EMBEDDED MINIMAL SPHERES IN 3-MANIFOLDS

by

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Abstract

Consider the following question: Does there exist a three-manifold M for which, given any riemannian metric on M there is an area bound for embedded minimal spheres? We answer this question negatively, and, in fact, find an open set of metrics for which this question is false. This generalizes similar results for surfaces of positive genus shown by W. Minicozzi, T. Colding and B. Dean. In addition, this paper provides some background on minimal surface theory related to the main theorem, as well as Colding and Minicozzi's proof for the torus. Some further directions for research are discussed in chapter 5.

To the reader. Keep doin' those exercises.

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Contents

List of Figures

Chapter 1

Introduction

From its sudsy beginnings as a geometric experiment by J. Plateau, the study of minimal surfaces has had a wide impact on the mathematical community. It was in the famous paper by Lebesgue ([Leb02]) which introduced the modern theory of integration which first referred to the problem of finding a minimal surface with given boundary data as the "problem of Plateau." Minimal surfaces are involved in the science of protein folding and nanotechnology, as well as the study of black holes. It was even in [SY79b] where R. Schoen and S. Yau provided the first known proof of the long conjectured positive mass theorem using minimal surface theory.

Although the plateau problem was finally solved independently in by J. Douglas in [Dou31] and T. Rado in [Rad33], for which Douglas was later awarded the fields medal for this work, existence problems for minimal surfaces is still an active area. For instance, in 1979, work by R. Schoen, S. Yau, M. Freedman, J. Hass, and P. Scott showed that, given an incompressible surface Σ in a mean convex manifold M, there exists an embedded minimal manifold which is close to Σ in terms of the image of the fundamental groups under injection. W. Meeks, L. Simons, and S. Yau have shown the existence of area minimizers for homotopy classes, allowing for some surgery. These particular examples are discussed in detail in chapters 2 and 3.

An existence problem of minimal surfaces which has been of recent interest has been to find examples of closed embedded minimal surfaces of arbitrarily large area. That is, given a manifold M, can one find a metric on M with the property that there is a sequence of surfaces Σ_n such that $\text{Area}(\Sigma_n) \to \infty$. This problem was originally discussed in [CM99a] where T. Colding and W. Minicozzi showed that there is, in fact, an open set of metrics on an arbitrary manifold M such that there is a sequence of tori with unbounded area for each metric in this set. In [Dea03], B. Dean showed that an analogous theorem held true for positive genus surfaces. These results also showed that it was possible to choose the sequence and set of metrics such that the sequence would also be stable.

The approaches to building the positive genus surfaces in [CM99a] and [Dea03] followed a simple structure. One considers a compact three-manifold with boundary which is thought of as a subset of the unit ball and thus any manifold. A sequence of surfaces is then constructed such that the area, allowing for certain variance of the surface, seems to approach infinity. The aforementioned result by Schoen, Yau, Freedman, Hass, and Scott (see theorem 2.3.2) is then used to show that there are stable embedded minimal surfaces which closely approximate each element in the sequence. These surfaces are then shown to have unbounded area. The genus zero case has remained an open problem since then, and appeared as a question in [CM04]. The main problem for generalizing this proof to the sphere is the fact that the fundamental group of a sphere is trivial. Specifically, the result of [SY79a] used in these constructions is only applicable if the embedded surface is not simply connected.

A notable result which solves a weaker version of the genus zero case is in the work of J. Hass, P. Norbury, and J. H. Rubinstein (see [HNR03]). In that paper, the authors were able to construct a metric on an arbitrary manifold M with a sequence of embedded minimal spheres of unbounded morse index. This necessarily implies unbounded area, however obtaining an open set of metrics from this example is not possible. It should also be noted here that the works of T. Colding and N. Hingston have also shown that there is a metric on any manifold for which there exist embedded minimal tori of arbitrary large morse index [CH03].

In this paper, we complete the genus zero case. The main result will be the following theorem

Theorem 1.0.1. *Let M^3 be a three-manifold (possibly with boundary). There exists* *an open, nonempty set of metrics on M for each of which there are stable embedded minimal two-spheres of arbitrarily large area. Here we use the C^2-topology on the space of metrics on a manifold.*

It should noted that the construction given in [HNR03] differs significantly from that presented here.

The appropriate replacement for theorem 2.3.2 is is a result by W. Meeks III, L. Simon, and S. T. Yau (see [MSY82]) in which one can find a set of closed stable embedded minimal surfaces which is obtained from the original by pinching off parts of the surface and varying each isotopically. This is discussed in detail in section 4.2. This result has the advantage of applying to the sphere and at the same time finds an actual minimizer in a isotopy class which is close in a geometrically intuitive way. A particularly interesting application of this result is a stronger version of theorem 2.3.2 is also presented in corollary 3.3.7.

In contrast of the main theorem, it should be noted that in [CW83], H. Choi and A. Wang proved that there is an open set of metrics, namely those with positive ricci curvature, for which there is a uniform area bound on compact embedded minimal surfaces depending only on the genus and the ambient metric. This set of metrics also has the property that there are no stable minimal surfaces, embedded or otherwise. The author also suspects that it may be possible to find an area bound for surfaces of a fixed genus for all but finitely many genera provided that the ambient manifold is diffeomorphic to the unit ball. See chapter 5 for more details.

In chapter 2, we discuss some basic results in minimal surface theory. This begins with the derivation of the graphical and geometric versions of the first fundamental form as well as some examples of minimal surfaces. We then discuss the stability operator and a few terms and results from the definition. We conclude this chapter with some existence theorems relevant to the results in this paper, with the notable exception of the theorem of Meeks, Simon and Yau, which is discussed in section 4.2.

In chapter 3 we provide Colding and Minicozzi's proof of the torus case. We also discuss the existence result of Meeks, Simon and Yau and provide an interesting corollary. Additionally, in this section, we show some results, due to the author which are used in chapter 4.

In chapter 4 we prove the main theorem. The method followed is related to, but noticeably different to the method provided in chapter 3. It should be noted that the result is mainly topological, as most of the regularity is shown in [MSY82].

Chapter 2

Background

In this chapter, we provide some basic minimal surface results. In particular we derive two forms of the minimal surface equation and discuss the concept of stability. In the final section, we discuss two important existence results relevant to the rest of the paper.

2.1 The First Variation Formula for Graphs

We will start by deriving the minimal surface equation for a graph, equation 2.1.3. Let $\Omega \subseteq \mathbb{R}^n$. We define the volume functional Vol over Ω as

$$\mathrm{Vol}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \tag{2.1.1}$$

Let $u \in C^2(\Omega)$ and assume $Vol(u) < \infty$. If u minimizes volume over all functions $\bar{u} \in C^2(\Omega)$ such that $\bar{u} \equiv u$ on $\partial\Omega$, u can be considered to be a critical point of Vol. Let $v \in C^2(\Omega)$ be compactly supported and assume $v \equiv 0$ on $\partial\Omega$. Consider $u_t = u + tv$. Since u minimizes volume, $\frac{\partial}{\partial t}\big|_{t=0} Vol(u_t) = 0$, and

$$\begin{aligned} 0 = \frac{\partial}{\partial t}\bigg|_{t=0} Vol(u_t) = \frac{\partial}{\partial t}\bigg|_{t=0} \int_\Omega \sqrt{1 + |\nabla(u + tv)|^2} = \int_\Omega \frac{\langle \nabla u, \nabla v \rangle}{\sqrt{1 + |u|^2}} \\ = -\int_\Omega v \,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \end{aligned} \tag{2.1.2}$$

where we used integration by parts and the fact that $v \equiv 0$ on $\partial\Omega$ in the last equality. Since v is arbitrary, we can conclude

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{2.1.3}$$

This is called the *minimal graph equation* and a solution to the minimal graph equation will be called a *minimal graph.* It should also be noted that if $n = 2$ this equation can be written in non-divergence form as

$$(1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0 \tag{2.1.4}$$

A few remarks. First of all, it can be noted that a graph satisfying the minimal graph equation is only a critical point for the volume functional. However, it can be shown (see [CM99b] for example) that the graph of u indeed minimizes volume amongst all manifolds in the cylinder $\Omega \times \mathbb{R}$ which agree with the graph on the boundary. Also, the minimal graph equation is a uniformly elliptic PDE and solutions to the minimal graph equation have many properties in common with solutions to elliptic PDE's. One such important result is the strong maximum principle

Theorem 2.1.5 (The strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a compact domain assume $u_1, u_2 \in C^2(\Omega)$ are minimal graphs with $u_1 \leq u_2$. If $u_1(x) = u_2(x)$ for some x in the interior of Ω, then $u_1 \equiv_2$*

Using this fact, it can also be shown (see [CM99b]) that if Ω is convex, then the graph of u minimizes volume amongst all surfaces which agree with the graph on the boundary.

The study of minimal surfaces in $\mathbb{R}^3$ is a rich field. In this paper, however, we will concern ourselves more with minimal surfaces in manifolds. Before switching to non-euclidean geometry, however we mention the following classical examples to the minimal graph equation for $\mathbb{R}^2$

1. The plane: $u = ax + by + c$
2. The catenoid: $u = \cosh^{-1}(\sqrt{x^2+y^2})$. First studied by L. Euler in [Eul44]

3. The helicoid: $u = \tan^{-1}(\frac{x_2}{x_1})$. First studied by J. B. Meusnier in [Meu76]

4. Scherk's surface: $u = \log \frac{cos(y)}{cos(x)}$. First studied by H. F. Scherk in [Sch34]

2.2 First and Second Variation for Submanifolds

The notion of volume is not restricted to graphs, but can be extended to differential manifolds as well, so it is natural to discuss the notion of a minimal surface in a differential geometry setting. We start by formulating equation 2.1.3 in terms of the second fundamental form

$$A(X, Y) = \nabla_X^{\perp} Y \tag{2.2.1}$$

where X and Y are tangent vectors to the minimal surface, ∇ is the covariant derivative for the ambient space, and $\nabla^{\perp}$ is the normal part of ∇.

Let M^k and N^n be C^2 manifolds with $n > k$. Let $\varphi : M \times (-\epsilon, \epsilon) \to N$ be a compactly supported C^2 variation of M, that is, φ is C^2, φ_0 is an isometric immersion, and there exists $K \subseteq M$ compact for which $\varphi_t|_{M-K} \equiv \phi_0|_{M-K}$. Let V be the variation vector field $\frac{\partial}{\partial t}\big|_{t=0} \varphi_t$. As tangential variations produce no change in area, it can be assumed that V is purely normal to $\varphi(M)$. Let E_i be an orthonormal frame for M, $E_i(t) = d\varphi_t(E_i)$, and $g_{ij}(t) = \langle E_j(t), E_j(t)\rangle$. Then we have

$$\frac{\partial}{\partial t}\mathrm{Vol}(\varphi_t) = \int_M \frac{\partial}{\partial t}\sqrt{det(g_{ij})} = \int_M \frac{\frac{\partial}{\partial t} g_{ij}}{2\sqrt{g_{ij}}} \tag{2.2.2}$$

To calculate $\frac{\partial}{\partial t} g_{ij}$ we will first prove the following lemma

Lemma 2.2.3. *Let $A(t)$ be a one parameter family of invertible $n \times n$ matrices such that A is differentiable. then*

$$\frac{\partial}{\partial t} det(A) = tr(A' A^{-1}) \det A$$

Proof. We will first show this in the case that $A(0) = I$. Let $A = I + tB$ and $B = (b_{ij})$.

Then

$$\frac{\partial}{\partial t}|_{t=0}\det(A) = \frac{\partial}{\partial t}|_{t=0}\det\begin{pmatrix}1+tb_{11} & \cdots & tb_{1n}\\ \vdots & \ddots & \vdots\\ tb_{n1} & \cdots & 1+tb_{nn}\end{pmatrix}$$
$$= \frac{\partial}{\partial t}|_{t=0}(1+tb_{11})(1+tb_{22})\dots(1+tb_{nn}) = \mathrm{tr}(B) = \mathrm{tr}(A'(0)) \tag{2.2.4}$$

Now, if $A(0)\neq I$ we can apply the above to $A(t)A^{-1}(t_0)$ to obtain the lemma at any time t_0 □

Using this lemma, we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0}\det(g_{ij}) &= \mathrm{tr}(g'_{ij}(0)) = \sum_i \left.\frac{\partial}{\partial t}\right|_{t=0}|E_i|^2 = \sum_i \langle\nabla_V E_i, E_i\rangle_N\\ &= \sum_i \langle\nabla_{E_i}V, E_i\rangle_N = -\sum_i \langle\nabla_{E_i}E_i, V\rangle_N\\ &= -\left\langle \mathrm{tr}(\nabla^{\perp}_{E_i}E_j), V\right\rangle_N = -\langle \mathrm{tr}(A(E_i,E_j), V\rangle\end{aligned} \tag{2.2.5}$$

The third equality comes from the fact that $[V,E_i]=0$ and the sixth is because V is perpendicular to M. We call $H=\mathrm{tr}(A(E_i,E_j))$ the *mean curvature vector*. We can determine

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0}\mathrm{Vol}(\varphi_t(M)) &= \left.\frac{\partial}{\partial t}\right|_{t=0}\int_M\sqrt{\det(g_{ij})} = \int_M \frac{\left.\frac{\partial}{\partial t}\right|_{t=0}\det(g_{ij})}{\sqrt{2\det(g_{ij}(0))}}\\ &= -\frac{1}{2}\int_M\langle H,V\rangle_N\end{aligned} \tag{2.2.6}$$

So, if $\varphi_0(M)$ is a critical point of the area functional, then

$$H = \mathrm{tr}(A(E_i,E_j)) = \mathrm{tr}(\nabla^{\perp}_{E_i}E_j) = 0 \tag{2.2.7}$$

for any orthonormal frame E_i. We will call this the *minimal surface equation for a manifold* and a surface M satisfying this equation will be called a *minimal immersion* or *minimal embedding* when appropriate. The following are some simple examples of minimal surfaces in noneuclidean manifolds.

1. Any great sphere in S^n is totally geodesic, and therefore minimal

2. The Clifford torus in S^4

3. Given a manifold M, M is minimally embedded in the manifold $N = M \times (-\epsilon, \epsilon)$ with the metric $\tilde{g}_N = (1 + \sin^2(t))g_N$ where g_N is the cross product metric.

4. In [Law70], B. Lawson constructs examples of orientable closed minimal surfaces any genus embedded in the round sphere S^3. Moreover, Lawson constructs minimal immersions of closed non-orientable surfaces of any genus with the exception of the projective plane.

It should be noted that of these examples, only (3) appears to be an area minimizer. As before, a minimal C^2 immersion φ is only a critical point of the area functional. To determine when a minimal surface is in fact a local minimum it is necessary to discuss the stability operator. Let M, N, φ, and V be as above. A calculation omitted here will show

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} Vol(\varphi_t(\Sigma)) = -\int_M \left\langle V, \Delta^{\perp}_{\varphi_0(M)} V + \mathrm{Ric}_N(V, E_i)E_i + \tilde{A}(V) \right\rangle_N \tag{2.2.8}$$

where $\tilde{A}$ is the Simons' operator

$$\tilde{A}(V) = \sum_{i,j=1}^{k} \left\langle \nabla^{\perp}_{E_i} E_j, V \right\rangle \nabla^{\perp}_{E_i} E_j \tag{2.2.9}$$

and $\Delta^N_{\varphi_0(M)}$ is the Laplacian on the normal bundle

$$\Delta^N_{\varphi_0(M)} V = \sum_{i=1}^{k} (\nabla_{E_i} \nabla_{E_i} V)^{\perp} + \sum_{i=1}^{k} (\nabla_{(\nabla_{E_i} E_i)^T} V)^{\perp} \tag{2.2.10}$$

We call $L = \Delta^{\perp}_{\varphi_0(M)} + \mathrm{Ric}_N(\cdot, E_i)E_i + \tilde{A}$ the *stability operator*. A case of particular importance is if $\varphi_0(M)$ is two sided in N and $k = n - 1$. We then have a well defined unit normal ν and we can write any normal variation as $V = f\nu$. In this case,

$$Lf = \Delta_{\Sigma} f + |A|^2 f + \mathrm{Ric}_M(\nu, \nu) f \tag{2.2.11}$$

Throughout the rest of this paper we will only concern ourselves with this last version of the stability operator. An eigenvector of L will refer to a solution of the equation

$$Lf + \lambda f = 0 \tag{2.2.12}$$

where λ is arbitrary. The number of negative eigenvalues with multiplicity will be called the *morse index* and the multiplicity of 0 as an eigenvalue will be called the *nullity.* Combining this definition with equation 2.2.8 we can see that M is a local minimum of the area functional if and only if the morse index of M is zero. In this case the manifold will be called *stable.*

For a stable manifold and a function $f : M \to \mathbb{R}$ with compact support, we get the inequality

$$-\int_M f\Delta_M f + (|A|^2 + \mathrm{Ric}(\nu))f^2 \geq 0 \tag{2.2.13}$$

which is also often written

$$\int_M (|A|^2 + \mathrm{Ric}(\nu))f^2 \leq \int_M |\mathrm{grad}_M(f)|^2 \tag{2.2.14}$$

Where grad_M is the gradient over M. This is known as the *stability inequality.* As an simple consequence of this equation, it can be seen that if N has a metric with positive ricci curvature, then taking $f \equiv 1$ shows that N contains no stable minimal surfaces, immersed or otherwise.

2.3 Area Minimizers; existence and embeddedness results

We end this chapter by stating some existence results used later on in this paper. Specifically, we aim to answer the following question: Let Σ^2 be an embedded surface in N^3. Under what conditions can we guarantee the existence of a smooth area minimizer in the homotopy class of Σ.

One common thread to all existence results is the notion of mean convexity. N will be called *mean convex* if $H \cdot \nu \geq 0$ where H is the mean curvature vector of the boundary ∂N and ν is the inward pointing unit normal. What this guarantees is that any minimal surface will stay away from the boundary. This can be seen by taking variations of the maximum principle for non-euclidean spaces, and is usually part of the hypothesis for any global existence theorem.

It is easy to construct examples where no area minimizer would exist (if Σ were homotopic to a point, for example). We would like to be able to say that Σ does not shrink enough to destroy any part of the topology. The following result, due to R. Schoen and S. Yau, states that if Σ is incompressible, we are able to guarantee a minimal immersion which is close to Σ in sense of π_1.

Theorem 2.3.1 (Main theorem of [SY79a]). *Let Σ^2 be a closed surface and $f : \Sigma \to M^3$ such that $f(\Sigma)$ is incompressible (i.e. $f_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective). Then there is a minimal immersion $h : \Sigma \to M$ such that $h_* = f_*$ and the area of h is leas among all maps with the same action on $\pi_1(\Sigma)$.*

This was improved by M. Freedman, J. Hass, and P. Scott to say that if Σ is embedded, then the surface provided in 2.3.1 is as well.

Theorem 2.3.2 ([FHS83]). *Let Σ, M, and f be as in theorem 2.3.1. If f is also homotopic to an embedding, then there is a minimal embedding $h : \Sigma \to M$ such that $h_* = f_*$ and the area of h is leas among all maps with the same action on $\pi_1(\Sigma)$.*

It should be noted here that these results do not actually find an area minimizer in the homotopy class and do not include the case where Σ is a sphere. The results of W. Meeks, L. Simon, and S. Yau discussed in section 3.3 (in particular corollary 3.3.7) show that, under certain topological conditions, one can indeed find an embedded minimal surface homotopic to Σ, with no a priori assumptions on the genus or orient ability of Σ.

Chapter 3

Compact embedded stable minimal surfaces of arbitrarily large area

In this chapter, we provide the proof for the torus case provided by Colding and Minicozzi. There is also a discussion of the Meeks-Simon-Yau Existence Theorem, which is necessary to prove the spherical case.

3.1 Notation

For the rest of this document, we will be using the following notation. $I = [0, 1] \subset \mathbb{R}$ will denote the unit interval. For a subset U of a topological space X, U° will be the interior and $\overline{U}$ will be the closure of U in X. For a continuous function $f : X \to Y$, f_* will denote the map induced on homotopy and $im(f) = f(X)$ will be the image of f.

Let N be an arbitrary three dimensional differentiable manifold. $U \subset N$ is said to be *homotopic* to $U' \subset N$ if there is a continuous homotopy $\varphi : I \times U \to N$ such that $\varphi_0(U) = U$ and $\varphi_1(U) = V$. A subset $U \subset N$ is said to be *homotopically non-trivial* if U is not homotopic to a point.

An *isotopy* (also known as an *ambient isotopy*) from a set $U \subseteq N$ to the set $U' \subseteq N$ is a continuous one parameter family of diffeomorphisms $\varphi : I \times N \to N$ such that $\varphi_0(U) = U$ and $\varphi_1(U) = U'$. $\mathcal{I}(U)$ will denote the class of sets in N isotopic to

U.

Let $\mathfrak{S}$ denote the space of closed subsurfaces of N. Then for $m \geq 0$, $m\Sigma = (m, \Sigma) \in \mathbb{Z} \times \mathfrak{S}$, if Σ has a smooth choice of unit normal, $\mathcal{I}(\Sigma_0 \cup (m\Sigma)) = \mathcal{I}(\Sigma_0 + m\Sigma)$ will denote the isotopy class of the union of Σ_0 and the graphs of the constant functions $k\epsilon/m$ over Σ for $k = 1, \ldots, m$ where ϵ is arbitrarily small. We define $\mathcal{I}(\Sigma_0 \cup 0\Sigma)$ to be $\mathcal{I}(\Sigma_0)$. A set U' is said to be iso(U) if $U' \in \mathcal{I}(U)$.

If N has a riemannian metric, the area of a subsurface Σ will be denoted $|\Sigma|$.

3.2 Colding and Minicozzi's Result for Tori

In this section we will prove the following theorem, originally due to T. Colding and W. Minicozzi

Theorem 3.2.1 ([CM99a]). *Let M^3 be a three-manifold. There exists an open non-empty set of metrics on M for which there are stable embedded minimal tori of arbitrarily large area.* ‘

We will also discuss the complications on generalizing this result to the sphere. It should also be noted that B. Dean proved a similar result for closed surfaces of higher genus in [Dea03]. The basic proof of both papers follows a similar pattern: construct a subset of a ball and a sequence of surfaces for which the area seems to be going to infinity, use an existence theorem to find embedded minimal surfaces of the same topological type close to each element in this sequence, and then show that the area approaches infinity.

Proof. Let Ω be a closed two dimensional disk minus three open subdisks. $\pi_1(\Omega)$ is generated by α, β, and γ going around each of the subdisks. We can also assume that there is an element of the free homotopy classes $[\alpha\beta]$, $[\alpha\gamma]$, and $[\beta, \gamma]$ which is embedded. Let C_n be an element of the free homotopy class

$$(\alpha\beta)^n\gamma(\alpha\beta)^{-n}\alpha^{-1}\beta\alpha \tag{3.2.2}$$

(see figure 3.2.3). As a side note, to prove the one dimensional analogue of theorem 3.2.1, as also shown in [CM99a], a result of M. H. Freedman, J. Hass, and P. Scott

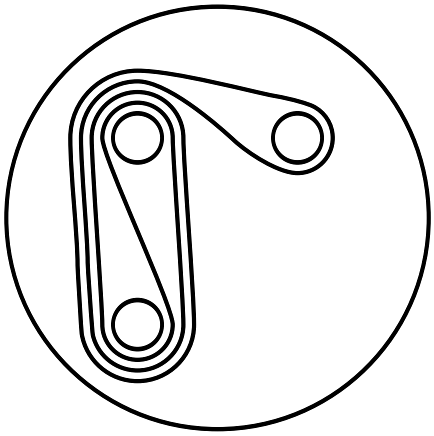

Figure 3.2.3: Omega and C^n

in [FHS82] shows that if the metric causes Ω to be strictly mean convex, then there is an connected closed geodesic freely homotopic to C_n. A calculation (see [Dea03]) shows that this set of metrics is both open and non empty.

For the torus, let $N = \Omega \times S^1$ and $\Sigma_n = C^n \times S^1$. Σ_n is incompressible, so theorem 2.3.2 tells us that if the metric causes N to be strictly mean convex, there is a surface Σ'_n with the same image as Σ_n for the induced map on the fundamental group. There is a sequence of closed curves $\alpha_n \subset \Sigma'_n$ which represent one of the generators of $\pi_1(\Sigma'_n)$ with the property that the infimum of length in the free homotopy class $\pi_1^*(N)$ goes to infinity. Let β be the other generator. Then we have the following estimate for area

$$\text{area}(\Sigma'_n) \geq \left(\inf_{\alpha' \in [\alpha]} \text{length}(\alpha') \right) \left(\inf_{\beta' \in [\beta]} \text{length}(\beta') \right) \tag{3.2.4}$$

The length of β has a lower bound depending only on N, so the right hand side, and therefore the area of Σ'^n goes to infinity. Again, the set of strictly mean convex metrics on N is open and non empty, so if we consider N to be a subset of the three manifold stated in the theorem, we have our desired set of metrics and sequence. □

This construction, was used extensively in [CH03] to find a metric on any manifold for which there are embedded minimal tori of arbitrarily large morse index.

The result of B. Dean for the higher genus case follows this pattern with some exceptions. First, because only the torus can be decomposed as the cross product of two one dimensional manifolds, generating the surface Σ_n must be done directly in the three manifold. Second, we were able to use the the decomposition of the torus as the cross product $S^1 \times S^1$ to find a lower bound for area. This second problem was solved using the following lemma, which we borrow to prove proposition 3.2.5

Lemma 3.2.5 (Lemma 1 in [Dea03])**.** *Let N^3 be a compact Riemannian manifold, and let $M_n \subset N$ be a sequence of stable, compact, connected, embedded minimal surfaces without boundary such that the following conditions hold:*

1. *there exists a constant $C_1 > 0$ such that Area($M_)$ $\leq C_1$ for all n.*
2. *there exists a constant $C_2 > 0$ such that $\sup_{M_n} |A_n|^2 \leq C_2$ for all n, where A_n is the second fundamental form of Mn .*

Then, a subsequence of M_n converges to a compact, connected, embedded minimal surface without boundary $M \subset N$ with finite multiplicity

The main problem in extending this result to the sphere is the fact that S^2 is simply connected. More specifically, theorem 2.3.2 does not apply to the sphere, and other methods are required. This is solved using a result by W. Meeks. L. Simon, and W. Yau which is discussed in the next section. There is also a problem which arises when we try to show that each element of the sequence constructed is in fact, homotopically distinct. Interestingly, the last part of this proof (showing that the area of the constructed sequence goes to infinity) relies on the fact S^2 only covers itself with multiplicity 1 and $\mathbb{RP}^2$. This fact does not extend to any other closed surface, and makes the spherical case particularly easy to solve.

3.3 The Existence Theorem of Meeks, Simon, and Yau

We will state in this section the main result of [MSY82]. We start by giving a formal definition of compression. Note that this definition is purely a topological concept and does not depend on the metric for N.

Definition 3.3.1. Let Σ and Σ' be an surface embedded in manifold N possibly with boundary. We say Σ' is a compression of Σ, or $\Sigma' \ll \Sigma$ if

1. Every connected component of $\tilde{\Sigma}$ is homotopically non-constant
2. $\Sigma - \tilde{\Sigma}$ is diffeomorphic to an open annulus.
3. $\tilde{\Sigma} - \Sigma$ consists of two components, each diffeomorphic to the open unit disk.
4. There exists $B \subseteq N$ diffeomorphic to the open unit three-ball such that B is disjoint from $\Sigma \cup \tilde{\Sigma}$ and $\partial B = \overline{\Sigma \Delta \tilde{\Sigma}}$ where $U \Delta V$ denote the symmetric difference $(U - V) \cup (V - U)$.

In other words, Σ' is obtained from doing surgery to Σ, replacing an annulus with disks to fill in the holes. It should be noted that the author of this paper chose to replace the concept of γ-reduction as defined in section 3 of [MSY82] with the concept of compression to simplify the statement of theorem 3.3.2. The consequence of the switch is a slightly weaker result which is still applicable to the current problem.

Theorem 3.3.2. *Let Σ_0 be an embedded surface in a mean convex three-manifold N and $\Sigma_k \in \mathcal{I}(\Sigma_0)$ a sequence of surface such that $|\Sigma_n| \to \inf_{\Sigma \in \mathcal{I}(\Sigma_0} |\Sigma|$. Then there is a subsequence (still called Σ_k) and compact embedded minimal surfaces $\Sigma^{(1)}, \ldots, \Sigma^{(R)}$ such that*

$$\Sigma_k \to m_1 \Sigma^{(1)} + \ldots + m_R \Sigma^{(R)} \tag{3.3.3}$$

in the Borel measure sense. In other words, for any $f \in C_0(N)$

$$\lim_{k \to \infty} \int_{\Sigma_k} f = \sum_{i=1}^{R} m_i \int_{\Sigma^{(i)}} f \tag{3.3.4}$$

Moreover, $m_1 \Sigma^{(1)} + \ldots + m_R \Sigma^{(R)}$ can be obtained from Σ_0 through a finite number of compressions, (i.e. there exists a sequence of compressions

$$\Sigma_0 = \Sigma^1 \gg \ldots \gg \Sigma^n \tag{3.3.5}$$

such that $\Sigma^n \in \mathcal{I}(m_1 \Sigma^{(1)} + \ldots + m_R \Sigma^{(R)})$. Also, if $g_j = genus(\Sigma^{(j)})$, then

$$\sum_{j \in \mathcal{U}} \frac{1}{2} m_j (g_j - 1) + \sum_{j \in \mathcal{O}} m_j g_j \leq genus(\Sigma_0) \tag{3.3.6}$$

where $\mathcal{U} = \{j : \Sigma^{(j)}$ *is one-sided in* $N\}$ *and* $\mathcal{O} = \{j : \Sigma^{(j)}$ *is two-sided in* $N\}$*. Also, if* Σ_0 *is two-sided, then each* $\Sigma^{(j)}$ *is stable.*

We will call $m_1\Sigma^{(1)} + \ldots + m_R\Sigma^{(R)}$ the *Meeks-Simon-Yau minimizer* or the *MSY-minimizer* for Σ_0.

In essence, one can find an area minimizer for each isotopy class if one allows for compressions of the original surface. What is not evident from this statement of the theorem, but is central in the proof in [MSY82] is that if Σ_k is close enough to the area minimizer in the isotopy class, the annuli of these compressions correspond to "necks" in Σ_k with small area. One does surgery to these surfaces, "pinching" off these annuli and stitching up with disks.

Notice that if Σ_0 is incompressible in the sense that $\pi_1(\Sigma_0)$ injects into $\pi_1(N)$, then the surface is also incompressible as in definition 3.3.1. As such, we can get the following corollary, which is a stronger version of theorem 2.3.2.

Corollary 3.3.7. *Let* M^2 *be a closed embedded incompressible* C^2 *subsurface of a mean convex manifold* N^3 *which is not* S^2 *or* $\mathbb{RP}^2$*. Then there exists an embedded surface* M' *isotopic to* M *that realizes the infimum of area in the isotopy class, that is*

$$|M'| = \inf_{\Sigma \in \mathcal{I}(M)} |\Sigma|$$

Proof. Since M is incompressible, the sequence 3.3.5 is trivial, therefore the MSY-minimizer is isotopic to M. $\square$

This result shows that elements of the sequences constructed in [CM99a] and [Dea03] are in fact isotopic to area minimizers.

We end this section by proving a convenient area result involved in compression. Basically, the proposition tells us that one cannot decrease area by compressing. This result will be used in section 4.3.

Proposition 3.3.8. *Let* $\mu' = \inf_{\hat{\Sigma} \in \mathcal{I}(\Sigma')} |\hat{\Sigma}|$ *and* $\mu = \inf_{\hat{\Sigma} \in \mathcal{I}(\Sigma)} |\hat{\Sigma}|$*. If* $\Sigma' \ll \Sigma$*, then* $\mu' \geq \mu$*.*

Proof. We start by making the following claim: Let $\Sigma' \ll \Sigma$ and $\epsilon > 0$. There is an isotopy φ of Σ which fixes Σ' and for which $|\varphi(\Sigma)| \leq |\Sigma'| + \epsilon$. This is true since $\overline{\Sigma'\Delta\Sigma}$ bounds a ball, therefore there is an isotopy which takes fixes Σ' and which takes $\Sigma - \Sigma'$ to a region which approximates $\Sigma' - \Sigma$ with a thin tube connecting both disks. The tube can be made arbitrarily small, and the resulting surface has area less than $|\Sigma'| + \epsilon$

Let φ be an isotopy of Σ' such that $|\varphi(\Sigma')| < \mu' + \epsilon$. $\varphi(\Sigma)$ is a compression of $\varphi(\Sigma')$, so by the claim, we can find $\hat{\Sigma}$ isotopic to $\varphi(\Sigma)$ with area less than $|\varphi(\Sigma')| + \epsilon$. So we have

$$\mu \leq |\hat{\Sigma}| \leq |\varphi(\Sigma')| + \epsilon \leq \mu' + 2\epsilon \qquad (3.3.9)$$

Allowing ϵ to go to zero, we have our desired result. □

Chapter 4

Embedded minimal spheres of arbitrarily large area

In this chapter we will prove theorem 1.0.1. Before starting on this, however, it is will be useful to discuss the construction. We will start by constructing a sequence of surfaces in an manifold which is a subset of the unit ball, and therefore can be embedded in any three-manifold. This sequence will seem to have the property that the area minimizer in the isotopy class for each element will seem to go to infinity. We will then show isotopic (in fact, homotopic) uniqueness of each element of the sequence and, in section 4.2, we will consider the possible compressions of these surfaces, the main result of which is proposition 4.2.2. We use this fact to prove proposition 4.3.10 to show that, under certain restraints of the metric, one can guarantee that there is a subsequence of the surfaces such that the MSY-minimizer for the elements of the sequence are in fact isotopic to the elements themselves, and that the area of such a sequence approaches infinity.

4.1 Construction of $M_{k/2}$

Consider the open unit ball $B_2 \subset \mathbb{R}^2$ and let $r(x, y) = (-x, y)$ where x, y are the usual Euclidean coordinates. Choose two disjoint closed sub-disks $D_1, D_2 \subset \{(x, y) \in B_0^\circ : x < 0\}$. Let Ω be the closure of $B_2 - (D_1 \cup D_2 \cup r(D_1 \cup D_2))$. Let

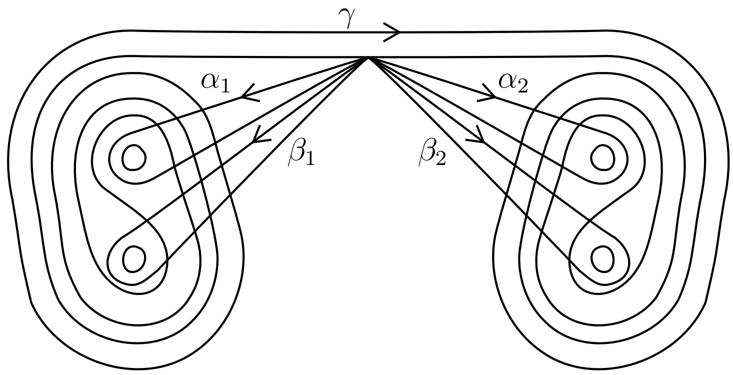

Figure 4.1.3: α_1, β_1, α_2, β_2, and γ_2

$L' = \{(x, y) \in B_2 : x = 0\} = \{(x, y) \in \Omega_2 : x = 0\}$ and $x_0 \in L'$. $\pi_1(\Omega, x_0)$ is generated by the loops α_1, β_1, $\alpha_2 = r_*(\alpha_1)$, and $\beta_2 = r_*(\beta_1)$ where α_1 and β_1 go around D_1 and D_2 counterclockwise respectively. For $n \in \mathbb{Z}$, define $\gamma_n, \gamma_{n+1/2} \in \pi_1(\Omega)$ as

$$\gamma_n = (\alpha_1\beta_1)^n \alpha_1^{-1} (\alpha_1\beta_1)^{-n} (\alpha_2\beta_2)^n \alpha_2 (\alpha_2\beta_2)^{-n} \tag{4.1.1}$$

$$\gamma_{n+1/2} = (\alpha_1\beta_1)^n \alpha_1 (\alpha_1\beta_1)^{-n-1} (\alpha_2\beta_2)^{n+1} \alpha_2^{-1} (\alpha_2\beta_2)^{-n} \tag{4.1.2}$$

(see figure 4.1.3).

For $k \in \mathbb{Z}$ choose an embedded C^2 curve $C_{k/2} \in \gamma_{k/2}$ such that $r(im(C_{k/2})) = im(C_{k/2})$. Define N to be the three manifold obtained by rotating Ω about the center line L'. That is $N = \Omega \times \mathbb{R}/\sim$ where $\sim$ represents the equivalence relations

i. $(x, \theta) \sim (x, \theta')$ for $x \in L'$ and $\theta, \theta' \in \mathbb{R}$

ii. $(x, \theta) \sim (x, \theta + 2k\pi)$ for $k \in \mathbb{Z}$

iii. $(x, \theta) \sim (r(x), \theta + (2k + 1)\pi)$ for $k \in \mathbb{Z}$

We define $M_{k/2}$ to be surface obtained by rotating $C_{k/2}$ about L'. Then N is diffeomorphic to a solid two-sphere minus two unlinked, unknotted tori, and $M_{k/2}$ are two-spheres hooking around the tori. We let $L = \{(x, \theta)/\sim : x \in L'\}$.

We also define $p : N \to \{(x, y) \in \overline{B}_2 : x \geq 0\}$ as $p(x, y, \theta) = (x, y)$ if $x \geq 0$ and $(-x, y) = r(x, y)$ if $x < 0$. This is well defined and continuous. We call a subset $U \subset N$ *rotationally symmetric* if $U = p^{-1}(V)$ for some subset V of the image of p. We also define a *euclidean ϵ-neighborhood of* $U \subset N$ to be the set $\{(x \in N : d(x,U) < \epsilon\}$ where d is the euclidean metric induced by this construction. Note that if U is rotationally symmetric, then so is any euclidean ϵ-neighborhood.

We also define two other surfaces which will come in use later. Let $\gamma_T = \alpha_1\beta_1$ and $\gamma_S = \alpha_1\beta_1\beta_2^{-1}\alpha_2^{-1}$ be elements of $\pi_1(\Omega)$. Choose an embedded C^2 curve $C_T \in \gamma_T$ such that C_T lies on one side of L' in Ω and choose an embedded C^2 curve $C_S \in \gamma_S$ such that C_S is symmetric with respect to reflection about L'. Then define T and S by rotating C_T and C_S respectively about L' as described above. Conceptually, T is a torus containing both of the torus components of ∂N and S is a sphere containing both of the tori which is homotopic to the spherical component of ∂N. We can assume, without loss of generality, we have chosen $M_{k/2}$ and S such that $M_{k/2} \cap S = \emptyset$.

Proposition 4.1.4. *If $M_{k/2}$ is homotopic to $M_{l/2}$, then $l = k$.*

This necessarily implies isotopic distinction as well. To prove this, we will need a few higher homotopy results. Let X be a path connected space. For a path $\gamma : I \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, we can define a natural change of base point map (also called γ) from $\pi_n(X, x_1)$ to $\pi_n(X, x_0)$ as follows: let $y_1, \ldots, y_n$ be coordinates for I^n and let $U = \{y \in I^n : 1/4 \leq y_i \leq 3/4 \text{ for } i = 1 \ldots n\}$. Choose a map $g : I^n - U \to I$ with $g(\partial I^n) = 0$ and $g(\partial U) = 1$. Then for $[f] \in \pi_n(X, x_1)$, let $f_\gamma : I^n \to X$ be defined as:

$$f_\gamma(y) = \begin{cases} \gamma \circ g(y) & \text{if } y \in (I^n - U) \\ f(2(y_1 - 1/4), \ldots, 2(y_n - 1/4)) & \text{if } y \in U \end{cases} \tag{4.1.5}$$

For the purposes of consistency, g will be fixed. We let $\gamma([f]) = [f_\gamma]$ (note that this is homotopically independent of our choice of g). If $x_0 = x_1$, then this represents an action of $G = \pi_1(X, x_0)$ on $\pi_n(X, x_0)$. For $A \in \pi_n(X, x_0)$, we call $GA = \{\gamma A : \gamma \in G\}$ the *orbit of* A. We let $\mathcal{O}_n(X)$ denote the set of orbits in $\pi_n(X)$.

In the following lemma, we will let $\mathcal{S}_n(X)$ the the set of homotopy classes of maps from S^n into X.

Lemma 4.1.6. *Let $q : I^n \to S^n$ be the quotient map taking ∂I^n to a single point and for $[f] \in \pi_n(X)$ let $\bar{f} : S^n \to X$ be the map such that $f = \bar{f} \circ q$. Let $\Theta : \pi_n(X) \to$*

$\mathcal{S}_n(X)$ *be the map* $\Theta([f]) = [\bar{f}]$. Θ *is surjective and if* $\Theta(A) = \Theta(B)$, *then* $B = \gamma A$ *for some* $\gamma \in G$. *That is,* Θ *gives a natural one to one correspondence between* $\mathcal{O}_n(X)$ *and* $\mathcal{S}_n(X)$.

Remark: The one dimensional analogue is the well known result that there is a one to one correspondence between conjugacy classes of $\pi_1(X)$ and homotopy classes of maps from S^1 into X.

Proof. Let $[\bar{f}] \in \mathcal{S}^n(X)$. Let $f = \bar{f} \circ q$ and let γ be a path connecting the base point x_0 of $\pi_n(X, x_0)$ and $f(\partial I^n)$. Then $\Theta(\gamma[f]) = [\bar{f}]$, and surjectivity is proven.

Assume $\Theta([f]) = \Theta([g])$. Then there exists a homotopy $\bar{\varphi}$ such that $\bar{\varphi}_0 = \bar{f}$ and $\bar{\varphi}_1 = \bar{g}$. Let $\gamma : I \times I \to X$ be a one parameter family of paths defined as $\gamma_t(s) = \bar{\varphi}_{st}(q(\partial I^n))$. Let φ be the homotopy $\varphi_t = (\bar{\varphi}_t(f))_{\gamma_t}$. This gives a homotopy between f and g_{γ_1}, thus we have $[f] = \gamma_1[g]$ □

We will call a basis $\mathfrak{B}$ for $\pi_n(X)$ a *natural basis* if for every basis element $A \in \mathfrak{B}$ and $\gamma \in \pi_1(X)$, $\gamma A \in \mathfrak{B}$. For $B \in \pi_n(X)$, define $\Phi(B)$ as the number of nonzero coefficients in the expression $B = \sum_{A \in \mathfrak{B}} \alpha_A A$. Clearly $\Phi(B) = \Phi(\gamma B)$ for $\gamma \in \pi_1(X)$, so Φ is constant on orbits. Lemma 4.1.6 gives a one to one correspondence between $\mathcal{O}_n(X)$ and $\mathcal{S}_n(X)$, so we can define unambiguously $\Phi(f) = \Phi(\Theta^{-1}([f]))$ for any $f : S^2 \to X$. Again, Φ is constant on homotopy classes.

Proof of proposition. N is homotopically equivalent to the wedge sum of two two-spheres with a line connecting the north pole of one with the south pole of the other through the connecting point. Call this space N' (see figure 4.1.8) and let $\phi : N \to N'$ be a homotopy equivalence. Also note that N' is homotopically equivalent to the wedge sum of two two-spheres and two copies of S^1. This means the universal covering space of N', $\tilde{N}'$, is homotopically equivalent to the universal covering space of $S^1 \wedge S^1$ (the well known "tree") with two two-spheres at each vertex. Since the universal cover is simply connected, $\pi_2(\tilde{N}')$ is homomorphic to the second homology $H_2(\tilde{N}')$ which is the direct sum of countably many copies of $\mathbb{Z}$, one for each sphere. Since the n'th homology of a space is homomorphic to n'th homology of any cover for $n \geq 2$, we get

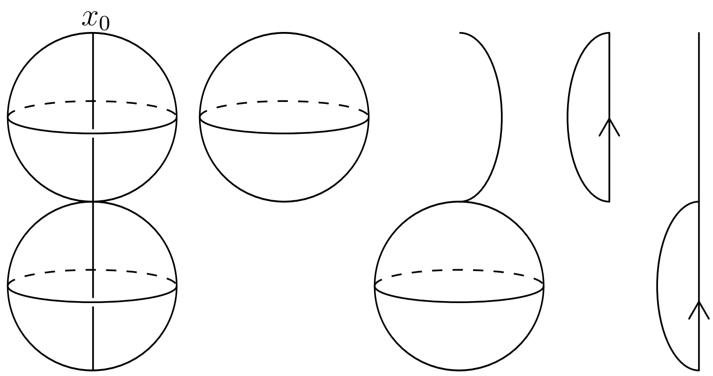

Figure 4.1.8: Respectively: N', A'_e, B'_e, α', and β'

that $\pi_2(N')$ is also the direct sum of countably many copies of $\mathbb{Z}$. More specifically

$$\pi_2(N') = \bigoplus_{\alpha \in \pi_1(N')} (\mathbb{Z} \oplus \mathbb{Z}) \tag{4.1.7}$$

We can find generators of $\pi_2(N')$ as follows: Let $I^2 = I \times I \subset \mathbb{R}^2$. Choose a base point $x_0 \in N'$ as the north pole of the top sphere. Let $A' \in \pi_2(N, x_0)$ be defined by mapping ∂I^2 to x_0 and I^2 over the top sphere. Define $B' \in \pi_2(N, x_0)$ by mapping ∂I^2 to x_0, stringing I^2 around the side of the top sphere and over the bottom sphere (see figure 4.1.8). For $\gamma \in \pi_1(N', x_0)$, we will define $A'_\gamma = \gamma A'$ and $B'_\gamma = \gamma B'$. We now have a set of generators for $\pi_2(N', x_0)$, namely $\left\{A'_\gamma, B'_\gamma | \gamma \in \pi_1(N', x_0)\right\}$. For convenience, we let $C' = A' + B'$ and $C'_\gamma = \gamma C'$. We let $A_\gamma = \phi_*^{-1}(A'_\gamma)$, $B_\gamma = \phi_*^{-1}(B'_\gamma)$, and $C_\gamma = \phi_*^{-1}(C'_\gamma)$. Without loss of generality, we can assume that we have chosen the orientation of A' and B' such that there is an element of C which is embedded in N.

Let α' and β' be generators of $\pi_1(N', x_0)$ such that α' goes around the side of the top sphere to the intersection point and back through to x_0 and β' goes through the middle top sphere, around the side of the bottom sphere, and back though the center of each (see figure 4.1.8). Also let $\gamma' = \alpha'\beta'$ be the path going around side of both spheres and back through the center. Again, let $\alpha = \phi_*^{-1}(\alpha')$ $\beta = \phi_*^{-1}(\beta')$ $\gamma = \phi_*^{-1}(\gamma')$.

Let $f_{k/2}$ be the inclusion map on $M_{k/2} \subset N$. We can see that $[f_{k/2}] = \Theta(D_{k/2})$

where

$$\begin{aligned} D_n &= C_e + C_\gamma + \ldots + C_{\gamma^{n-1}} - A_{\gamma^n\alpha^{-1}} - C_{\gamma^n\alpha^{-1}\gamma^{-1}} - \ldots - C_{\gamma^n\alpha^{-1}\gamma^{-n}} \\ &= -A_{\gamma^n\alpha^{-1}} + \sum_{k=1}^{n} \left[C_{\gamma^{k-1}} - C_{\gamma^n\alpha^{-1}\gamma^{-k}}\right] \end{aligned} \tag{4.1.9}$$

$$\begin{aligned} D_{n+1/2} &= C_e + C_\gamma + \ldots + C_{\gamma^{n-1}} + A_{\gamma^n} - C_{\gamma^n\alpha} - \ldots - C_{\gamma^n\alpha\gamma^{-n-1}} \\ &= A_{\gamma^n} + \sum_{k=1}^{n} \left[C_{\gamma^{k-1}} - C_{\gamma^n\alpha\gamma^{-k}}\right] - C_{\gamma^n\alpha\gamma^{-n-1}} \end{aligned} \tag{4.1.10}$$

for $n \in \mathbb{Z}$. (Note the similarity between the last term in each and equations 4.1.1 and 4.1.2) So $\Phi(f_{k/2}) = 2k + 1$ and since Φ is constant on homotopy classes, all $f_{k/2}$ are all homotopically distinct. □

4.2 $M_{k/2}$ and Compression

We would like to be able to find an area minimizer in the isotopy class of $M_{k/2}$. To do so, we use theorem 3.3.2, however it will be necessary to discuss what possible compressions one can obtain from $M_{k/2}$. The main result of this section is proposition 4.2.2.

For an example of compression on M, take D to be the disk with boundary along the "lip" of $M_{k/2}$ and consider the surface obtained by "cutting" $M_{k/2}$ along D (i.e., replacing the annulus $A = \{x \in M_{k/2} | d(x, \partial D) < \epsilon\}$ with D moved in both normal directions by a distance of ϵ). This reduction is iso($M_{(k-1)/2} \cup S$). As it turns out, this is one of only two isotopically distinct compressions of $M_{k/2}$. To show this, we will first show that every compression of $M_{k/2}$ can be done such that one of the components of the compression is rotationally symmetric. We in order to show this, we will need the following lemma.

Lemma 4.2.1. *Let $\Sigma \ll M_{k/2}$. Then there exists a surface $\Sigma' \in \mathcal{I}(\Sigma)$ such that $\Sigma' \ll M_{k/2}$ and for which on of the components of Σ' is rotationally symmetric.*

To prove this lemma, we will introduce the following notion of a surface containing another set. We will say a surface $M \subset N$ *encapsulates* a set U if for all $x \in U$ any path connecting x to the spherical component of ∂N intersects M non-trivially. Note, in particular, that any surface $\Sigma_{k/2} \in \mathcal{I}(M_{k/2})$ encapsulates one and only one of the torus components of ∂N. Moreover, for all integers n, all M_n encapsulate the same torus and all $M_{n+1/2}$ encapsulate the other.

Also notice that any embedded sphere which encapsulates both torus components of ∂N is isotopic to S. to see this, notice that because the unit three ball B is contractible, any smooth S^2 embedded in B bounds a ball, and the region in between the embedded sphere and ∂B is diffeomorphic to $S^2 \times (-\epsilon, \epsilon)$, and thus the sphere is homotopic to the ∂B. Now, assume Σ is a sphere encapsulating both torus components of ∂N. Since $N \subset B$ and the torus components of ∂N are encapsulated by Σ, Σ is again homotopic to ∂B and isotopic to S.

Proof of lemma 4.2.1. Let Σ_1 and Σ_2 be the disjoint spherical components of Σ and let $D_1 = \overline{\Sigma_1 - M_{k/2}}$, $D_2 = \overline{\Sigma_2 - M_{k/2}}$, and $A = \overline{M_{k/2} - \Sigma}$. Let $B \subset N$ be the open ball with $\partial B = A \cup D_1 \cup D_2$.

Because $\pi_1(M_{k/2}) = 0$ and $M_{k/2}$ is rotationally symmetric, we can find an isotopy ψ of Σ such that $\psi_1(M_{k/2}) = M_{k/2}$ and $\psi_1(\Sigma)$ is another compression of $M_{k/2}$ such that $\psi(\partial D_1)$ and $\psi(\partial D_2)$ are rotationally symmetric circles in M_n. Without loss of generality, we will assume that ∂D_1 and ∂D_2 are rotationally symmetric.

Each component of Σ encapsulates at least one torus component of ∂N. Otherwise, one of the components would bound a ball in N and be homotopically trivial, contradicting the definition of compression. Moreover, one component of Σ encapsulates the other. If not, then each would encapsulate a separate torus, B would lie in the region outside both spheres, and Σ would have resulted from a compression of a surface which encapsulates both tori. Since $M_{k/2}$ encapsulates only one, this cannot be. Say Σ_2 encapsulates Σ_1.

If Σ_2 encapsulates only one torus, then the region between Σ_2 and Σ_1 is homeomorphically $S^2 \times (-\epsilon, \epsilon)$. B would have to lie inside this region. $M_{k/2}$ would then bound a ball in N and be homotopically trivial, a contradiction.

Thus, Σ_2 encapsulates both tori and is iso(S). Therefore we can find an isotopy which fixes $\Sigma' - D_2$ and takes Σ_2 to a rotationally symmetric surface. □

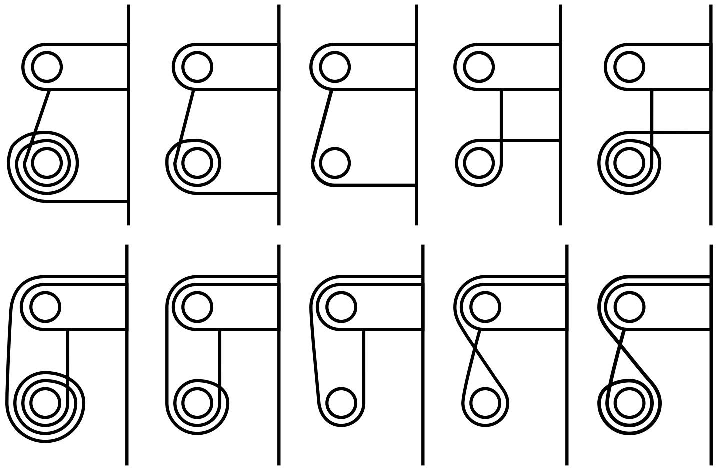

Figure 4.2.3: Examples for γ_D

Remark: Since D_1 is homotopic to $A \cup D_2$, there is an isotopy of Σ which takes D_1 to a rotationally symmetric disk in the euclidean ϵ-neighborhood of $A \cup D_2$, thus we can, in fact, choose both components of Σ' to be rotationally symmetric. The full result, however, is unnecessary to prove the following lemma.

Proposition 4.2.2. *If Σ is a compression of $M_{k/2}$ then Σ has two connected components, one of which is iso(S) and the other is either iso$(M_{(k+1)/2})$ or iso$(M_{(k-1)/2})$*

Proof. Throughout this proof, if γ is a curve, we will use γ and $\mathrm{im}(\gamma)$ interchangeably.

By lemma 4.2.1, there exists a compression $\Sigma' \in \mathcal{I}(\Sigma)$ for which one of the components is rotationally symmetric. Let $D_2' = \overline{\Sigma_2' - M_{k/2}}$ where Σ_2' is the component of Σ' which is rotationally symmetric. Now consider $\gamma_M = p(M_{k/2})$ and $\gamma_D = p(D_2')$ as subsets of $\Omega_{1/2} = im(p)$. Also let $L' = p(L)$. Since $M_{k/2}$ and D_2' are embedded and rotationally symmetric, γ_M and γ_D are embedded 1-dimensional curves. Moreover, they are connected so we can consider $\gamma_M, \gamma_D : I \to \Omega_{1/2}$.

γ_D has one endpoint in L' and another in γ_M. The endpoints of γ_M split L' into three line segments. Σ_2' is iso(S) so γ_D must lie in the component of $\Omega_{1/2} - \gamma_M$ containing $p(S)$ and the endpoint of γ_D in L' must lie in one of the two outer subsegments of $L' - \gamma_M$.

Let $\mathfrak{L}$ be the set of embedded curves mapping $I \to \Omega_{1/2}$ in the component of

$\Omega_{1/2} - \gamma_M$ containing γ_D with one endpoint in γ_M and the other in L'. Define the equivalence relation $\sim$ on $\mathfrak{L}$ as follows: $\delta \sim \delta'$ if there exists a homotopy of δ such that the endpoint of δ which is in γ_M remains in γ_M and the endpoint in L' remains in L' for the entire homotopy. The component of $\Omega_{1/2} - \gamma_M$ containing γ_D is a homeomorphically a disk minus a sub-disk so π_1 of this component is $\mathbb{Z}$. Since the endpoint of γ_D in L' is contained in one of two path connected components, there is a one to two correspondence between $\mathbb{Z}$ and $\mathfrak{L}/\sim$. Representatives of these are shown in figure 4.2.3 with M_0 chosen for simplicity. We can see that there are only two possibilities for which γ_D (and thus D_2') is embedded, both of which correspond to the compressions described. □

4.3 Minimal Spheres

Now we consider the case where N has a metric. Let $S_{k/2}$ be a MSY-minimizer for $M_{k/2}$. Let S_{min} and T_{min} be the MSY-minimizers for S and T respectively. We let $\mu_{k/2} = |S_{k/2}|$, $\sigma = |S_{min}|$, and $\tau = |T_{min}|$. In this section, we show that, under certain restrictions on the metric, we can guarantee a subsequence of $S_{k/2}$ which are iso($M_{k/2}$).

We start off this section by proving the following simple, yet important consequence of proposition 4.2.2

Lemma 4.3.1. *There exists an integer m such that*

$$S_{k/2} \in \mathcal{I}(M_{(k+m)/2} \cup |m| S_{min}) \tag{4.3.2}$$

Proof. By equation 3.3.5, we know that there is a finite sequence $M_{k/2} = \Sigma_0 \gg \ldots \gg \Sigma_n$ such that $\Sigma_n \in \mathcal{I}(S_{k/2})$. If $n = 0$, then $m = 0$ and we are done. Assume $n \geq 1$. Proposition 4.2.2 tells us that $\Sigma_k^1 \in \mathcal{I}(M_{(k\pm1)/2} \cup S)$ so if $n = 1$, then $m = \pm 1$ and we are also done.

So assume $n > 2$. First notice that, by proposition 3.3.8, we have

$$\mu_{k/2} = \inf_{\Sigma \in \mathcal{I}(\Sigma_0)} |\Sigma| \geq \inf_{\Sigma \in \mathcal{I}(\Sigma_1)} |\Sigma| \geq \ldots \geq \inf_{\Sigma \in \mathcal{I}(\Sigma_n)} |\Sigma| = \mu_{k/2} \tag{4.3.3}$$

The last equality comes from the fact that $S_{k/2}$ is the MSY-minimizer for $M_{k/2}$. Therefore

$$\inf_{\Sigma \in \mathcal{I}(\Sigma_l)} |\Sigma| = \mu_{k/2} \text{ for all } 0 \le l \le n \tag{4.3.4}$$

If $\Sigma_l \in \mathcal{I}(M_{i/2} \cup \hat{S})$ where $l < n$, i is a constant and $\hat{S}$ is some surface, then Σ_{l+1} is isotopic to either $M_{(i\pm1)/2} \cup S \cup \hat{S}$ or $M_{i/2} \cup \hat{S}'$ where $\hat{S}' \ll S$. Inductively we get that for $0 \le l \le n$, $\Sigma_l \in \mathcal{I}(M_{l_i/2} \cup \hat{S})$ for some $\hat{S}$ which is obtained from compressions of possibly multiple copies of S. Note that l_{i+1} is equal to either $l_i \pm 1$ or l_i

We claim that l_i is either non-decreasing or non-increasing. To show this, assume that for some $0 \le i < j \le n$ we have $l_i = l_j$. Assume there is an intermediate term $i < \alpha < j$ such that $l_\alpha \ne l_i$. without loss of generality we can assume that that $l_{i+1} \ne l_i$. Let $\Sigma_i = M \cup \tilde{S}$ where $M \in \mathcal{I}(M_{l_i/2})$, then we can say $\Sigma_{i+1} = \hat{M} \cup \hat{S} \cup \tilde{S}$ where $\hat{M} \cup \hat{S} \ll M$ and $\Sigma_j = M' \cup \hat{S}' \cup \tilde{S}'$ where M', $\hat{S}'$, and $\tilde{S}'$ derive from a chain of compressions of M, $\hat{S}$ and $\tilde{S}$ respectively. To put it neatly

$$\Sigma_i = M \cup \tilde{S} \gg \hat{M} \cup \hat{S} \cup \tilde{S} \gg \ldots \gg M' \cup \hat{S}' \cup \tilde{S}' = \Sigma_j \tag{4.3.5}$$

Using the fact that $M' \in \mathcal{I}(M_{l_i/2})$, we get

$$\inf_{\Sigma \in \mathcal{I}(\Sigma_i)} |\Sigma| = \mu_{l_i/2} + \inf_{\Sigma \in \mathcal{I}(\tilde{S})} |\Sigma| \tag{4.3.6}$$

$$\inf_{\Sigma \in \mathcal{I}(\Sigma_j)} |\Sigma| = \mu_{l_i/2} + \sigma + \inf_{\Sigma \in \mathcal{I}(\tilde{S})} |\Sigma| \tag{4.3.7}$$

Since $\sigma \ne 0$, this contradicts equation 4.3.4, so for all $i < \alpha < j$, $l_\alpha = l_j$ and l_i is either non-increasing or non-decreasing. So we have $S_{k/2} \in \mathcal{I}(M_{(k+m)/2} \cup \hat{S})$ for some $\hat{S}$.

To figure the exact nature of $\hat{S}$, consider again the fact that if $\Sigma_i \in \mathcal{I}(M_{l_i/2} \cup \tilde{S})$ then Σ_{i+1} is isotopic to either $M_{(l_i\pm1)/2} \cup S \cup \tilde{S}$ or to $M_{l_i/2} \cup \tilde{S}'$ where $\tilde{S}' \ll \tilde{S}$. Since every step of the chain is of one of these forms, we see that $\hat{S}$ must have resulted from compressions of the "leftover" components in the former reduction (i.e., the components isotopic to S). Since steps of the first type happens exactly $|m|$ times, $\hat{S}$ must have resulted from a compression of a surface which was iso$(|m|S)$, and since S_{min} minimizes area in this isotopy class, we get that $\hat{S} \in \mathcal{I}(|m|S_{min})$. □

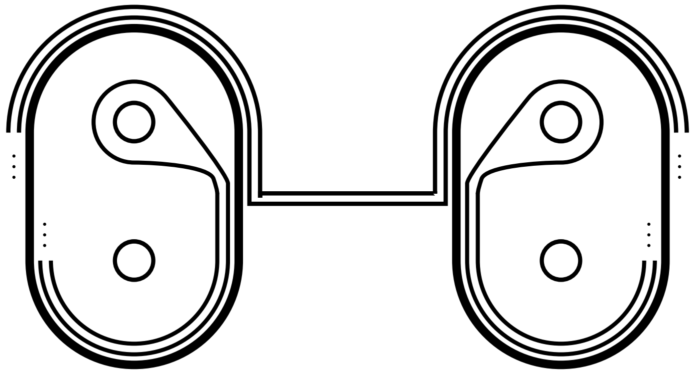

Figure 4.3.9: The cross section of a surface approximating T_{min} (bold) and M'

The next lemma gives an linear upper bound for $\mu_{k/2}$ in terms of k. This result, as well as the above lemma, combined to give us proposition 4.3.10

Lemma 4.3.8. *There exists a constant c such that* $\mu_{k/2} \leq k\tau + c$

Proof. $M_{k/2}$ is isotopic to a surface M' which hooks around one of the torus components of ∂N, loops around in a region arbitrarily close in area to kT, and caps off with disks in the middle (see figure 4.3.9). If we allow the hook and the disks to remain fixed, we find a surface with area less than $k\tau + c + \epsilon$ where c is the area of the disk and the hook, and ϵ is arbitrarily small. Letting $\epsilon \to 0$, we get our desired result. □

We are now in a position to describe the conditions on the metric for which we can ensure the existence of the desired sequence of surfaces. Intuitively, when the rings taken out of N are close enough together in relation to the radii of the tori, then we can say there is a subsequence of $M_{k/2}$ for which compression increases area, guaranteeing an area minimizer in the true isotopy class.

Proposition 4.3.10. *If N is mean convex and $\tau < \sigma$, then there is a sequence k_i such that $S_{k_i/2} \in \mathcal{I}(M_{k_i/2})$. Moreover $\mu_{k_i/2}$ is unbounded.*

To show that the area of these surfaces becomes unbounded, we will need the following result originally due to R. Schoen in [Sch83], and restated in the following form in [CM02]

Theorem 4.3.11 (Schoen Curvature Estimate). *Let $M^2 \subset B_{r_0} = B_{r_0}(y) \subset N^3$ be an immersed stable minimal surface with trivial normal bundle. Also assume $|sec_N| \leq k$, $\partial M \subset \partial B_{r_0}$, and r_0 is sufficiently small (depending only on k). Then there exists a constant K depending only on k, such that for all $0 < r \leq r_0$*

$$\sup_{B_{r_0 - r}} |A|^2 \leq \frac{K}{r^2}$$

where A is the second fundamental form of M

Proof. Lemma 4.3.1 states that for each k there is an $m_k \in Z$ such that $S_{k/2} \in \mathcal{I}(M_{(k-m_k)/2} \cup |m_k|S)$. If there is no such sequence then $|k - m_k| < K$ for some large K and all k, hence $\mu_{k/2} = \mu_{(k-m_k)/2} + m_k\sigma$. Then we have

$$\begin{aligned} \sigma = \lim_{k\to\infty} \frac{|k-K|\sigma}{k} &\leq \lim_{k\to\infty} \frac{\mu_{(k-m_k)/2} + m_k\sigma}{k} \\ = \lim_{k\to\infty} \frac{\mu_k}{k} &\leq \lim_{k\to\infty} \frac{k\tau + c}{k} = \tau \end{aligned} \tag{4.3.12}$$

Where the first inequality comes from lemma 4.3.8 and the second comes from the fact that $\{\mu_k : |k| < K\}$ is a finite set. But this violates our assumption that $\tau < \sigma$, therefore such a sequence must exist.

Assume there is a bound for area. By theorem 4.3.11, there is a constant C such that for small enough r and $\rho \in (0, r]$,

$$\sup_{B_{r-\rho}} |A_i|^2 \leq \frac{C}{\rho^2} \tag{4.3.13}$$

where $B_{r-\rho}$ is a ball of radius $r - \rho$ and A_i is the second fundamental form $S_{k_i/2}$. Since $S_{k_i/2}$ are all without boundary, there is a uniform bound on $|A|$ for all k_i. Therefore, by lemma 3.2.5, there is a compact connected embedded minimal surface without boundary, Σ, to which the sequence converges with finite multiplicity.

Because the $|A|$ is uniformly bounded, we can see that for large i, $S_{k_i/2}$ are coverings of Σ, and since the $\mathbb{RP}^2$ does not embed into $\mathbb{R}^3$ (and thus N), Σ must be a sphere and the multiplicity of convergence must be 1. Therefore, for large i, $\mathcal{S}_{k_i/2} \in \mathcal{I}(\Sigma)$, which contradicts the fact that $M_{k/2}$ are isotopically distinct. □

Remark: proposition 4.3.10 does not show that there is an area minimizer for each $\mathcal{I}(M_{k/2})$, but rather that there is a subsequence with an area minimizer in the isotopy class. It is possible to find an area minimizer in each isotopy class after a point, however the process of showing this is rather tedious. The weaker result was done to simplify the process needed to achieve the end goal: to find a sequence of surfaces of unbounded area. This last lemma insures that the set of metrics which satisfy this the hypothesis of 4.3.10 is both open and non empty.

Lemma 4.3.14. *The set of metrics on N for which N is strictly mean convex and $\tau < \sigma$ is open and non empty*

Proof. Let g be a metric on N for which $\tau < \sigma$. Let $\tilde{g} = fg$ for some $0 < f \in C^2(N)$ and let $\tilde{\tau}$ and $\tilde{\sigma}$ be the counterparts of τ and σ in the new metric. Also, we let $|\Sigma|_g$ and $|\Sigma|_{\tilde{g}}$ be the area of Σ in the respective metrics. Note that $f_m|\Sigma|_g \leq |\Sigma|_{\tilde{g}} \leq f_M|\Sigma|_g$ where f_m and f_M are the minimum and maximum of f respectively.

$$\begin{aligned}\tilde{\tau} - \tilde{\sigma} &= \inf_{\Sigma \in \mathcal{I}(T)} |\Sigma|_{\tilde{g}} - \inf_{\Sigma \in \mathcal{I}(S)} |\Sigma|_{\tilde{g}} \leq f_M \inf_{\Sigma \in \mathcal{I}(T)} |\Sigma|_g - f_m \inf_{\Sigma \in \mathcal{I}(S)} |\Sigma|_g \\ &= f_M\tau - f_m\sigma\end{aligned} \tag{4.3.15}$$

Choose a constant ϵ such that $x\tau < \sigma$ for all $x < 1 + \epsilon$. Then if $\frac{f_M}{f_m} < 1 + \epsilon$, the rightmost term of the above expression is negative and we get $\tilde{\tau} < \tilde{\sigma}$. Thus the set of metrics for which $\tau < \sigma$ is open. Proposition (1) in [Dea03] states that the set of metrics for which N is mean convex is both open and non empty, so our desires set of metrics is open. We need now only show the intersection of these two classes is non-empty.

Let $x = (\rho, \theta, z)$ be cylindrical coordinates for $\mathbb{R}^3$ and for $r < 1$, let

$$\begin{aligned} T_r^+ &= \{x \in \mathbb{R}^3 : (\rho - 1)^2 + (z - \frac{r}{2})^2 < \frac{r^2}{16}\} \\ T_r^- &= \{x \in \mathbb{R}^3 : (\rho - 1)^2 + (z + \frac{r}{2})^2 < \frac{r^2}{16}\} \\ B &= \{x \in \mathbb{R}^3 : \|x\| \leq 2\},\ N = B - (T_r^+ \cup T_r^-) \\ T_r &= \{x \in \mathbb{R}^3 : (\rho - 1)^2 + z^2 = r\} \\ C &= \{x \in B : \rho < \frac{1}{2}\} \end{aligned} \tag{4.3.16}$$

So N is as before. Also, note that $T_r \in \mathcal{I}(T)$ and $|T_r| = 4\pi^2 r$. Let g be the euclidean metric on N. Proposition 1 in [Dea03] tells us that we can find $f \in C^2(N)$ which has support in an ϵ neighborhood of ∂N for which N with the metric $\tilde{g} = e^{2f} g$ is mean convex. Notice that for any surface $\Sigma \in \mathcal{I}(S)$, $\Sigma \cap C$ contains at least two disks with area at least $\pi/4$, so $\sigma \geq \pi/2$. So, if $r < 1/8\pi$, we have

$$\tau \leq 4\pi^2 r < \pi/2 \leq \sigma \tag{4.3.17}$$

and N with $\tilde{g}$ has the desired properties. □

Proof of Main Theorem. Let M^3 be a three-manifold. N embeds into a three dimensional ball which can in turn be embedded into M, so we will consider N to be a subset of M. The open set of metrics is that described in lemma 4.3.14, for each of which there is a sequence of stable, embedded two-spheres with arbitrarily large area as described in proposition 4.3.10. □

Chapter 5

Future Directions

With the spherical case settled, we can now consider some other directions for work.

5.1 Two Dimensional Study

Theorem 1.0.1, [CM99a], and [Dea03] all provide a way of embedding closed stable large area minimal surfaces in a manifold, given certain restrictions on the metric. In particular, the construction provided in [Dea03] lends itself to an interesting question: When can we find a metric a manifold M for which there are embedded minimal surfaces with no genus specific area bounds? When M is non-compact this can be easily done (even an open set of metrics can be found) by finding a sequence of balls B_i for which any sequence x_i with $x_i \in B_i$ does not converge and choosing a metric on each of these balls such that there are large embedded minimal surfaces of genus i in B_i. However, when M is compact, we cannot go about the problem in this manner.

Question 5.1.1. For every manifold, does there exist an open non empty set of metrics such that given any metric in this set and non negative integer γ there exists a sequence $\Sigma_{\gamma,k}$ of closed embedded stable minimal surfaces of genus γ such that $|\Sigma_{\gamma,i}| \to \infty$?

If we could get the existence of such surfaces to say something about the curvature of the ambient manifold, we might be able to construct a singularity in M. It should also be noted that the condition of embedding is essential. If we only consider immersed surfaces, using a construction like the one shown in [Dea03] it seems very possible to show the following

Conjecture 5.1.2. Given a manifold M there exist an open non empty set of metrics such that given any metric in this set and non negative integer γ there exists a sequence $\Sigma_{\gamma,k}$ of closed immersed stable minimal surfaces of genus γ such that $|\Sigma_{\gamma,k}| \to \infty$. Moreover, these sequences can be chosen such that if $\text{im}(\Sigma_{\gamma,k}) = \text{im}(\Sigma_{\gamma',k'})$, then $\gamma = \gamma'$ and $k = k'$.

Idea of proof. Let $C_0 = S^1$ and let C_k be disjoint copies of S^1 intersection C_0 at distinct points for $k = 1, \dots, n$. Let $C = \cup_{k=0}^{n} C_k$ and let M_r to be a tubular neighborhood for some small constant r. Let H_r be the tubular neighborhood corresponding to C_k for $k = 1, \dots n$. Let T_1 and T_2 be two solid tori with disjoint closures contained in the same handle H_1 such that $\overline{T}_1, \overline{T}_2 \subset M_r - M_{r/2}$. Also consider these tori to be embedded in a nice way (i.e. not knotted). Let $M = M_r - T_1 - t_2$, and give M a metric such that M and $M - \cup_{k=1}^{n} H_k$ are mean convex.

Consider the covering space $\tilde{M}$ obtained by unwinding the main ring. For any positive integer γ, there is a mean convex subset of $\tilde{M}$ which corresponds precisely with those constructed in [Dea03]. This can be done by cutting off the ring after the desired number of handles. We can then construct a sequence of closed embedded stable minimal surfaces $\Sigma_{\gamma,k}$ of genus γ which then immerses into M through the covering map. □

As for the embedded case, it has been suggested to me by Saul Schleimer that even in the case where M is compact, we can find an M with odd enough topology that there will indeed be no genus specific area bound on closed embedded stable minimal surfaces. To this end, I provide the following question:

Question 5.1.3. Let M be a three ball with metric g. Does there exist an infinite set of γ_i such that there are no area bounds $C = C(g, \gamma_i)$ for closed embedded stable minimal surfaces of genus γ_i?

If this is solved affirmatively, then one can provide an adequate set of conditions for question 5.1.1, to be solved.

5.2 Higher Dimensions

It is also natural to consider the higher dimensional case. In particular, we would like to know the answer to the following question:

Question 5.2.1. Let M^n and N^{n+k} be differentiable manifolds and assume M is closed. Under what conditions is it possible to give N a metric such that there is a sequence of stable minimal embeddings of M into N?

There are two main problems to this. First, it is hard to classify all compact higher dimensional manifolds. Although it is near to impossible to do so in most dimensions, the construction provided in chapter 4 suggests that it might be possible to work locally. That is, if $M \in N$ is embedded, minimal, and strictly stable (i.e. no non positive eigenvalues for the stability operator) and $x \in M$ and $B_r(x) \cap M$ is a disk, it may be possible to alter the metric of N only in g such that we are able to get large surfaces.

The second problem is the lack of a Meeks-Simon-Yau type existence theorem, and such a theorem would need to be proven if existence were to be proven. Even if such a theorem were to be possible, one benefits of theorem 3.3.2 is that the only singularities one encounters when close to the area minimizer in the isotopy class correspond to compressions. This restriction became necessary in proving lemma 4.2.1 and proposition 4.2.2. In higher dimensions, such a regular result may not be possible.

Despite this drawback, it may be possible to prove an existence theorem in the following way: First, show that if $M_k \in \mathcal{I}(M_0)$ be such that $|M_k| \to \inf_{M\in\mathcal{I}(M_0)} |M|$ and $|A|^2 \leq C$ for some C then some subsequence M_k converges to some limiting surface $M \in \mathcal{I}(M)$. Second, show that if M_k satisfies the conditions above but without the curvature bound, then there is some very large L such that, after performing some type of surgery on M_k to get rid of the components of M_k with $|A|^2 \geq L$ then some subsequence of the surfaces with surgery $\tilde{M}_k$ are all isotopic. Third, apply the first result to the new sequence and obtain a convergent sequence.

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Vita

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