{2}} N^{-m} r^{-|j|} \right\}\right) &\geq Prob\left(\{ |\alpha_j| < 1\}\right) \\
&> \frac{1}{2} \\
&\geq e^{-N\log(2)}
\end{aligned}$$

Further, $Prob(\{|\alpha_0| > N\}) = e^{-1}$. Hence, $Prob(\Omega) \geq e^{-c_{r,m}N^{m+1}}$

$\square$

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Vita

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