BILINEAR STRICHARTZ ESTIMATES IN TWO DIMENSIONAL COMPACT MANIFOLDS AND CUBIC NONLINEAR SCHRÖDINGER EQUATIONS

by

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ABSTRACT

In this thesis, we establish bilinear Strichartz estimates for Schrödinger operators in 2 dimensional compact manifolds without boundary and with boundary. Then we use estimates on manifold with boundary to prove the local well-posed of cubic nonlinear Schrödinger equation in H^s for every $s > \frac{2}{3}$ on it.

Readers:Professor Christopher Sogge (Advisor)

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DEDICATION

I dedicate this dissertation to my wife Yi-Chuan Tsai without her unwavering love and support it would never have been completed.

Contents

1 Introduction

1.1 Background

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Consider the Schrödinger equation

$$D_t u + \Delta_g u = 0, \qquad u(0,x) = f(x) \tag{1.1}$$

where Δ_g denotes the Laplace-Beltrami operator on manifold and $D_t = i^{-1}\partial_t$. Strichartz estimates are a family of dispersive estimates on solutions $u(t,x) : [0,T] \times M \to \mathbb{C}$ which state

$$\|u\|_{L^p([0,T];L^q(M))} \leq C\|f\|_{H^s(M)} \tag{1.2}$$

where H^s denotes the L^2 Sobolev space over M, and $2 \leq p, q \leq \infty$ satisfies

$$\tfrac{2}{p} + \tfrac{n}{q} = \tfrac{n}{2} \qquad (n,p,q) \neq (2,2,\infty).$$

In Euclidean space, one can take $T = \infty$ and $s = 0$; see for example Strichartz [22], Ginibre and Velo [12], Keel and Tao [16] and references therein. Such estimates have been a key tool in the study of nonlinear Schrödinger equations. In a compact manifold (M, g) without boundary Burq, Gérard and Tzvetkov [10] proved the finite time scale estimates (1.2) for the Schrödinger operators with a loss of derivatives $s = \frac{1}{p}$ in their estimates when compared to the case of flat geometries.

In the case of manifolds with boundary, one considers Dirichlet or Neumann boundary conditions in addition to (1.1)

$$u(t,x)|_{\partial M} = 0 \text{ (Dirichlet)}, \qquad N_x \cdot \nabla u(t,x)|_{\partial M} = 0 \text{ (Neumann)}$$

where N_x denotes the unit normal vector field to ∂M. Here one except further loss of derivatives due to Rayleigh whispering galley modes. Recently, Anton [4] showed that the estimates (1.2) hold on general manifolds with boundary if $s > \frac{3}{2p}$ which arguments work equally well for a manifold without boundary equipped with a Lipschitz metric. Then Blair, Smith and Sogge [5] built estimates (1.2) with less loss of derivatives $s = \frac{4}{3p}$ in manifolds with boundary.

We consider bilinear estimates for the Schrödinger operators in compact manifolds of the form

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C(\min(\Lambda, \Gamma))^{s_0} \|f\|_{L^2(M)} \|g\|_{L^2(M)} \tag{1.3}$$

where Λ, Γ are large dyadic numbers, and f, g are supposed to be spectrally localized on dyadic intervals of order Λ, Γ respectively, namely

$$\mathbb{I}_{\Lambda \leq \sqrt{-\Delta} \leq 2\Lambda}(f) = f \quad , \quad \mathbb{I}_{\Gamma \leq \sqrt{-\Delta} \leq 2\Gamma}(g) = g.$$

Here $\mathbb{I}_{\Lambda \leq \sqrt{-\Delta} \leq 2\Lambda}$ denotes the spectral projection operator

$$\sum_{\Lambda \leq \lambda_j \leq 2\Lambda} E_j f = \sum_{\Lambda \leq \lambda_j \leq 2\Lambda} e_j \int_M f e_j \ ,$$

while $\{\lambda_j^2\}$ and $\{e_j\}$ are eigenvalues and corresponding eigenfunctions of $-\Delta_g$. Such kind of estimates were established and used on Schrödinger equation on manifolds with flat metric; see Klainerman-Machedon-Bourgain-Tataru [17], Bourgain [6] [7] and Tao [23] . Then Burq,Gérard and Tzvetkov [11] established the bilinear estimates in sphere and Zoll surface with $s_0 > \frac{1}{4}$.

For the manifold with boundary, Anton [3] proved (1.3) and the following

$$\|(\nabla_x e^{it\Delta} f)e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C\Lambda(\min(\Lambda,\Gamma))^{s_0}\|f\|_{L^2(M)}\|g\|_{L^2(M)} \tag{1.4}$$

with $s_0 > \frac{1}{2}$ on three dimensional balls with Dirichlet boundary condition and radial data. Using these she proved the local well-posed of cubic nonlinear Schrödinger equation with Dirichlet boundary condition and radial data in H^s for every $s > \frac{1}{2}$ on such manifolds.

1.2 Results

Consider Strichartz estimates on manifolds without boundary obtained by Burq, Gérard and Tzvetkov [10]. If $n = 2$, $(p, q) = (4, 4)$ is admissible, we have

$$\|e^{it\Delta} f\|_{L^4([0,1]\times M)} \leq \|f\|_{H^{1/4}(M)}.$$

Using Littlewood-Paley theory, let $f_\Lambda = \mathbb{I}_{\Lambda\leq\sqrt{-\Delta}\leq 2\Lambda}(f)$, this is equivalent to say

$$\|e^{it\Delta} f_\Lambda\|_{L^4([0,1]\times M)} \leq \Lambda^{1/4}\|f_\Lambda\|_{L^2(M)}$$

holds for all dyadic number Λ, which is implied by bilinear estimates (1.3) with $s_0 = \frac{1}{2}$. However we established the following estimates with $s_0 > \frac{1}{2}$.

Theorem 1.1. *Let (M, g) be a 2 dimensional compact manifold without boundary. For any $f,\ g \in L^2(M)$ satisfies*

$$\mathbb{I}_{\Lambda\leq\sqrt{-\Delta}\leq 2\Lambda}(f) = f \quad , \quad \mathbb{I}_{\Gamma\leq\sqrt{-\Delta}\leq 2\Gamma}(g) = g$$

Then for any $s_0 > \frac{1}{2}$, there exists a $C > 0$ such that

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C(\min(\Lambda,\Gamma))^{s_0}\|f\|_{L^2(M)}\|g\|_{L^2(M)}. \tag{1.5}$$

Also we extend the Strichartz estimats of Blair, Smith and Sogge [5], which read

$$\|e^{it\Delta} f_\Lambda\|_{L^4([0,1]\times M)} \leq \Lambda^{1/3}\|f_\Lambda\|_{L^2(M)}$$

for $(n, p, q) = (2, 4, 4)$, to bilinear estimates (1.3) and (1.4) for the manifolds with boundary with $s_0 > \frac{2}{3}$ in (1.3) and (1.4).

Theorem 1.2. *Let (M, g) be a 2 dimensional compact manifold with boundary. For any $f,\ g \in L^2(M)$ satisfies*

$$\mathbb{I}_{\Lambda\leq\sqrt{-\Delta}\leq 2\Lambda}(f) = f \quad \mathbb{I}_{\Gamma\leq\sqrt{-\Delta}\leq 2\Gamma}(g) = g$$

Then for any $s_0 > \frac{2}{3}$,there exists a $C > 0$ such that

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C(\min(\Lambda, \Gamma))^{s_0}\|f\|_{L^2(M)}\|g\|_{L^2(M)} \tag{1.6}$$

Theorem 1.3. *Let (M, g) be a 2 dimensional compact manifold with boundary. For any $f,\ g \in L^2(M)$ satisfies*

$$\mathbb{I}_{\Lambda\leq\sqrt{-\Delta}\leq 2\Lambda}(f) = f \quad \mathbb{I}_{\Gamma\leq\sqrt{-\Delta}\leq 2\Gamma}(g) = g$$

Then for any $s_0 > \frac{2}{3}$,there exists a $C > 0$ such that

$$\|(\nabla_x(e^{it\Delta} f))e^{it\Delta} g\|_{L^2([0,1]\times M)} \leq C\Lambda(\min(\Lambda, \Gamma))^{s_0}\|f\|_{L^2(M)}\|g\|_{L^2(M)} \tag{1.7}$$

As an application of above theorems, we consider the following Cauchy problem in 2-dimensional compact manifolds with boundary:

$$
\begin{cases}
i\partial_t u + \triangle u &= \alpha|u|^2 u, \text{ on } \mathbb{R}\times M \\
u|_{t=0} &= u_0, \text{ on } M \\
u|_{\partial M} &= 0 \text{ (Dirichlet)}, \quad \text{(or)} \quad \mathrm{N_x}\cdot\nabla \mathrm{u}|_{\partial \mathrm{M}} = 0 \text{ (Neumann)}
\end{cases}
\tag{1.8}
$$

where $\alpha = \pm 1$. When $\alpha = 1$, the equation is defocusing. When $\alpha = 1$, the equation is focusing. We consider the local well-posedness property of (1.8).

Definition 1.4. *Let s be a real number. We shall say that the Cauchy problem (1.8) is uniformly well-posed in $H^s(M)$ if, for any bounded subset B of $H^s(M)$, there exists $T > 0$ such that the flow map*

$$u_0 \in C^\infty(M) \cap B \mapsto u \in C([-T,T], H^s(M))$$

is uniformly continuous when the source space is endowed with H^s norm, and when the target space is endowed with

$$\|u\|_{C_T H^s} = sup_{|t|\le T}\|u(t)\|_{H^s(M)}$$

For manifolds without boundary, we only consider first two equations of (1.8). The first result was due to Bourgain [9] who built the local well-posedness result in H^s for $s > 0$ on the flat torus. Recently, Burq,Gérard and Tzvetkov [10] established local well-posedness of cubic nonlinear Schrödinger equation in $H^s(M)$ for $s > \frac{1}{2}$ on manifold without boundary. In [11] they proved the local well-posed property in $H^s(M)$ for $s > \frac{1}{4}$ on sphere and Zoll surface.

For manifolds with boundary, it is natural to except more loss of derivative due to Rayleigh whispering galley modes. Here we get the local well-posedness result for $s > \frac{2}{3}$. Even though the estimates are not know to be sharp, the result is still natural given the current scope of parametrix constructions. In the case of domains of $\mathbb{R}^2$ the local well-posedness for (1.8) with Dirichlet boundary condition and $s = 1$ were proved by Anton [4]. Here, we will prove the following results.

Theorem 1.5. *If (M, g) is a 2 dimensional manifold with boundary, then the Cauchy problem for (1.8) is uniformly well-posed in $H^s(M)$ for every $s > \frac{2}{3}$.*

Now we discuss the methods for proving local well-posedness. For 2 dimensional manifolds without boundary (only consider first two equations of (1.8)), Burq, Gérard and Tzvetkov [10] proved the local well-posed property in $H^s(M)$ for $s > \frac{1}{2}$ by combining Strichartz inequality (1.2) (in that case $s = \frac{1}{p}$) and Sobolev embedding theorem. The key ingredient there is knowing that

$$u(t, x) \in L^p([-T, T], L^\infty(M)) \quad \text{for} \quad p > 2. \tag{1.9}$$

Using this method one can also get Theorem 1.5. However, we should use the bilinear estimates (1.6) and (1.7) to prove Theorem 1.5. Bilinear estimates have advantage of showing interaction of large and small frequencies , which is useful in dealing with nonlinear terms, see Bourgain [8]. They also reflect more geometry information, Burq,Gérard and Tzvetkov [11] established the bilinear estimates in sphere and Zoll surface with $s_0 > \frac{1}{4}$ by which they infer local well-posedness of (1.8.) in $H^s(M)$ for $s > \frac{1}{4}$. That is better than $s_0 > \frac{1}{2}$ in general 2-dimensional manifold without boundary. In the cases of flat torus and sphere, we know eigenvalues of the Laplacian precisely. Using the arithmetic property of these eigenvalues, the bilinear Strichartz estimates are reduced to bilinear eigenfunctions estimates. For general manifolds, our poor knowledge of spectrums does not allow us to use the same technique.

The thesis is organized as followings: in section 2.1 we deal with manifold without boundary, reduce Theorem 1.1 to a crucial dispersive estimate which will be proved in section 2.2. Then in section 2.3 we see that Theorem 1.2 for manifold with boundary can also be reduced to the same dispersive estimate by paying loss of derivatives, in that section we also prove Theorem 1.3. Then we discuss the Cauchy problem in section 3.1 and introduce the Bourgain space in section 3.2 in order to establish equivalent bilinear estimates in such space. Using these bilinear estimates in Bourgain space we are able to do nonlinear analysis in section 3.3. In section 3.4, we use Sobolev imbedding to show that on three dimensional manifolds with boundary, cubic nonlinear Schrödinger equation is local well-posed of in H^s for every $s > \frac{7}{6}$ on it.

Notation. In what follows d will denote the gradients operators which maps scalar functions to vector fields and vector fields to matrix functions in the natural way. The expression $X \lesssim Y$ means that $X \leq CY$ for some C depending on each occurrence. The notation $a \ll b$ means a is much less than b. We also use Japanese bracket $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ for the convenience.

2 Proof For Bilinear Strichartz Estimates

2.1 Manifolds Without Boundary

We start with the proof of Theorem 1.1. The Laplace-Beltrami operators on M will take the following form in local coordinates

$$(Pf)(x) = \rho^{-1} \sum_{i,j=1}^{n} \partial_i(\rho(x) g^{ij}(x) \partial_j f(x)) \tag{2.1}$$

Assume $\mathbb{I}_{\Lambda \le \sqrt{-\Delta} \le 2\Lambda}(f) = f$, $\mathbb{I}_{\Gamma \le \sqrt{-\Delta} \le 2\Gamma}(g) = g$ and $\Lambda < \Gamma$. Then

$$\begin{aligned}
\|e^{it\Delta} f \; e^{it\Delta} g\|_{L^2([0,1]\times M)} &\lesssim \|v\|_{L^\infty([0,1];L^2(M))} \|u\|_{L^2([0,1];L^\infty(M))} \\
&\lesssim \|g\|_{L^2(M))} \|u\|_{L^2([0,1];L^\infty(M))},
\end{aligned}$$

where we have used the conservation of mass for the free Schrödinger operator in the last inequality.

We define Sobolev spaces on M using the spectral resolution of P,

$$\|f\|_{H^s(M)} = \|\langle D_p \rangle^s f\|_{L^2(M)} \;\;,\;\; \langle D_p \rangle = (1-P)^{\frac{1}{2}}$$

By elliptic regularity (e.g [[14], Theorem 8.10]) the space H^s coincide with the Sobolev spaces defined using local coordinates, provided $0 \le s \le 2$.

Let $r = \frac{1}{2} + \varepsilon > \frac{1}{2}$, $s = r - 1$. Then we need to establish

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|f\|_{H^r(M)} \approx (\Lambda)^r \|f\|_{L^2(M)},$$

or equivalently,

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s f\|_{H^1(M)}$$

By conservation law of free Schrödinger operator which is equivalent to

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s u\|_{L^\infty([0,1];H^1(M))} \tag{2.2}$$

Although $(2,2,\infty)$ is not Schrödinger admissible, we should see that once we localize both time and frequency we can still get desired type of Strichartz estimates.

By taking a finite partition of unity, it suffices to prove that

$$\|\psi\ u\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|\Lambda^s u\|_{L^\infty([0,1];H^1(M))} \tag{2.3}$$

for each smooth cutoff ψ supported in a suitably chosen coordinate chart. After multiplying $\rho(x)$ by a constant and rescaling variables if necessary, we may choose coordinate charts such that the image contains the unit ball, and

$$\|g^{ij} - \delta_{ij}\|_{C^2(B_1(0))} \leq c_0 \ \ , \ \ \|\rho - 1\|_{C^2(B_1(0))} \leq c_0 \tag{2.4}$$

for c_0 to be taken suitably small. We may then extend g^{ij} and ρ globally, preserving condition (2.4), so that P is defined globally on $\mathbb{R}^2$ and such that

$$g^{ij} = \delta^{ij} \ \ , \ \ \rho(x) = 1 \quad \text{for} \ \ |x| > \tfrac{3}{4}.$$

We will use the notation $u = u_k$, to address that the solution is localized to frequency $\Lambda = 2^k$. Hence it is convenient to rewrite (2.3) as

$$\|\psi\ u_k\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}. \tag{2.5}$$

Let $\{\beta_j(D)\}_{j\geq 0}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, and let

$v_j = \beta_j(D)(\psi u_k)$, $v_j^s = (2^j)^s v_j$, then we will see that it is equivalent to show for each j,

$$\|v_j\|_{L_t^2 L_x^\infty} \lesssim \|v_j^s\|_{L_t^\infty H_x^1} + (2^j)^s \|(D_t + P)v_j\|_{L_t^\infty L_x^2} \tag{2.6}$$

is true. [1] Here all norms are taken over $[0,1] \times \mathbb{R}^2$. Note that for any $\varepsilon > 0$

$$\|\psi u_k\|_{L_t^2 L_x^\infty} \lesssim \|2^{j\varepsilon} v_j\|_{L_t^2 L_x^\infty l_2^j} \lesssim \|2^{j\varepsilon} v_j\|_{l_2^j L_t^2 L_x^\infty}.$$

Since ε in the above inequalities can be absorbed by s in (2.6), thus we only have to deal with $\|v_j\|$ instead of $\|2^{j\varepsilon} v_j\|$ in (2.6). On the other hand,

$$\|v_j^s\|_{L^\infty([0,1];H^1(\mathbb{R}^2))} \lesssim \min\{(2^j)\|v_j^s\|_{L^\infty([0,1];L^2(\mathbb{R}^2))}, (2^j)^{-1}\|v_j^s\|_{L^\infty([0,1];H^2(\mathbb{R}^2))}\}$$

$$\lesssim \min\{(2^j)^{1+s}\|u_k\|_{L^\infty([0,1];L^2(M))}, (2^j)^{-1+s}\|u_k\|_{L^\infty([0,1];H^2(M))}\}$$

To sum up $\|v_j^s\|_{L_t^\infty H_x^1}$ over j, we dominate those terms with $j \le k$ by the first term inside minimum bracket, dominate those terms with $j \ge k$ by the second term inside minimum bracket. The series is then bounded by a finite sum plus a geometric series. So the summation over j of first terms in the right hand side of (2.6) is bounded by sum of those two series. They are

$$(2^k)^{1+s}\|u_k\|_{L^\infty([0,1];L^2(M))} + (2^k)^{-1+s}\|u_k\|_{L^\infty([0,1];H^2(M))} \lesssim (2^k)^s\|u_k\|_{L^\infty([0,1];H^1(M))}$$

$$\approx \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

For the second term in the right hand side of (2.6), we note that $[P, \beta_j(D)\psi]$:

[1] It is not obvious now, but we can see later from the proof that only j=k component is significant.

$H^1 \to L^2$, hence we have

$$\|(D_t + P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim \|u_k\|_{L^\infty([0,1];H^1(M))}. \tag{2.7}$$

Thus

$$(2^j)^s\|(D_t + P)v_j\|_{L^\infty([0,1];H^1(\mathbb{R}^2))} \lesssim (2^{(j-k)})^s\|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

$$\lesssim \min\{(2^{(j-k)})^s 2^k\|\Lambda^s u_k\|_{L^\infty([0,1];L^2(M))}, (2^{(j-k)})^s 2^{-k}\|\Lambda^s u_k\|_{L^\infty([0,1];H^2(M))}\}$$

To sum up $(2^j)^s\|(D_t+P)v_j\|_{L^\infty([0,1];H^1(\mathbb{R}^2))}$ over j, we dominate those terms with $j \geq k$ (since $s < 0$) by the first term inside minimum bracket, dominate those terms with $j \leq k$ by the second term inside minimum bracket. The series is then bounded by a finite sum plus a geometric series. So the summation over j of seccond terms in the right hand side of (2.6) is bounded by sum of those two series. They are

$$2^k\|\Lambda^s u_k\|_{L^\infty([0,1];L^2(M))} + 2^{-k}\|\Lambda^s u_k\|_{L^\infty([0,1];H^2(M))} \lesssim \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

Now rewrite (2.6) as

$$\|v_j\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim 2^{j(1/2+\varepsilon)}(\|v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))}+2^{-j}\|(D_t+P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))})$$

Let $\lambda = 2^j$ and $v_j = w_\lambda$, it is now

$$\|w_\lambda\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \lambda^{\frac{1}{2}+\varepsilon}(\|w_\lambda\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} + \lambda^{-1}\|(D_t + P)w_\lambda\|_{L^\infty([0,1];L^2(\mathbb{R}^2))})$$

which is implied by showing for each interval I_λ with length λ^{-1}, we all have

$$\|w_\lambda\|_{L^2(I_\lambda;L^\infty(\mathbb{R}^2))} \lesssim \lambda^\varepsilon(\|w_\lambda\|_{L^\infty(I_\lambda;L^2(\mathbb{R}^2))} + \|(D_t+P)w_\lambda\|_{L^1(I_\lambda;L^2(\mathbb{R}^2))})$$

For a natural reason, we want to localize the coefficients of P by setting

$$g_\lambda^{ij} = S_{\lambda^{1/2}}(g^{ij}) \quad \rho_\lambda = S_{\lambda^{1/2}}(\rho)$$

where $S_{\lambda^{1/2}}$ denotes a truncation of a function to frequencies less than $\lambda^{\frac{1}{2}}$. Let P_λ be the operator with coefficients g_λ^{ij} and ρ_λ. Then

$$\|(P-P_\lambda)w_\lambda\|_{L^1(I_\lambda;L^2(\mathbb{R}^2))} \lesssim \|w_\lambda\|_{L^\infty(I_\lambda;L^2(\mathbb{R}^2))}$$

since we know $|g_\lambda^{ij} - g^{ij}| \lesssim \lambda^{-1}$ and similarly for ρ.

Thus we will conclude Theorem 1.1 by showing following Theorem 2.1 which will be proved by using wave packet method.

Theorem 2.1. *Suppose that $u(t,x)$ is localized to frequencies $|\xi| \in [\frac{1}{4}\lambda, 4\lambda]$ and solves*

$$(D_t + \textstyle\sum_{1\le i,j\le n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{1\le i\le n} b^i(x)\partial_{x_i})u = F$$

Assume also that the metric satisfies

$$\|a^{ij}-\delta_{ij}\|_{C^2} \ll 1, \qquad \|b^i\|_{C^1} \lesssim 1$$

$$\operatorname{supp}(\widehat{a^{ij}}),\ \operatorname{supp}(\widehat{b^i}) \subset B_{\lambda^{1/2}}(0).$$

Then the following estimate holds

$$\|u\|_{L^2([0,\lambda^{-1}];L^\infty(\mathbb{R}^2))} \lesssim (\log\lambda)^{\frac{1}{2}}(\|u\|_{L^\infty([0,\lambda^{-1}];L^2(\mathbb{R}^2))} + \|F\|_{L^1([0,\lambda^{-1}];L^2(\mathbb{R}^2))})$$

To prove Theorem 2.1, we need some notations for wave packet transform. We fix a real, radial Schwartz function $g(x) \in \mathcal{S}(\mathbb{R}^2)$, with $\|g\|_{L^2} = (2\pi)^{-1}$, and assume its Fourier transform $h(\xi) = \hat{g}(\xi)$ is supported in the unit ball $\{|\xi| < 1\}$. For $\lambda \geq 1$, we define $T_\lambda : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^4)$by

$$(T_\lambda f)(x,\xi) = \lambda^{\frac{1}{2}} \int e^{-i\langle \xi, z-x\rangle} g(\lambda^{\frac{1}{2}}(z-x)) f(z) dz.$$

A simple calculation shows that

$$f(y) = \lambda^{\frac{1}{2}} \int e^{i\langle \xi, y-x\rangle} g(\lambda^{\frac{1}{2}}(y-x))(T_\lambda f)(x,\xi) dx d\xi,$$

so that $T_\lambda^* T_\lambda = I$. In particular,

$$\|T_\lambda f\|_{L^2(\mathbb{R}^4_{x,\xi})} = \|f\|_{L^2(\mathbb{R}^2_x)}.$$

Let

$$D_t + A(x,D) + B(x,D) = D_t + \sum_{1\leq i,j\leq n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{1\leq i\leq n} b^i \partial_{x_i}.$$

we conjugate $A(x,D)$ by T_λ and take a suitable approximation to the resulting operator. Define the following differential operator over (x,ξ)

$$\widetilde{A} = -i d_\xi a(x,\xi)\cdot d_x + i d_x a(x,\xi)\cdot d_\xi + a(x,\xi) - \xi\cdot d_\xi a(x,\xi)$$

By the argument from wave packet methods (Lemmas 3.1-3.3 in Smith [19]), we have that if $\widetilde{\beta}_\lambda$ is a Littlewood-Paley cutoff truncating to frequencies $|\xi| \approx \lambda$ then

$$\|T_\lambda A(\cdot,D)\widetilde{\beta}_\lambda(D) - \widetilde{A} T_\lambda \widetilde{\beta}_\lambda(D)\|_{L^2_x \to L^2_{x,\xi}} \lesssim \lambda$$

This yields that, if $\tilde{u}(t,x,\xi) = (T_\lambda u(t,\cdot))(x,\xi)$, then $\tilde{u}$ solves the equation

$$(\partial_t + d_\xi a(x,\xi)\cdot d_x - d_x a(x,\xi)\cdot d_\xi + i a(x,\xi) - i\xi\cdot d_\xi a(x,\xi))\tilde{u}(t,x,\xi) = \tilde{G}(t,x,\xi)$$

where $\tilde{G}$ satisfies

$$\int_0^{\lambda^{-1}} \|\tilde{G}(t,x,\xi)\|_{L^2_{x,\xi}} dt \lesssim \|u\|_{L^\infty([0,\lambda^{-1}];L^2])} + \|F\|_{L^1([0,\lambda^{-1}];L^2)}$$

Given an integral curve $\gamma(r) \in \mathbb{R}^4_{x,\xi}$ of the vector field

$$\partial_t + d_\xi a(x,\xi) \cdot d_x - d_x a(x,\xi) \cdot d_\xi$$

with $\gamma(t) = (x,\xi)$, we denote $\chi_{s,t}(x,\xi) = (x_{s,t}, \xi_{s,t}) = \gamma(s)$. Also define

$$\sigma(x,\xi) = a(x,\xi) - \xi \cdot d_\xi a(x,\xi), \quad \psi(t,x,\xi) = \int_0^t \sigma(\chi_{r,t}(x,\xi)) dr$$

This allows us to write

$$\tilde{u}(t,x,\xi) = e^{-i\psi(t,x,\xi)} \tilde{u}_0(\chi_{0,t}(x,\xi)) + \int_0^t e^{-i\psi(t-r,x,\xi)} \tilde{G}(r, \chi_{r,t}(x,\xi)) dr$$

where $\tilde{u}$ is an integrable superposition over r of functions invariant under the flow of $\tilde{A}$, truncated to $t > r$.

Since $u(t,x) = T_\lambda^* \tilde{u}(t,x,\xi)$ it thus suffices to obtain estimates

$$\|\tilde{\beta}_\lambda(D) W_t f\|_{L^2_t L^\infty_x} \lesssim (\log \lambda)^{\frac{1}{2}} \|f\|_{L^2_{x,\xi}} \tag{2.8}$$

where W_t acts on function $f(x,\xi)$ by the formula

$$(W_t f)(y) = T_\lambda^* (e^{-i\psi(t,x,\xi)} f(\chi_{0,t}(\cdot)))(y) \tag{2.9}$$

In order to get the desired estimates by TT^* method, we investigate the kernel $K(t,y,s,x)$ of $W_t W_s^*$ which is

$$\lambda \int e^{-i\langle \zeta, x-z\rangle - i\int_s^t \sigma(\chi_{r,t}(z,\zeta)) + i\langle \zeta_t, y - z_t\rangle} g(\lambda^{\frac{1}{2}}(y - z_{t,s})) g(\lambda^{\frac{1}{2}}(x-z)) dz d\zeta$$

Recall that $\text{supp}(\hat{g}) \subset B_1(0)$. We are concerned with $\tilde{\beta}_\lambda W_t W_s^* \tilde{\beta}_\lambda$, thus we can inserted a cutoff $S_\lambda(\zeta)$ into the integrand which is supported in a set $|\zeta| \approx \lambda$.

Also note that the Hamiltonian vector field is independent of time, that is $\chi_{t,s} = \chi_{t-s,0}$. We denote it by $\chi_{t-s,0}(z,\zeta) = \chi_{t-s}(z,\zeta) = (z_{t-s}, \zeta_{t-s})$. It then suffices to consider $s = 0$, and the kernel $K(t,x,0,y)$ as

$$\lambda \int e^{-i\langle\zeta,x-z\rangle - i\psi(t,z,\zeta) + i\langle\zeta_t, y-z_t\rangle} g(\lambda^{\frac{1}{2}}(y - z_{t,s}))g(\lambda^{\frac{1}{2}}(x-z))S_\lambda(\zeta)dzd\zeta$$

We will built the estimates (2.8) by considering the estimate for time variable between $[0, \lambda^{-2}]$ and $[\lambda^{-2}, \lambda^{-1}]$ respectively. That is we will prove

$$\|\tilde{\beta}_\lambda(D)W_t f\|_{L^2([0\ ,\ \lambda^{-2}];L^\infty(\mathbb{R}^2))} \lesssim \|f\|_{L^2_{x,\xi}} \tag{2.10}$$

and

$$\|\tilde{\beta}_\lambda(D)W_t f\|_{L^2([\lambda^{-2}\ ,\ \lambda^{-1}];L^\infty(\mathbb{R}^2))} \lesssim (\log\lambda)^{\frac{1}{2}}\|f\|_{L^2_{x,\xi}} \tag{2.11}$$

The inequality (2.10) is easy to prove , note that when $t \in [0, \lambda^{-2}]$, it is easy to see that

$$|K(t,x,0,y)| \approx \lambda \cdot (\lambda^{-\frac{1}{2}})^2 \cdot \lambda^2 = \lambda^2.$$

The term $(\lambda^{-\frac{1}{2}})^2$ came from the size of g and λ^2 from S_λ. Then the the estimates follows from applying Schwartz inequality to time variables.

The inequality (2.11) comes from establishing

$$|K(t,x,0,y)| \lesssim \frac{1}{t} \tag{2.12}$$

for $t \in [\lambda^{-2}, \varepsilon\lambda^{-1}]$ with ε chosen sufficient small and independent of λ. Then by Schwartz inequality, we get

$$\|\tilde{\beta}_\lambda W_t W_s^* \tilde{\beta}_\lambda\|_{L^2\to L^2} \lesssim \int_{\lambda^{-2}}^{\lambda^{-1}} \tfrac{1}{t}dt = \log\lambda.$$

Note that taking partial derivatives to the spatial variables of wave packet transformation in (2.9) will gain λ factors. Thus we get the following result as corollary of Theorem 2.1. which will be used to prove Theorem 1.3.

2.2 The Dispersive Estimate

The dispersive estimate (2.12) we are going to prove here is actually proved in the Theorem 2.1 of Blair,Smith and Sogge [5]. For the reason of completeness, we will include their proof here.

First, we need derivative estimates on the transformation $\chi_t(z,\zeta)$. In addition, we suppose that $t \leq \varepsilon\lambda^{-1}$ with ε chosen sufficiently small and independent of λ.

Lemma 2.2. *Consider the solutions $(z_t(z,\zeta), \zeta_t(z,\zeta))$ to Hamilton's equations*

$$\partial_t z_t = d_\zeta a(z,\zeta) \ , \ \ \partial_t \zeta_t = -d_z a(z,\zeta) \ , \ \ (z_0,\zeta_0) = (z,\zeta). \tag{2.13}$$

We then have the following estimates on the first partial derivatives of (z_t, ζ_t) when $|\zeta| \in [\frac{1}{4}\lambda, 4\lambda]$ and $|t| < \lambda^{-1}$

$$\begin{aligned} |d_z z_t - I| \lesssim \lambda t \qquad & |d_\zeta z_t| \lesssim t \\ |d_z \zeta_t| \lesssim \lambda^2 t \qquad & |d_\zeta \zeta_t - I| \lesssim \lambda t \end{aligned} \tag{2.14}$$

$$|d_\zeta z_t - \int_0^t (d_\zeta^2 a)(\chi_s(z,\zeta))ds| \lesssim \lambda t^2 \tag{2.15}$$

The higher partial derivatives satisfy for $j + k \geq 2$

$$\lambda |d_z^j d_\zeta^k z_t| + |d_z^j d_\zeta^k \zeta_t| \lesssim \lambda^{2-k} t < \lambda^{\frac{3}{2}} t >^{j+k-1} . \tag{2.16}$$

Proof. . If $|\zeta| \approx 1$, then we can write the Hamilton equations as:

$$(z_t, \zeta_t) = (z, \zeta) + \int_0^t v(z_s, \zeta_s) ds$$

where the vector field v satisfies

$$|d^k_{z,\zeta} v| \lesssim \lambda^{\frac{1}{2}(k-1)} \quad , \quad k \geq 1.$$

Differentiating the equation and using induction yields the bound,

$$|d^k_{z,\zeta}(z_t, \zeta_t) - d^k_{z,\zeta}(z, \zeta)| \lesssim t < \lambda^{\frac{1}{2}} t >^{k-1} \quad , \quad |t| < 1.$$

Estimates (2.14) and (2.16) now follow by the rescaling property

$$(z_t(z, \zeta), \zeta_t(z, \zeta)) = (z_{\lambda t}(z, \lambda^{-1}\zeta), \lambda \zeta_{\lambda t}(z, \lambda^{-1}\zeta))$$

Estimates (2.15) follows by differentiating Hamilton's equations as above and applying the bounds (2.14). □

We take a partition of unity $\{\phi_m\}_{m \in \mathbb{R}^2}$ over $\mathbb{R}^2$ with $\phi_m(\zeta) = \phi(t^{\frac{1}{2}}(\zeta - t^{-\frac{1}{2}})m)$ for some ϕ smooth and compactly supported. We then can write

$$K(t, y, 0, x) = \sum_{m \in \mathbb{R}^2} K_m(t, y, x)$$

where $K_m(t, y, x)$ is defined by

$$\lambda \int e^{-i\langle \zeta, x - z \rangle - i\psi(t, z, \zeta) + i\langle \zeta_t, y - z_t \rangle} g(\lambda^{\frac{1}{2}}(y - z_{t,s})) g(\lambda^{\frac{1}{2}}(x - z)) S_\lambda(\zeta) \phi_m(\zeta) dz d\zeta$$

The key estimates is that ,for $\xi_m = t^{-\frac{1}{2}} m$,

$$|K_m(t, y, x)| \lesssim t^{-1}(1 + t^{-\frac{1}{2}} |y - x_t(x, \xi_m)|)^{-N}. \tag{2.17}$$

Estimates (2.15) and the fact that

$$\|d^2_\xi a(x, \xi) - 2I\| = 2\|a^{ij} - \delta^{ij}\| \ll 1$$

yields for $l, m \in \mathbb{Z}^2$ and $t \leq \varepsilon\lambda^{-1}$,

$$|x_t(x, \xi_m) - x_t(x, \xi_l)| \approx t|\xi_m - \xi_l| = t^{\frac{1}{2}}|m - l|.$$

This now yields

$$\textstyle\sum_{m\in\mathbb{Z}^2} |K_m(t, x, y)| \lesssim t^{-1} \sum_{m\in\mathbb{Z}^2}(1 + |m|)^{-N}.$$

Since the sum on the right converges for N large this establishes the dispersive estimates.

To prove (2.17), note that

$$\partial_{\zeta_i}(\textstyle\int_0^t a(z_r, \zeta_r) - \zeta_r \cdot (d_\zeta a)(z_r, \zeta_r)dr) + \zeta_t \cdot \partial_{\zeta_i} z_t = 0$$

The expression vanishes at $t = 0$ since $d_\zeta z_0 = 0$, and Hamilton's equations show that the derivative of the expression with respect to t vanishes.

As in the Theorem 5.4 of Smith-Sogge [20], we define the differential operator

$$L = \frac{1 + it^{-1}(x - z - d_\zeta\zeta_t \cdot (y - z_t) \cdot d_\zeta)}{1 + t^{-1}|x - z - d_\zeta\zeta_t(y - z_t)|^2}.$$

Observe that L preserve the phase function in the definition of K_m. The estimates (2.14) and (2.15) show that, if p is any one of the functions $\phi_m(\zeta)$, $t^{-\frac{1}{2}}z_t$, $\lambda^{\frac{1}{2}z_t}$, $S_\lambda(\zeta)$, $\lambda^{-\frac{1}{2}}t^{-\frac{1}{2}}\zeta_t$, then for $\lambda^{-2} \leq t \leq \lambda^{-1}$,

$$|(t^{-\frac{1}{2}}\partial_\zeta)^k p| \lesssim 1.$$

Integration by parts yields the following upper bound on $K_m(t, x, y)$

$$\lambda \int_{\mathbb{R}^2\times\mathrm{supp}(\phi_m)} (1 + t^{-1}|(x - z) - d_\zeta\zeta_t \cdot (y - z_t)|^2)^{-N}$$

$$\times (1 + \lambda^{\frac{1}{2}}|x - z|)^{-N}(1 + \lambda^{\frac{1}{2}}|y - z_t|)^{-N} dz d\zeta$$

We conclude by showing that

$$t^{-\frac{1}{2}}|(x-z) - d_\zeta \zeta_t \cdot (x_t - z_t)| \lesssim 1 + \lambda |x-z|^2 \tag{2.18}$$

where x_t denotes $x_t(x, \xi_m)$. This implies that the integrand is dominated by

$$(1 + t^{-1}|d_\zeta \zeta_t \cdot (y - x_t)^2|)^{-N}(1 + \lambda^{\frac{1}{2}}|x - z|)^{-N}.$$

Since $|d_\zeta \zeta_t - I| \lesssim \varepsilon$, this establishes the estimates (2.17), since the z decay and compact ζ support imply that the integral is essential over a region in phase space of volume rough $(t\lambda)^{-1}$.

To establish (2.18), we employ a Taylor expansion and (2.16) to obtain

$$\begin{aligned} t^{-\frac{1}{2}}|x_t - z_t &- (d_z z_t)(x - z) - (d_\zeta z_t)(\xi_m - \zeta)| \\ &\lesssim t^{\frac{1}{2}} \langle \lambda^{\frac{3}{2}} t \rangle (\lambda |x - z|^2 + |x - z||\xi_m - \zeta| + \lambda^{-1}|\xi_m - \zeta|^2) \\ &\lesssim 1 + \lambda |x - z|^2 \end{aligned}$$

where the last inequality uses the fact that $\lambda^{-2} \leq t \leq \lambda^{-1}$ and $|\xi_m - \zeta| \lesssim t^{-\frac{1}{2}}$. In addition, by (2.14)

$$t^{-\frac{1}{2}}|(d_\zeta z_t)(\xi_m - \zeta)| \lesssim t(t^{-\frac{1}{2}})^2 = 1.$$

Since $\chi_t(z, \zeta)$ is a symplectomorphism, we have

$$\partial_{\zeta_i} \zeta_t \cdot \partial_{z_j} z_t - \partial_{\zeta_i} z_t \cdot \partial_{z_j} \zeta_t = \delta_{ij}$$

where $\cdot$ pairs the z_t and ζ_t indices. And by (2.14)

$$t^{-\frac{1}{2}}|d\zeta z_t||d_z \zeta_t||x - z| \lesssim \lambda^2 t^{\frac{3}{2}} \leq \lambda^{\frac{1}{2}}|x - z|.$$

These facts now combine to yield the estimate (2.18).

2.3 Manifold With Boundary

To prove Theorem 1.2, we will reduce it to Theorem 2.1 again. As the reduction (2.5) in Section 2.1, we only need to prove for $r = \frac{2}{3} + \varepsilon > \frac{2}{3}$

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \Lambda^r \|f\|_{L^2(M)},$$

Let $s = r - 1$, it is equal to prove

$$\|u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s f\|_{H^1(M)}, \tag{2.19}$$

For manifold with boundary, we work in boundary normal coordinates for the Riemannian metric g_{ij} that is dual g^{ij} of (2.1). Let $x_2 > 0$ define the manifold M, and x_1 is a coordinate function on ∂M which we choose so that ∂_{x_1} is of unit length along ∂M. In these coordinates,

$$g_{22}(x_1, x_2) = 1 \quad g_{11}(x_1, 0) = 1 \quad g_{12}(x_1, x_2) = 0$$

We now extend the coefficient g^{11} and ρ in an even manner across the boundary, so that

$$g^{11}(x_1, -x_2) = g^{11}(x_1, x_2) \quad \rho(x_1, -x_2) = \rho(x_1, x_2).$$

The extended functions are then piecewise smooth, and of Lipschitz regularity across $x_2 = 0$. Because g is diagonal, the operator P is preserved under the reflection $x_2 \to -x_2$. Eigenspaces for the extended operator $\tilde{P}$ decompose into symmetric and antisymmetric functions; these correspond to extensions of eigenfunctions for P satisfying Dirichlet (resp. Neumann) conditions. These eigenfunctions are of $C^{1,1}$ across the boundary. The Schrödinger flow for P is thus extended to $\tilde{P}$.

Hence matters reduces to considering the Schrödinger evolution on the manifold without boundary with Lipschitz metrics. And we have to show

$$\|u\|_{L^2[0,1],L^\infty(M)} \lesssim \|\Lambda^s u\|_{L^\infty[0,1];H^1(M)}$$

By taking a finite partition of unity, it suffices to prove that

$$\|\psi u\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|\Lambda^s u\|_{L^\infty([0,1];H^1(M))}$$

for each smooth cutoff ψ supported in a suitably chosen coordinate charts. We will choose coordinate charts such that the image contains the unit ball, and

$$\|g^{ij} - \delta_{ij}\|_{Lip(B_1(0))} \leq c_0 \ \ , \ \ \|\rho - 1\|_{Lip(B_1(0))} \leq c_0$$

for c_0 to be taken suitably small. We take ψ supported in the unit ball, and assume g^{ij} and ρ are extended so that the above holds globally on $\mathbb{R}^2$.

We denote $u = u_k$ to address that it's frequency being localized to $\Lambda = 2^k$, the equation is now

$$\|\psi u_k\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}.$$

Let $\{\beta_j(D)\}_{j\geq 0}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, and let $v_j = \beta_j(D)(\psi u_k)$, $v_j^s = (2^j)^s v_j$, then we will see that it is equivalent to show for each j,

$$\|v_j\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \|v_j^s\|_{L^\infty([0,1];H^1(\mathbb{R}^2))} + (2^j)^{s-1/3}\|(D_t + P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))}) \tag{2.20}$$

is true. Note that for any $\varepsilon > 0$

$$\|\psi u_k\|_{L_t^2 L_x^\infty} \lesssim \|2^{j\varepsilon} v_j\|_{L_t^2 L_x^\infty l_2^j} \lesssim \|2^{j\varepsilon} v_j\|_{l_2^j L_t^2 L_x^\infty}.$$

Here ε can be absorbed by s in (2.20), thus we only have to deal with $\|v_j\|$ instead of $\|2^{j\varepsilon} v_j\|$ in in (2.20).

On the other hand,

$$\|v_j^s\|_{L^\infty([0,1];H^1(\mathbb{R}^2))} \lesssim \min\{(2^j)\|v_j^s\|_{L^\infty([0,1];L^2(\mathbb{R}^2))}, (2^j)^{-1}\|v_j^s\|_{L^\infty([0,1];H^2(\mathbb{R}^2))}\}$$

$$\lesssim \min\{(2^j)^{1+s}\|u_k\|_{L^\infty([0,1];L^2(M))}, (2^j)^{-1+s}\|u_k\|_{L^\infty([0,1];H^2(M))}\}$$

So the summation over j of first terms in the right hand side of (2.20) is bounded by

$$(2^k)^{1+s}\|u_k\|_{L^\infty([0,1];L^2(M))} + (2^k)^{-1+s}\|u_k\|_{L^\infty([0,1];H^2(M))} \lesssim (2^k)^s\|u_k\|_{L^\infty([0,1];H^1(M))}$$

$$\approx \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

For the second term in the right hand side of (2.20), we note that for a Lipschitz function a, $[\beta_j(D), a] : H^{s-1} \to H^s$, $s = 0, 1$. Hence $[P, \beta_j(D)\psi] : H^1 \to L^2$, by Coilfman-Meyer commutator theorem (see also Proposition 3.6B of [26]). Therefore we have

$$\|(D_t + P)v_j\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} \lesssim \|u_k\|_{L^\infty([0,1];H^1(M))}. \tag{2.21}$$

Thus

$$(2^j)^{s-1/3}\|(D_t + P)v_j\|_{L^\infty([0,1];H^1(\mathbb{R}^2))}$$

$$\lesssim (2^{(j-k)})^{s-1/3}\|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

$$\lesssim \min\{(2^{(j-k)})^{s-1/3}2^k\|\Lambda^s u_k\|_{L^\infty([0,1];L^2(M))}, (2^{(j-k)})^{s-1/3}2^{-k}\|\Lambda^s u_k\|_{L^\infty([0,1];H^2(M))}\}$$

To sum up $(2^j)^{s-1/3}\|(D_t + P)v_j\|_{L^\infty([0,1];H^1(\mathbb{R}^2))}$ over j, we dominate those terms with $j \geq k$ (since $s - 1/3 < 0$) by the first term inside minimum bracket, dominate those terms with $j \leq k$ by the second term inside minimum bracket. The series is then bounded by a finite sum plus a geometric series. So the summation over j of seccond terms in the right hand side of (2.20) is bounded by summation of summing up those two series. They are

$$2^k \|\Lambda^s u_k\|_{L^\infty([0,1];L^2(M))} + 2^{-k}\|\Lambda^s u_k\|_{L^\infty([0,1];H^2(M))} \lesssim \|\Lambda^s u_k\|_{L^\infty([0,1];H^1(M))}$$

Now let $\lambda = 2^j$, $w_\lambda = v_j$, (2.20) can be written as

$$\|w_\lambda\|_{L^2([0,1];L^\infty(\mathbb{R}^2))} \lesssim \lambda^{\frac{2}{3}+\varepsilon}(\|w_\lambda\|_{L^\infty([0,1];L^2(\mathbb{R}^2))} + \lambda^{-\frac{4}{3}}\|(D_t+P)w_\lambda\|_{L^\infty([0,1];L^2(\mathbb{R}^2))})$$

which is implied by showing for each interval I_λ with length $\lambda^{-\frac{4}{3}}$, we all have

$$\|w_\lambda\|_{L^2(I_\lambda;L^\infty(\mathbb{R}^2))} \lesssim (\lambda)^\varepsilon(\|w_\lambda\|_{L^\infty(I_\lambda;L^2(\mathbb{R}^2))} + \|(D_t+P)w_\lambda\|_{L^1(I_\lambda;L^2(\mathbb{R}^2))})$$

This reduction is quite close to that in manifold without boundary expect that the operator P here is rough. Thus we regularize the coefficients of P by setting

$$g^{ij}_\lambda = S_{\lambda^{2/3}}(g^{ij}), \quad \rho_\lambda = S_{\lambda^{2/3}}(\rho)$$

where $S_{\lambda^{2/3}}$ denotes a truncation of a function to frequencies less than $\lambda^{\frac{2}{3}}$. Let P_λ be the operator with coefficients g^{ij}_λ and ρ_λ. Then

$$\|(P-P_\lambda)w_\lambda\|_{L^1(I_\lambda;L^2(\mathbb{R}^2))} \lesssim \|w_\lambda\|_{L^\infty(I_\lambda;L^2(\mathbb{R}^2))}$$

since we know $|g^{ij}_\lambda - g^{ij}| \lesssim \lambda^{-\frac{2}{3}}$ and similarly for ρ.

Then we rescale the problem by letting $\mu = \lambda^{\frac{2}{3}}$ and define

$$u_\mu(t,x) = w_\lambda(\lambda^{-\frac{2}{3}}t, \lambda^{-\frac{1}{3}}x), \quad Q_\mu = P_\lambda(\lambda^{-\frac{1}{3}}x, D)$$

The function $u_\mu(t,\dot{)}$ is localized to frequencies of size μ, and the coefficients of Q_μ are localized to frequencies of the size less than $\mu^{\frac{1}{2}}$. This implies the following estimates of the coefficients of Q_μ

$$\|\partial_x^\alpha g_\lambda^{ij}(\lambda^{-\frac{1}{3}}x)\| + \|\partial_x^\alpha \rho_\lambda(\lambda^{-\frac{1}{3}}x)\| \leq C_\alpha \mu^{\frac{1}{2}\max(0,|\alpha|-2)}.$$

The time interval I_λ scales to μ^{-1}. Also note that by our reduction $\|g_\lambda^{ij} - \delta^{ij}\|_{C^2} \ll 1$. Thus we have reduced the proof of Theorem 1.2 to Theorem 2.1 again.

Next we will prove theorem 1.3. If $\Lambda > \Gamma$, we can prove as following

$$\begin{aligned}\|\nabla(e^{it\triangle}f)e^{it\triangle}g\|_{L^2([0,1]\times M)} &\lesssim \|\nabla e^{it\triangle}f\|_{L^\infty([0,1];L^2(M))}\|e^{it\triangle}g\|_{L^2([0,1]L^\infty(M))} \\ &\lesssim \Lambda\|e^{it\triangle}f\|_{L^\infty([0,1];L^2(M))}\Gamma^s\|g\|_{L^2(M)} \\ &\lesssim \Lambda\Gamma^s\|f\|_{L^2(M)}\|g\|_{L^2(M)},\end{aligned}$$

where we have used the fact Riesz transform $\nabla(-\triangle)^{-1/2}$ is bounded on $L^2(M)$ (see [18]) and then apply Hörmander multiple theorem (see [27]) in the second inequality.

If $\Lambda < \Gamma$, as the reduction (2.5), Let $r = \frac{5}{3} + \varepsilon$, $s = r - 1$. Then we need to prove that

$$\|\nabla u\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s f\|_{H^1(M)}$$

is true. Again we write it as

$$\|\nabla u_k\|_{L^2([0,1];L^\infty(M))} \lesssim \|\Lambda^s u_k\|_{H^1(M)} \tag{2.22}$$

for denoting that it's frequency being localized to $\Lambda = 2^k$. By making use of the following inequality

$$\|\nabla u_k\|_{L^2([0,1];L^\infty(M))} \lesssim \Lambda\|u_k\|_{L^2([0,1];L^\infty(M))} \tag{2.23}$$

and estimate (2.19) we conclude the result.

To see (2.23) is true, we will use an argument concerning finite speed of propagation of wave equation (see for example [21], [27]) and the following gradient estimate of unit band spectral projection operator. The unit band spectral projection operator is defined as

$$\chi_\lambda f(x) = \sum_{\lambda \le \lambda_k < \lambda+1} E_k f(x) = \sum_{\lambda \le \lambda_k < \lambda+1} e_k(x) \int_M f(y) e_k(y) dy$$

Theorem 2.3 ([27] Theorem 1)**.** *Fix a compact Riemannian manifold* (M, g) *with boundary and* $\dim M = n$*, for both Dirichlet Laplacian and Neumann Laplacian on* M*, there is a uniform constant* C *such that*

$$\|\nabla \chi_\lambda f\|_{L^\infty(M)} \le C\lambda^{(n+1)/2} \|f\|_{L^2(M)} \tag{2.24}$$

Let $\{\beta_j\}_{j\ge0}$ be a Littlewood-Paley partition on $\mathbb{R}$. Since Littlewood -Paley operator commute with Schrodinger operator, estimate (2.23) will be a consequence of

$$\|\nabla \beta_k(D) f\|_{L^\infty(M)} \lesssim \lambda \|f\|_{L^\infty(M)} \tag{2.25}$$

where $2^k = \lambda$ and f is spectrally localized to on dyadic interval of order λ.

Recall that $\beta_j(\cdot) = \beta(\frac{\cdot}{2^j})$, $j \ge 1$ for some $\beta \in C_0^\infty(1/2, 2)$. We may assume it is an even function on $\mathbb{R}$. Write

$$\nabla \beta(\frac{P}{\lambda}) f(x) = \frac{1}{2\pi} \nabla \int_{\mathbb{R}} \lambda \widehat{\beta}(\lambda t) e^{itP} f(x) dt.$$

Note that proving (2.25) is equivalent to considering

$$T_\lambda f(x) = \nabla \int_{\mathbb{R}} \lambda \widehat{\beta}(\lambda t) \cos tPf(x)dt,$$

and proving

$$\|T_\lambda f\|_{L^\infty(M)} \lesssim \lambda \|f\|_{L^\infty(M)} \tag{2.26}$$

Here $P = \sqrt{-\triangle}$ and

$$\cos tPf(x) = \sum_{k=1}^{\infty} \cos t\lambda_k E_k(f)(x) = u(t,x)$$

is the cosine transform of f. It is the solution of wave equation

$$(\partial_t^2 - \triangle_g)u = 0\ ,\ u(0,\cdot) = f\ ,\ u_t(o,\cdot) = 0.$$

We shall use the finite propagation speed for solutions to the wave equation. Specifically, if f is supported in a geodesic ball $B(x_0, R)$ centered at x_0 with radius R, then $x \longrightarrow \cos tPf$ vanishes outside of $B(x_0, 2R)$ if $0 \leq t \leq R$.

Let $1 = \eta(t) + \sum_{j=1}^{\infty} \rho(2^{-j}t)$ be a Littlewood-Paley partition of $\mathbb{R}$. Denote

$$T_\lambda^0 f = \nabla \int_{\mathbb{R}} \eta(\lambda t)\lambda \widehat{\beta}(\lambda t) \cos tPf dt \tag{2.27}$$

and

$$T_\lambda^j f = \nabla \int_{\mathbb{R}} \rho(2^{-j}\lambda t)\lambda \widehat{\beta}(\lambda t) \cos tPf dt \tag{2.28}$$

We will prove T_λ satisfies (2.26) by showing T_λ^0 and $\sum_{j\geq 1} T_\lambda^j$ both satisfy (2.26).

Now

$$
\begin{aligned}
T_\lambda^0 f(x) &= \nabla \int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t) \cos tPf(x)dt \\
&= \nabla \int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t) \sum_{\lambda \le \lambda_k \le 2\lambda} \cos t\lambda_k e_k(x) \int_M e_k(y)f(y)dydt \\
&= \int_M \{\int_{\mathbb{R}} \eta(\lambda t)\lambda\widehat{\beta}(\lambda t) \sum_{\lambda \le \lambda_k \le 2\lambda} \cos t\lambda_k \nabla e_k(x)e_k(y)dt\} f(y)dy \\
&= \int_M K_\lambda^0(x,y)f(y)dy
\end{aligned}
$$

Because the support property of $\eta(\lambda t)$ and finite propagation speed for solutions to the wave equation, we know $K_\lambda^0(x,y) = 0$ if $\text{dist}(x,y) > \frac{1}{\lambda}$. Thus we only have to check the case when the support of f is contained in $B(x, \frac{1}{\lambda})$ when proving (2.27) has property (2.26) in $L^\infty(M)$. In order to use (2.24), we write $f = \sum_l f_l$ with each f_l being spectrally localized to unit band. Thus we have

$$
\begin{aligned}
\|T_\lambda^0 f\|_{L^\infty(M)} = \|\sum_l T_\lambda^0 f_l\|_{L^\infty(M)} &\le \lambda^{1/2}(\sum_l \|T_\lambda^0 f_l\|^2_{L^\infty(M)})^{1/2} \\
&\lesssim \lambda^{1/2}(\sum_l \|\nabla \chi_l f\|^2_{L^\infty(M)})^{1/2} \\
&\lesssim \lambda^{1/2}\lambda^{3/2}(\sum_l \|f_l\|^2_{L^2(M)})^{1/2} = \lambda^2 \|f\|_{L^2(M)} \\
&\lesssim \lambda \|f\|_{L^\infty(M)}.
\end{aligned}
$$

Note that in the last inequality, we exploited the support of f is contained in a ball with radius less than $\frac{1}{\lambda}$ which is implied by the property of the operator

T_λ^0 .

Similar,

$$
\begin{aligned}
T_\lambda^j f(x) &= \nabla \int_{\mathbb{R}} \rho(2^{-j}\lambda t)\lambda\widehat{\beta}(\lambda t)\cos tPf(x)dt \\
&= \int_M \{\int_{\mathbb{R}} \rho(2^{-j}\lambda t)\lambda\widehat{\beta}(\lambda t) \sum_{\lambda\leq\lambda_k\leq 2\lambda} \cos t\lambda_k \nabla e_k(x)e_k(y)dt\}f(y)dy \\
&= \int_M K_\lambda^j(x,y)f(y)dy
\end{aligned}
$$

has the property that $K_\lambda^j(x,y) = 0$ if $\mathrm{dist(x,y)} \neq [\frac{2^{j-2}}{\lambda}, \frac{2^{j+1}}{\lambda}]$. Also note that β is a Schwartz function, hence the $\|T_\lambda^j\|_\infty$ is bounded by $\lambda^2(\lambda t)^{-N}(\frac{2^j}{\lambda})\|f\|_\infty$ with $t \approx \frac{2^j}{\lambda}$ and N be a large enough positive integer. Thus we have

$$\|T_\lambda^j\|_{L^\infty} \lesssim \lambda 2^{-jN}\|f\|_{L^\infty}$$

which forms a geometric series and thus their sum enjoy the property (2.26).

3 Cubic NLS

3.1 Cauchy Problem

In the following, we establish the well-posedness of the cubic nonlinear Schrödinger equation in manifolds (M, g) with boundary. The equations we are interested in is following.

$$\begin{cases} i\partial_t u + \triangle u &= \alpha |u|^2 u, \text{ on } \mathbb{R} \times M \\ u|_{t=0} &= u_0, \text{ on } M \\ u|_{\partial M} &= 0 \text{ (Dirichlet)}, \quad \text{(or)} \quad \mathrm{N_x \cdot \nabla u|_{\partial M} = 0} \text{ (Neumann)} \end{cases} \tag{3.1}$$

where $\alpha = \pm 1$.

Definition 3.1. *Let s be a real number. We shall say that the Cauchy problem (3.1) is uniformly well-posed in $H^s(M)$ if, for any bounded subset of $H^s(M)$, there exists $T > 0$ such that the flow map*

$$u_0 \in C^\infty(M) \cap B \mapsto u \in C([-T, T], H^s(M))$$

is uniformly continuous when the source space is endowed with H^s norm, and when the target space is endowed with

$$\|u\|_{C_T H^s} = sup_{|t| \leq T} \|u(t)\|_{H^s(M)}$$

The remain part of thesis is to prove the following local well-posdness results.

Theorem 3.2. *If (M, g) is a 2 dimensional manifold with boundary, then the Cauchy problem for (3.1) is uniformly well-posed in $H^s(M)$ for every $s > \frac{2}{3}$.*

Theorem 3.3. *If (M,g) is a 3 dimensional smooth compact Riemannian manifold with boundary, then the Cauchy problem for is uniformly local well-posed in* $H^s(M)$ *for every* $s > \frac{7}{6}$.

The Cauchy problem (3.1) is quite different in 2 and higher dimensional cases. In the 2 dimensional manifold more results are know. For manifold without boundary(only consider first two equations of (5.1)), Burq, Gérard and Tzvetkov [11] proved the local well-posed property in $H^s(M)$ for $s > \frac{1}{2}$ by combining Strichartz inequality (1.2) (in that case $s = \frac{1}{p}$) and Sobolev embedding theorem. The key ingredient there is knowing that

$$u(t,x) \in L^p([-T,T], L^\infty(M)) \quad \text{for} \quad p > 2. \tag{3.2}$$

We should follow the same method to get Theorem 3.3. We call this method A. Using method A can also get Theorem 3.2. However, we should use the bilinear estimates (1.6) and (1.7) to prove Theorem 3.2. which we call it method B. Bilinear estimates have advantage of showing interaction of large and small frequencies , which is useful in dealing with nonlinear terms, see Bourgain [8]. They also reflect more geometry information, Burq,Gérard and Tzvetkov [11] established the bilinear estimates in sphere and Zoll surface with $s_0 > \frac{1}{4}$ by which they infer local well-posed of (3.1) in $H^s(M)$ for $s > \frac{1}{4}$. That is better than $s_0 > \frac{1}{2}$ in general 2-dimensional manifold without boundary.

In the 3 or higher dimensional manifold less results are know. Burq, Gérard and Tzvetkov [10] proved the local well posed of (3.1) in $H^s(M)$ for $s \geq 1$ on manifold without boundary by method A and extended it to global result by conservation of energy. For manifold with boundary Anton [3] proved the local well posed property in $H^s(M)$ on the ball with Dirichlet boundary condition and radial data for $s > \frac{1}{2}$, where she used the method B to deal with the nonlinear term. While in [2] ,she proved the global well posed property on the exterior of non-trapping domains for $s = 1$ by method A.

In the 3 (or high) dimensional manifolds, we note that $(p, q) = (4, 4)$ does not fit the Strichartz admissible condition

$$\tfrac{2}{p} + \tfrac{n}{q} = \tfrac{n}{2} \qquad (n, p, q) \neq (2, 2, \infty).$$

Hence the bilinear estimates in $\mathbb{R}^3$ for $\lambda < \mu$

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \tfrac{\lambda}{\mu^{1/2}} \|f\|_2 \|g\|_2 \lesssim \lambda^{1/2} \|f\|_2 \|g\|_2.$$

have loss of derivatives. Even we localize the time to interval $[0, 1]$, we can not get better estimates. It is not surprising then both method A and B will lead us to well-posed of $s > \frac{1}{2}$ in $\mathbb{R}^3$. Recall that we proved bilinear Stirchartz estimates in manifolds following the approach of proving Strichartz [5] by using the fact that the wave behaves in short time like it does in flat metric space. By adding up short time estimates then gives us the desired estimates. Thus the estimates could get from this approach in 3 dimensional manifold with boundary will be

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^2([0,T]\times M)} \lesssim \lambda \mu^{1/6} \|f\|_{L^2(M)} \|g\|_{L^2(M)}.$$

Here the frequency was given by $\frac{\lambda}{u^{1/2}} \mu^{\frac{2}{3}} = \lambda \mu^{\frac{1}{6}}$. Also the power $1 + \frac{1}{6}$ correspond to $\frac{7}{6}$ in Theorem 3.3. But this estimate will not help us in dealing interaction of large and small frequencies. However we do not know if it is optimal. It is interesting to find estimates eliminate high frequency , then we can employ them to do analysis for nonlinear term.

The remaining part of thesis was organized as following: In section 3.2 and 3.3 we outlined the details of method B and built bilinear estimates in Bourgain space which are equivalent to our bilinear estimates, then we were able to infer Theorem 3.2. In section 3.4 we used method A to built Theorem 3.3.

3.2 Bourgain Spaces

In order to prove the local well-posedness of cubic nonlinear Schrödinger equation on manifolds with boundary. We introduce Bourgain space $X^{s,b}$. Our definition follows from Burq, Gérard and Tzvetkov [11] using the spectral projectors on manifolds.

Let (e_k) be a $L^2(M)$ orthonormal basis of eigenfunctions of Dirichlet(or Neumann) Laplacian $-\triangle_g$ with eigenvalues μ_k^2, E_k be the orthogonal projector along e_k. The Sobolev space $H^s(M)$ is associated to $(I-\triangle)^{1/2}$, equipped with the norm

$$\|u\|_{H^s(M)}^2 = \sum_k \langle\mu_k\rangle^{2s}\|E_k u\|_{L^2(M)}^2$$

where $\langle\mu_k\rangle = (1+\mu_k^2)^{\frac{1}{2}}$.

Definition 3.4. *The space $X^{s,b}(\mathbb{R}\times M)$ is the completion of $C_0^\infty(\mathbb{R}_t; H^s(M))$ with the norm*

$$\|u\|_{X^{s,b}(\mathbb{R}\times M)}^2 = \sum_k \|\langle\tau+\mu_k^2\rangle^b\langle\mu_k\rangle^s \widehat{E_k u}(\tau)\|_{L^2(\mathbb{R}_\tau;L^2(M))}^2 \tag{3.3}$$

$$= \|e^{-it\triangle}u(t,\cdot)\|_{H^b(R_t;H^s(M))}^2 \tag{3.4}$$

where $\widehat{E_k u}(\tau)$ *denote the Fourier transform of* $E_k u$ *with respect to the time variable.*

In fact ,if $s \geq 0$ and $u \in \mathcal{S}'(\mathbb{R}, L^2(M))$. Let $F(t,\cdot) = e^{-it\triangle}u(t,\cdot)$, then $F(t,\cdot) \in \mathcal{S}'(\mathbb{R}, L^2(M))$ and $E_k(F(t,\cdot)) = e^{it\mu_k^2}E_k(u(t,\cdot))$. Hence $\widehat{E_k(F)}(\tau) = \widehat{E_k(u)}(\tau - \mu_k^2)$. Applies this to (3.3), we conclude

$$\|u\|^2_{X^{s,b}(\mathbb{R}\times M)} = \|e^{-it\triangle}u(t,\cdot)\|^2_{H^b(R_t;H^s(M))}.$$

We also note that if $b > \frac{1}{2}$, $H^b(\mathbb{R}, H^s(M)) \hookrightarrow C(\mathbb{R}, H^s(M))$, since $u(t,\cdot) = e^{it\triangle}F(t,\cdot)$, we have $u \in C(\mathbb{R}, H^s(M))$.

In order to use a contraction mapping argument to obtain local existence. We need to define local in time version of $X^{s,b}(\mathbb{R}\times M)$. For $T > 0$ we denoted by $X^{s,b}_T(M)$ the space of restrictions of elements of $X^{s,b}(\mathbb{R}\times M)$ endowed with the norm

$$\|u\|_{X^{s,b}_T} = \inf\{\|\tilde{u}\|_{X^{s,b}(\mathbb{R}\times M)}\ ,\ \tilde{u}|_{(-T,T)\times M} = u\}$$

Now we can reformulate the bilinear estimates in the $X^{s,b}$ content. The following lemma should refer to the lemma 2.3 of [11].

Lemma 3.5. *Let* $s \in \mathbb{R}$. *The following statements are equivalent:*
(1) For any $u_0\ ,\ v_0 \in L^2(M)$ *satisfying*

$$1_{\lambda\leq\sqrt{-\triangle}\leq 2\lambda}u_0 = u_0 \quad , \quad 1_{\mu\leq\sqrt{-\triangle}\leq 2\mu}v_0 = v_0$$

one has

$$\|e^{it\triangle}u_0\ e^{it\triangle}v_0\|_{L^2((0,1)_t\times M)} \leq C(\min(\lambda,\mu))^s\|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)} \tag{3.5}$$

(2)For any $b > \frac{1}{2}$ *and any* $f, g \in X^{0,b}(\mathbb{R}\times M)$ *satisfying*

$$1_{\lambda\leq\sqrt{-\triangle}\leq 2\lambda} f = f \quad , \quad 1_{\mu\leq\sqrt{-\triangle}\leq 2\mu} g = g$$

one has

$$\|fg\|_{L^2(\mathbb{R}\times M)} \leq C(\min(\lambda,\mu))^s \|f\|_{X^{0,b}(\mathbb{R}\times M)} \|g\|_{X^{0,b}(\mathbb{R}\times M)} \tag{3.6}$$

Proof. If $u(t) = e^{-it\triangle}u_0$ then for any $\psi \in C_0^\infty(\mathbb{R})$ and any b , $\psi(t)u(t) \in X^{0,b}(\mathbb{R}_t \times M)$ with

$$\|\psi u\|_{X^{0,b}(\mathbb{R}\times M)} \leq C\|u_0\|_{L^2(M)}$$

which shows that (3.6) implies (3.5).

Suppose that $f(t)$ and $g(t)$ are supported in time in the interval $(0,1)$ and write

$$f(t) = e^{it\triangle}e^{-it\triangle}f(t) = e^{it\triangle}F(t) \ , \ g(t) = e^{it\triangle}e^{-it\triangle}g(t) = e^{it\triangle}G(t)$$

Then

$$f(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau}e^{it\triangle}\widehat{F}(\tau)d\tau \ , \ g(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\tau}e^{it\triangle}\widehat{G}(\tau)d\tau$$

and hence

$$(fg)(t) = \tfrac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(\tau+\sigma)}e^{it\triangle}\widehat{F}(\tau)e^{it\triangle}\widehat{G}(\sigma)d\tau d\sigma.$$

Ignoring the oscillating factors $e^{it(\tau+\sigma)}$, using (3.5) and the Cauchy-Schwartz inequality in (τ,σ) (in this places we use that $b > \frac{1}{2}$ to get the needed integrability) yields

$$
\begin{aligned}
\|fg\|_{L^2((0,1)\times M)} &\leq C(\min(\lambda,\mu))^s \int_{\tau,\sigma} \|\widehat{F}(\tau)\|_{L^2(M)}\|\widehat{G}(\sigma)\|_{L^2(M)} d\tau d\sigma \\
&\leq C(\min(\lambda,\mu))^s \|\langle\tau\rangle^b \widehat{F}(\tau)\|_{L^2(\mathbb{R}_\tau\times M)}\|\langle\sigma\rangle^b \widehat{G}(\sigma)\|_{L^2(\mathbb{R}_\sigma\times M)} \quad (3.7) \\
&= C(\min(\lambda,\mu))^s \|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)}
\end{aligned}
$$

Finally, by decomposing $f(t) = \sum_{n\in\mathbb{Z}} \psi(t-\frac{n}{2})f(t)$ and $g(t) = \sum_{n\in\mathbb{Z}} \psi(t-\frac{n}{2})g(t)$ with a suitable $\psi \in C_0^\infty(\mathbb{R})$ supported in (0,1), the general case for $f(t)$ and $g(t)$ follows from the considered particular case of $f(t)$ and $g(t)$ supported in time in the interval $(0,1)$. Thus (3.5) implies (3.6). □

A similar proof for the gradient bilinear estimates should refer to Anton [3].

Lemma 3.6. *Let $s \in \mathbb{R}$. The following statements are equivalent:*
(1) For any u_0 , $v_0 \in L^2(M)$ satisfying

$$1_{\lambda\leq\sqrt{-\triangle}\leq 2\lambda} u_0 = u_0 \quad , \quad 1_{\mu\leq\sqrt{-\triangle}\leq 2\mu} v_0 = v_0$$

one has

$$\|(\nabla e^{it\triangle}u_0)\, e^{it\triangle}v_0\|_{L^2((0,1)_t\times M)} \leq C\lambda(\min(\lambda,\mu))^s \|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)} \qquad (3.8)$$

(2)For any $b > \frac{1}{2}$ and any $f, g \in X^{0,b}(\mathbb{R}\times M)$ satisfying

$$1_{\lambda\leq\sqrt{-\triangle}\leq 2\lambda} f = f \quad , \quad 1_{\mu\leq\sqrt{-\triangle}\leq 2\mu} g = g$$

one has

$$\|(\nabla f)g\|_{L^2(\mathbb{R}\times M)} \leq C\lambda(\min(\lambda,\mu))^s \|f\|_{X^{0,b}(\mathbb{R}\times M)}\|g\|_{X^{0,b}(\mathbb{R}\times M)} \qquad (3.9)$$

Denote by $S(t) = e^{it\triangle}$ the free evolution. Using the Duhamel formula , we know that to solve is equivalent to solve the integral equation

$$u(t) = S(t)u_0 - i\alpha \int_0^t S(t-\tau)\{|u(\tau)|^2 u(\tau)\} d\tau$$

To deal with it , we need the following lemmas:

Lemma 3.7. *Let b , $s > 0$ and let $u_0 \in H^s(M)$. Then*

$$\|S(t)u_0\|_{X_T^{s,b}} \lesssim T^{\frac{1}{2}-b} \|u_0\|_{H^s} \tag{3.10}$$

Lemma 3.8. *Let $0 < b' < \frac{1}{2}$ and $0 < b < 1 - b'$. Then for all $F \in X_T^{s,-b'}(M)$,*

$$\| \int_0^t S(t-\tau)F(\tau)d\tau \|_{X_T^{s,b}(M)} \lesssim T^{1-b-b'} \|F\|_{X_T^{s,-b'}(M)} \tag{3.11}$$

Lemma 3.9. *For $s > s_0$, there exists $(b, b') \in \mathbb{R}^2$, satisfying*

$$0 < b' < \frac{1}{2} < b \ , \ \ b + b' < 1, \tag{3.12}$$

and $C > 0$ such that for every triple $(u_j), j = 1, 2, 3$ in $X^{s,b}(\mathbb{R} \times M)$

$$\|u_1 u_2 u_3\|_{X^{s,-b'}(\mathbb{R}\times M)} \leq C \prod_{j=1}^{3} \|u_j\|_{X^{s,b}(\mathbb{R}\times M)}. \tag{3.13}$$

Lemma 3.7 is easy to see.

Proof. Let $\varepsilon > 0$ and $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi = 1$ on $(-T-\varepsilon, T+\varepsilon)$. Then $\|S(t)u_0\|_{X_T^{s,b}} \leq \|\varphi(t)S(t)u_0\|_{X^{s,b}} \leq \|\varphi(t)u_0\|_{H^b(\mathbb{R},H^s(M))} \leq cT^{\frac{1}{2}-b}\|u_0\|_{H^s(M)}$.

□

The lemma 3.8 is due to Bourgain [7], we also refer to Ginibre [13] for a simpler proof.

The proof of lemma 3.9 will rely on the bilinear estimates (3.6) and (3.9). However we will postpone this proof and see how can we proof theorem 3.2 by these there lemmas first.

Proof. (of Theorem 3.2)To solve NLS equation is equivalent to solve the integral equation with Dirichlet (or Neumann) boundary conditions

$$u(t) = S(t)u_0 - i\alpha \int_0^t S(t-\tau)\{|u(\tau)|^2 u(\tau)\}d\tau$$

We denote by $\Phi(u)$ by the left hand side of the equation.

Consider $(b, b') \in \mathbb{R}^2$ given by lemma 3.8 and let $R > 0$ and $u_0 \in H^s(M)$ such that $\|u_0\|_{H^s} \leq R$. We show that there exists $R' > 0$ and $0 < T < 1$ depending on R such that Φ is a contracting map from the ball $B(0, R') \subset X_T^{s,b}(M)$ onto itself.

From the linear estimate (3.10) we know that $\|S(t)u_0\|_{X_1^{s,b}(M)} \leq c\|u_0\|_{H^s}$. From the definition of $X_T^{s,b}$ spaces we know that $T_1 < T_2$ implies $X_{T_2}^{s,b} \subset X_{T_1}^{s,b}$. Therefore for $T < 1$, $\|S(t)u_0\|_{X_T^{s,b}(M)} \leq c_0\|u_0\|_{H^s}$.

Define $R' = 2c_0R$. From estimates (3.11), we obtain for $T < 1$,

$$\|\Phi(u)\|_{X_T^{s,b}(M)} \leq c_0\|u_0\|_{H^s} + c_1 T^{1-b-b'}\|u\bar{u}u\|_{X_T^{s,-b'}(M)}$$

Combine this with (3.13) gives

$$\|\Phi(u)\|_{X_T^{s,b}(M)} \leq c_0\|u_0\|_{H^s} + c_2 T^{1-b-b'}\|u\|^3_{X_T^{s,b}(M)}.$$

Taking $T < 1$ such that $T^{1-b-b'}c_2R'^3 \leq c_0R$, we ensure $\Phi : B(0, R') \subset X_T^{s,b} \to$

$B(0,R') \subset X_T^{s,b}$. In addition Φ is a contraction, let $u_1, u_2 \in B(0,R') \subset X_T^{s,b}$, then

$$\|\Phi(u_1) - \Phi(u_2)\|_{X_T^{s,b}(M)} \leq c_2 T^{1-b-b'} \||u_1|^2 u_2 - |u_2|^2 u_1\|_{X_T^{s,b}(M)}.$$

Use the decomposition $|u_1|^2 u_1 - |u_2|^2 u_2 = u_1^2(\overline{u}_1 - \overline{u}_2) + \overline{u}_2(u_1 - u_2)(u_1 + u_2)$,(3.11) and (3.13), we get

$$\|\Phi(u_1) - \Phi(u_2)\|_{X_T^{s,b}(M)} \leq c_3 T^{1-b-b'} R'^2 \|u_1 - u_2\|_{X_T^{s,b}(M)}.$$

By choosing $T < 1$ sufficient small , we know Φ is a contraction. Thus there exists an uniqueness $u \in X_T^{s,b}(M)$ such that $\Phi(u) = u$. Since $b > \frac{1}{2}$, $u \in C((-T,T), H^s(M))$.

The flow $u_0 \in B(0,R) \subset H^s(M) \to u \in X_T^{s,b}(M)$ is Lipschitz. For if u , v are two solutions with initial data u_0 , v_0, we have as above

$$\|u - v\|_{X_T^{s,b}} \leq c\|u_0 - v_0\|_{H^s} + c_3 T^{1-b-b'} R'^2 \|u - v\|_{X_T^{s,b}}.$$

By choosing T small enough , we have

$$\|u - v\|_{X_T^{s,b}} \leq c\|u_0 - v_0\|_{H^s}$$

□

3.3 Nonlinear Analysis

Now we only owe to prove Lemma 3.9. We will use a decomposition of the spectrum of functions $u_j \in X^{s,b}(\mathbb{R} \times M)$.

The duality argument leads to the following equivalence: $u \in X^{s,b}(\mathbb{R} \times M)$, $\Leftrightarrow$ for all $u_0 \in X^{\infty,\infty}(\mathbb{R} \times M) = \cap_{s>0, b\in\mathbb{R}} X^{s,b}(\mathbb{R} \times M)$ we have

$$|< u, u_0 >| \leq c\|u_0\|_{X^{-s,-b}(\mathbb{R}\times M)}$$

where $<,>$ denote the bracket pairing $\mathcal{S}'$ and $\mathcal{S}$. Thus (3.13) is implied by

$$|\int_{\mathbb{R}}\int_{M} u_0 u_1 u_2 u_3 dxdt| \leq c \prod_{j=1}^{3} \|u_j\|_{X^{s,b}(\mathbb{R}\times M)} \|u_0\|_{X^{-s,b'}(\mathbb{R}\times M)} \tag{3.14}$$

holding for all $u_0 \in X^{\infty,\infty}(\mathbb{R}\times M)$. We will prove a similar result for spectrally localized functions and then sum over all frequencies.

For $j \in \{0,1,2,3\}$ and $N_j \in 2^{\mathbb{N}}$. We denote by $u_{jN_j} = 1_{\sqrt{-\triangle}\in[N_j,2N_j]} u_j$. Using the definition of $X^{s,b}(\mathbb{R}\times M)$ spaces the following equivalence holds

$$\|u_j\|^2_{X^{s,b}(\mathbb{R}\times M)} \cong \sum_{N_j\in 2^{\mathbb{N}}} \|u_{jN_j}\|^2_{X^{s,b}(\mathbb{R}\times M)} \cong \sum_{N_j\in 2^{\mathbb{N}}} N_j^{2s} \|u_{jN_j}\|^2_{X^{0,b}(\mathbb{R}\times M)}. \tag{3.15}$$

We denote by $\underline{N} = (N_0, N_1, N_2, N_3)$ the quadruple of 2^n numbers, $n \in \mathbb{N}$. Also

$$I(\underline{N}) = \int_{\mathbb{R}\times M} \prod_{i=0}^{3} u_{jN_j} dxdt$$

We will establish the two estimates about $I(\underline{N})$ in the following lemma by using (3.6) and (3.9) respectively. With aid of these two estimates,we can built Lemma 3.9.

We also need the fact that

$$\|f\|_{L^4(\mathbb{R},L^2(M))} \leq \|f\|_{X^{0,\frac{1}{4}}(\mathbb{R}\times M)}. \tag{3.16}$$

This is due to conservation of L^2 norm by the linear Schrödinger flow and Sobolev embedding $H^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$, thus

$$\|f\|_{L^4(\mathbb{R},L^2(M))} = \|e^{it\triangle} f\|_{L^4(\mathbb{R},L^2(M))} \leq \|e^{it\triangle} f\|_{H^{\frac{1}{4}}(\mathbb{R}\times L^2(M))} = \|f\|_{X^{0,\frac{1}{4}}(\mathbb{R}\times M)}.$$

Lemma 3.10. *If (3.5) and (3.8) hold for $s > s_0$, then for all $s' > s_0$ there exists*

$0 < b' < \frac{1}{2}$, $c > 0$ *such that, assuming* $N_3 \le N_2 \le N_1$, *the following estimates hold:*

$$|I(\underline{N})| \le c(N_2N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}(\mathbb{R}\times M)} \tag{3.17}$$

$$|I(\underline{N})| \le c(\frac{N_1}{N_0})^2 (N_2N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}(\mathbb{R}\times M)} \tag{3.18}$$

Proof. Use Holder inequality, we get

$$|I(\underline{N})| \le \|u_{3N_3}\|_{L^4(L^\infty_x)} \|u_{2N_2}\|_{L^4(L^\infty_x)} \|u_{1N_1}\|_{L^4(L^2_x)} \|u_{0N_0}\|_{L^4(L^2_x)}$$

$$\le c(N_2N_3)^{1+\varepsilon} \prod_{j=0}^{3} \|u_{jN_j}\|_{L^4(L^2_x)}$$

$$\le c(N_2N_3)^{1+\varepsilon} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,\frac{1}{4}}(\mathbb{R}\times M)} \tag{3.19}$$

In the second inequality, we use Sobolev embedding $\|u_{N_j}\|_{L^\infty(M)} \le cN_j^{1+\varepsilon}\|u_{N_j}\|_{L^2(M)}$. The third inequality came from (3.16).

Use Cauchy inequality and (3.6)(which is implied by (3.5)), we obtain that for any $b_0 > \frac{1}{2}$ there exists $c_0 > 0$ such that

$$|I(\underline{N})| \le \|u_{0N_0}u_{2N_2}\|_{L^2(\mathbb{R}\times M)} \|u_{1N_1}u_{3N_3}\|_{L^2(\mathbb{R}\times M)}$$

$$\le c_1(N_2N_3)^{s_0} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b_0}(\mathbb{R}\times M)} \tag{3.20}$$

We need further decomposition $u_{jN_j} = \sum_{K_j} u_{jN_jK_j}$ for interpolation, where $u_{jN_jK_j} = 1_{K_j \le \langle i\partial_t + \triangle\rangle \le 2K_j} u_{jN_j}$ and the sum is taken over 2^n numbers , for $n \in$

$\mathbb{N} : K_j \in 2^{\mathbb{N}}$. Let us denote $I(\underline{N}, \underline{K}) = \int_{\mathbb{R}\times M} \prod_{j=0}^{3} u_{jN_jK_j}$. Estimates (3.19) and (3.20) give

$$|I(\underline{N}, \underline{K})| \leq c(N_2N_3)^{\alpha}(\prod_{j=0}^{3} K_j)^{\beta} \prod_{j=0}^{3} \|u_{jN_jK_j}\|_{L^2(\mathbb{R}\times M)}$$

where (α, β) equals $(1+\varepsilon, \frac{1}{4})$ or (s_0, b_0). For $s_0 < s < 1$ we can choose $\varepsilon > 0$, $b_0 > \frac{1}{2}$ and $0 < b_1 < \frac{1}{2}$ such that by interpolation we have the same estimates for $(\alpha, \beta) = (s', b_1)$.

Taking $b' \in (b_1, \frac{1}{2})$, this reads

$$|I(\underline{N}, \underline{K})| \leq c(N_2N_3)^{s'} \prod_{j=0}^{3} K_j^{b_1-b'} \|u_{jN_jK_j}\|_{X^{0,b'}(\mathbb{R}\times M)}.$$

Summing up over $\underline{K} \in (2^{\mathbb{N}})^4$, by geometric series and using Cauchy Schwartz, we obtain

$$|I(\underline{N})| \leq c(N_2N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}(\mathbb{R}\times M)}$$

which conclude the proof of (3.17).

For the proof of (3.18), we start with Green formula:

$$\int_M \triangle f g - f \triangle g dx = \int_{\partial M} \frac{\partial f}{\partial v} g - f \frac{\partial g}{\partial v} d\sigma$$

If e_k are eigenfunctions of the Dicichlet(or Neumann) Laplacian assoiated with eigenvalues λ_k^2. The $u_{0N_0} = \sum_{\lambda_k \sim N_0} c_k e_k$, where $c_k = (u_{0N_0}, e_k)$. We write

$$u_{0N_0} = -\frac{\Delta}{N_0^2} \sum_{\lambda_k \sim N_0} c_k (\frac{N_0}{\lambda_k})^2 e_k.$$

Define $Tu_{0N_0} = \sum_{\lambda_k \sim N_0} c_k (\frac{N_0}{\lambda_k})^2 e_k$ and $Vu_{0N_0} = \sum_{\lambda_k \sim N_0} c_k (\frac{\lambda_k}{N_0})^2 e_k$. Then we have $TVu_{0N_0} = VTu_{0N_0} = u_{0N_0}$ and $\|Tu_{0N_0}\|_{H^s} \sim \|u_{0N_0}\|_{H^s}$ for all s. Use this notation

$$u_{0N_0} = -\frac{\Delta}{(N_0)^2} Tu_{0N_0}.$$

Apply it to green formula and using $u_{jN_j}|_{\partial M} = 0$ (or $N_x \cdot \nabla u|_{\partial M} = 0$), we obtain

$$I(\underline{N}) = \frac{1}{N_0^2} \int_{\mathbb{R}\times M} T u_0 N_0 \Delta(u_{1N_1} u_{2N_2} u_{3N_3})$$

By Leibniz's law, we have to deal with summation of terms of the forms

$$\frac{1}{N_0^2} J_{11}(\underline{N}) = \frac{1}{N_0^2} \int_{\mathbb{R}\times M} T u_{0N_0} (\Delta u_{1N_1}) u_{2N_2} u_{3N_3}$$

and

$$\frac{1}{N_0^2} J_{12}(\underline{N}) = \frac{1}{N_0^2} \int_{\mathbb{R}\times M} T u_{0N_0} (\nabla u_{1N_1})(\nabla u_{2N_2}) u_{3N_3}.$$

As we will see soon, they are always the largest terms in each sum. Use $\triangle u_{2N_2}$ we get $J_{11}(\underline{N}) = -N_1^2 \int_{\mathbb{R}\times M} T u_{0N_0} V u_{1N_1} u_{2N_2} u_{3N_3}$. Thus by (3.17) and $\|u_{jN_j}\|_{H^s} \sim \|T u_{jN_j}\|_{H^s} \sim \|V u_{jN_j}\|_{H^s}$, we have

$$\frac{1}{N_0^2} |J_{11}(\underline{N})| \leq c \frac{N_1^2}{N_0^2} (N_2 N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}} (\mathbb{R} \times M).$$

To estimates $J_{12}(\underline{N})$, we note that $\|\nabla u_{jN_j}\|_{L^2(M)} \leq c N_j \|u_{jN_j}\|_{L^2(M)}$. Use the same process as in the proof of (3.17), then (3.19) and (3.20) correspond to

$$|J_{12}(\underline{N})| \leq c (N_1 N_2)(N_2 N_3)^{1+\varepsilon} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,\frac{1}{4}}(\mathbb{R})\times M}$$

and

$$|J_{12}(\underline{N})| \leq c (N_1 N_2)(N_2 N_3)^{s_0} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b_0}(\mathbb{R})\times M}.$$

In fact, we just got an additional term $N_1 N_2$ in these new estimates. Therefore the interpolation argument leads to

$$\frac{1}{N_0^2} |J_{12}(\underline{N})| \leq c \frac{N_1 N_2}{N_0^2} (N_2 N_3)^{s'} \prod_{j=0}^{3} \|u_{jN_j}\|_{X^{0,b'}} (\mathbb{R} \times M).$$

Since $N_1 N_2 \leq N_1^2$, we are done. □

Now we can use Lemma 3.10 to prove Lemma 3.9.

Proof. (Proof of Lemma 3.9)

Our goal is to prove (3.14). Use the same notation as above, we consider $I(\underline{N}) = \int_{\mathbb{R}\times M}\prod_{i=0}^{3} u_{jN_j}dxdt$. Without loss of generality, we may assume $N_3 \le N_2 \le N_1$.

Let $\frac{2}{3} < s' < s$. Using (3.17) in Lemma 3.10 and (3.15), we have

$$|\sum_{N_0<cN_1} I(\underline{N})| \le c \sum_{N_0<cN_1} (N_2N_3)^{s'-s}(\frac{N_0}{N_1})^s \|u_{0N_0}\|_{X^{-s,b'}\mathbb{R}\times M} \prod_{j=1}^{3} \|u_{jN_j}\|_{X^{s,b'}(\mathbb{R}\times M)}.$$

Using Cauchy Schwartz inequality and (3.15), we have

$$|\sum_{N_0<cN_1} I(\underline{N})| \le c\|u_2\|_{X^{s,b'}(\mathbb{R}\times M)}\|u_3\|_{X^{s,b'}(\mathbb{R}\times M)} \sum_{N_0\le CN_1} (\frac{N_0}{N_1})^s\alpha(N_0)\beta(N_1).$$

where $\alpha(N_0) = \|u_{0N_0}\|_{X^{-s,b'}(\mathbb{R}\times M)}$ and $\beta(N_1) = \|u_{1N_1}\|_{X^{s,b'}(\mathbb{R}\times M)}$. Thus we have

$$\textstyle\sum_{N_0}\alpha(N_0)^2 \cong \|u_0\|^2_{X^{-s,b'}} \ , \ \sum_{N_1}\beta(N_1)^2 \cong \|u_1\|^2_{X^{s,b'}}.$$

Since N_0 , N_1 are both dyadic numbers, we write $N_1 = 2^l N_0$ and $N_0 \ge N(l) = \max(1, 2^{-l})$, where l is an integer, $l \ge -l_0$ for some $l_0 \in \mathbb{N}$ depending on c. Thus

$$\sum_{N_0<cN_1} (\frac{N_0}{N_1})^s\alpha(N_0)\beta(N_1) = \sum_{l\ge -l_0}\sum_{N_0\ge N(l)} 2^{-sl}\alpha(N_0)\beta(2^l N_0)$$

$$\le \sum_{l>-l_0} 2^{-sl}(\sum_{N_0}\alpha(N_0)^2)^{\frac{1}{2}}(\sum_{N_0>N(l)}\beta(2^lN_0)^2)^{\frac{1}{2}}$$

$$\le c\|u_0\|_{X^{-s,b'}(\mathbb{R}\times M)}\|u_1\|_{X^{s,b'}(\mathbb{R}\times M)}$$

Since $\|u\|_{X^{s,b'}} \le \|u\|_{X^{s,b}}$ for $b' < b$, we conclude that

$$\textstyle|\sum_{N_0<cN_1} I(\underline{N})| \le c\|u_0\|_{X^{-s,b'}} \prod_{j=1}^{3} \|u_j\|^{X^{s,b}}.$$

For $N_0 \geq cN_1$, we use (3.18) of Lemma 3.10 to get:

$$|\sum_{N_0 \geq cN_1} I(\underline{N})| \leq c \sum_{N_0 \geq cN_1} (N_2 N_3)^{s'-s} (\frac{N_1}{N_0})^{2-s} \|u_{0N_0}\|_{X^{-s,b'}\mathbb{R}\times M} \prod_{j=1}^{3} \|u_{jN_j}\|_{X^{s,b'}(\mathbb{R}\times M)}.$$

This is just an exchange the role of N_0 and N_1 in the previous argument. Thus we obtain again

$$|\sum_{N_0 \geq cN_1} I(\underline{N})| \leq c\|u_0\|_{X^{-s,b'}(\mathbb{R}\times M)} \|u_1\|_{X^{s,b'}(\mathbb{R}\times M)} \|u_2\|_{X^{s,b'}(\mathbb{R}\times M)} \|u_3\|_{X^{s,b'}(\mathbb{R}\times M)}$$

□

3.4 Proof of Theorem 3.3

In this section, we will make use of Strichartz estimates obtained by Blair, Smith and Sogge [5] and Sobolev imbedding theorem to built the well-posed property in 3 dimensional manifolds with boundary.

Theorem 3.11. [5]. *Let (M,g) be either a smooth compact Riemannian manifold with boundary, or a manifold without boundary equipped with Lipschitz metric g. Then the following Strichartz estimate holds for any Strichartz pair*

$$\|e^{it\triangle} f\|_{L^p([-T,T];L^q(M))} \leq C(p,T)\|f\|_{H^{\frac{4}{3p}}(M)}$$

By using Minkowski inequality, we also get

Corollary 3.12. *Let (M,g) as Theorem 8.1, $f \in L^1([-T,T],\ H^{\frac{4}{3p}}(M))$ then*

$$\|\int_{-T}^{t} e^{i(t-\tau)\triangle} f(\tau) d\tau\|_{L^p([-T,T],L^q(M))} \leq C(p,T)\|f\|_{L^1([-T,T],H^{\frac{4}{3p}}(M))}$$

Proof. The left hand side equals

$$I = \| \int_{-T}^{T} F_\tau d\tau \|_{L^p([-T,T],L^q(M))} \ , \ F_\tau(t) = 1_{\tau \le t} e^{i(t-\tau)\Delta} f(\tau).$$

then we have

$$\begin{aligned} I &\le \int_{-T}^{T} \|F_\tau\|_{L^p([-T,T],L^q(M))} d\tau \\ &\le \int_{-T}^{T} \|e^{i(t-\tau)\Delta} f(\tau)\|_{L^p([-T,T],L^q(M))} d\tau \\ &\lesssim \int_{-T}^{T} \|f(\tau)\|_{H^{\frac{4}{3p}}} d\tau \qquad (3.21) \end{aligned}$$

□

Use Sobolev imbedding theorem, we get the following

Corollary 3.13. *Let $2 < p < \infty$ and dim M=3, $s > \frac{7}{6}$. For any $u_0 \in H^s(M)$ and $f \in L^1([-T,T], H^s(M))$.*

$$\|e^{it\triangle} u_0\|_{L^p([-T,T],L^\infty(M))} \le C(p,T) \|u_0\|_{H^s(M)}$$

$$\| \int_0^t e^{i(t-\tau)\triangle} f(\tau) d\tau \|_{L^p([-T,T],L^\infty(M))} \le C(p,T) \|f\|_{L^1([-T,T],H^s(M))}$$

Proof. Let $s > \frac{3}{2} - \frac{2}{3p} > \frac{7}{6}$, $\sigma = s - \frac{4}{3p} > \frac{3}{2} - \frac{2}{p} = \frac{3}{q}$.

$$\begin{aligned}
\|e^{it\triangle}u_0\|_{L^p([-T,T],L^\infty(M))} &\lesssim \|e^{it\triangle}u_0\|_{L^p([-T,T],W^{\sigma,q})} \\
&= \|(I-\triangle)^{\frac{\sigma}{2}}e^{it\triangle}u_0\|_{L^p([-T,T],L^q)} \\
&\leq C(p,T)\|u_0\|_{H^{\sigma+\frac{4}{3p}}(M)} = C(p,T)\|u_0\|_{H^s(M)}
\end{aligned}$$

□

In order to prove theorem 3.3 , we also need the following lemma

Lemma 3.14. [1] *If u , $v \in L^\infty \cap H^s$ $(s > 0)$,then so is uv, and*

$$\|uv\|_{H^s} \lesssim \|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty}$$

For the convenience of readers, we restate Theorem 3.3 here.

Theorem 3.3. *If (M,g) is a 3 dimensional smooth compact Riemannian manifold with boundary, then the Cauchy problem (3.1) is uniformly local well-posed in $H^s(M)$ for $s > \frac{7}{6}$.*

Proof. Let $Y_T = C([-T,T],H^s(M)) \cap L^p([-T,T],L^\infty(M))$, where $s > \frac{6}{7}$ is a fixed number and $p > 2$. This is a complete Banach space for the following norm

$$\|u\|_{Y_T} = \max_{|t|\leq T}\|u(t)\|_{H^s} + \|u\|_{L^p([-T,T];L^\infty)}.$$

We will use the contraction mapping theorem to prove the existence and uniqueness of the local solution. According to Duhamel formula, we need to prove the functional

$$\Phi(u)(t) = e^{it\triangle}u_0 - i\int_0^t e^{i(t-\tau)\triangle}(|u^2|u)(\tau)d\tau.$$

is a contraction mapping on some ball of Y_T centered at the origin.

$$\begin{aligned}
\|\Phi(u)(t)\|_{H^s} &\lesssim \|u_0\|_{H^s} + \int_0^T \||u(\tau)|^2u(\tau)\|_{H^s}d\tau \\
&\lesssim \|u_0\|_{H^s} + c\int_0^T \|u(\tau)\|_{L^\infty}^2\|u(\tau)\|_{H^s}d\tau \\
&\lesssim \|u_0\|_{H^s} + cT^{1-\frac{2}{p}}\|u\|_{L^p(L^\infty)}^2\|u\|_{L_T^\infty(H^s)} \\
&\lesssim \|u_0\|_{H^s} + cT^{1-\frac{2}{p}}\|u\|_{Y_T}^3
\end{aligned}$$

We used lemma 3.14 in the second inequality, and Hölder inequality in the third inequality.

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^p(L^\infty)} &\lesssim \|e^{it\triangle}u_0\|_{L^p(L^\infty)} + \|\int_0^T e^{i(t-\tau)\triangle}|u(\tau)|^2u(\tau)d\tau\|_{L^p(L^\infty)} \\
&\lesssim \|u_0\|_{H^s} + c\int_0^T \||u(\tau)|^2u(\tau)\|_{H^s}d\tau \\
&\lesssim \|u_0\|_{H^s} + cT^{1-\frac{2}{p}}\|u\|_{L^p(L^\infty)}^2\|u\|_{L_T^\infty(H^s)} \\
&\lesssim \|u_0\|_{H^s} + cT^{1-\frac{2}{p}}\|u\|_{Y_T}^3
\end{aligned}$$

Therefore

$$\|\Phi(u)\|_{Y_T} \leq c(\|u_0\|_{H^s} + T^{1-\frac{2}{p}}\|u\|_{Y_T}^3) \tag{3.22}$$

Similarly, we have

$$\begin{aligned}\|\Phi(u)(t) - \Phi(v)(t)\|_{H^s} &\leq \int_0^T \||u(\tau)|^2 u(\tau) - |v(\tau)|^2 v(\tau)\|_{H^s} d\tau \\ &\leq cT^{1-\frac{2}{p}}\|u-v\|_{Y_T}(\|u\|_{Y_T}^2 + \|v\|_{Y_T}^2) \end{aligned} \tag{3.23}$$

and

$$\|\Phi(u) - \Phi(v)\|_{L^pL^\infty} \leq cT^{1-\frac{2}{p}}\|u-v\|_{Y_T}(\|u\|_{Y_T}^2 + \|v\|_{Y_T}^2) \tag{3.24}$$

The existence of the solution can be obtained by (3.23), (3.24) and (3.25). □

4 Future Research Goals

My research goals include the following directions:

(1)Study the bilinear Strichartz estimates and their applications in higher dimensional manifolds.

In the 3 dimensional manifold , Burq, Gérard and Tzvetkov [10] proved the local well posed of (1.1) in $H^s(M)$ for $s \geq 1$ on manifold without boundary, then extended it to global result for defocusing case by conservation of energy. For manifold with boundary, Anton [3] proved the local well posed property in $H^s(M)$ on the ball with Dirichlet boundary condition and radial data for $s > \frac{1}{2}$. While in [2], she proved the global well posed property on the exterior of non-trapping domains for $s = 1$.

One of the difficulty of building desirable bilinear Strichartz estimates in 3 or higher dimensional manifold is that $(p, q) = (4, 4)$ is not admissible for $n \geqslant 3$. Hence the higher frequency will not disappear in the bilinear estimates like $n = 2$. It will be interesting to find a method to overcome this issue.

(2)Study the optimality of Strichartz estimates by Burq,Gérard and Tzvetkov [11] for manifold without boundary, by Blair,Smith and Sogge [5] for manifolds with boundary then extend the possible results to bilinear case.

In Encliden space, one can take $T = \infty$ and $s = 0$ in Strichartz estimates (1.2); see for example Strichartz [22], Ginibre and Velo [12], Keel and Tao [16] and references therein. Such estimates have been a key tool in the study of non-linear Schrödinger equations. In a compact manifold (M, g) without boundary Burq, Gérard and Tzvetkov [10] proved the finite time scale estimates (1.2) for the Schrödinger operators with a loss of derivatives $s = \frac{1}{p}$ in their estimates when compared to the case of flat geometries. Blair,Smith and Sogge [5] adopted the same idea to manifolds with boundary case and built the estimates with a loss of derivatives $s = \frac{4}{3p}$. However,these results are not sure to be optimal. I will like to study these problems. Also recently Ivanovici [15] proved the Strichartz estimates without loss of derivatives ($s = 0$) outside strictly convex obstacles. I will like to extend her result to bilinear case and find its applications.

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