SYMMETRIC SARKISOV LINKS OF FANO THREEFOLDS

by

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Abstract

In this thesis, examples of type II Sarkisov links between complex projective Fano threefolds with Picard number one are provided. To show examples of these links, I study smooth weak Fano threefolds with Picard number two whose pluri-anticanonical morphism contracts only a finite number of curves. I focus only on the case when the Mori extremal contraction of the smooth weak Fano is divisorial and of the same type both before and after a flop. The numerical classification of these particular types of smooth weak Fano threefolds with Picard number two is completed and the existence of some numerical cases is proven.

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I dedicate this dissertation to my family: Mom, Dad, Andrew and Liz. I also dedicate this dissertation to Anna and Spunkers. Thank you all for your constant love, support, and cookies.

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Contents

Chapter 1

Introduction

1.1 Introduction

The study and classification of weak Fano threefolds is important in the framework of the Mori minimal model program as well as a branch of algebraic geometry called Sarkisov theory.

Starting with a $\mathbb{Q}$-factorial variety Y with at most terminal singularities, the end result of running the Mori minimal model program on Y is either a minimal model or a Mori fibration. If the canonical divisor K_Y is nef then Y is defined to be a *minimal model*. A *Mori fibration* (equivalently, a *Mori fiber space*) is defined as a contraction $\phi : Y \to S$, where S is a normal projective variety such that the relative Picard number $\rho(Y/S) = 1$, dim $Y >$ dim S, and $-K_Y$ is ϕ-ample. These latter varieties will be our primary objects of study.

Given a birational map between two threefold Mori fiber spaces, it is natural to ask whether this map decomposes as a composition of elementary birational morphisms. Kawamata has shown that any birational map between minimal models of a threefold decomposes as a finite sequence of flops ([Ka08]). This result is true for higher dimensional minimal models as well ([ShC10]). In [Sar89], Sarkisov introduced elementary links (certain birational maps) between Mori fiber spaces and in [Cor95] Corti proved that any birational map between threefold Mori fiber spaces is a composition of these elementary links. In addition, Corti showed that there are only four kinds of links that can exist, thus completing the Sarkisov program in dimension three. They are appropriately called links of type I, II, III and IV.

Any Sarkisov link has a central variety. If the link contains a flop, both varieties on either side of the flop are central varieties. If the link does not contain a flop, the central variety is unique. Fixing a central variety, there are a series of small modifications (a flop and flips in both directions) which preserve terminal singularities and $\mathbb{Q}$-factorality, followed by divisorial or fibered contractions. Sarkisov links will be further explained in Section 2.1.

I study the particular situation when the central varieties in Sarkisov links are smooth weak Fano threefolds. For a link whose central variety is a smooth weak Fano threefold, the composition of small modifications does not include flips, so only a flop needs to be considered. The primary links of interest in this thesis are those of type II with central variety a smooth weak Fano threefold. I study the type II links that occur specifically when the Mori contraction of a smooth weak Fano threefold is divisorial both before and after a flop. The image of the divisorial contraction is then a Fano threefold, the contraction of which to a point is a Mori fibration. These are very specific examples of what are more generally known as global links.

In dimension two, the possible links between two Mori fibrations are well known and the Sarkisov program leads to a new proof of the classic Castelnuovo-Noether theorem. This theorem states that the Cremona group (the group of birational self-maps of $\mathbb{P}^2$) is generated by the projective transformations together with a fixed quadratic transformation. It is hoped that in higher dimensions the Sarkisov program can be used to classify birational self-maps of $\mathbb{P}^n$ or other Mori fibrations.

The Sarkisov program is expected to provide a powerful tool to study the birational structure of varieties with Kodaira dimension -∞. In particular, it helps study the rationality of certain threefolds. For example, the dimension three Sarkisov Program provides an alternate proof of the classic result of Iskovskikh and Manin that the smooth quartic threefold $X_4 \subset \mathbb{P}^4$ is not rational [Mat10].

The study of weak Fano varieties leads to new examples of Sarkisov links. In this thesis, I study links of type II. The study of weak Fano threefolds also provides new examples of $\mathbb{Q}$-Fano varieties, which arise after an $E3, E4$ or $E5$ contraction (defined in the next section). The properties and classification of $\mathbb{Q}$-Fano varieties, especially in dimension three, are the subject of current research of many prominent algebraic geometers.

It is important to distinguish between numerical classification and geometric classification, two notions used throughout this paper. Using relations among intersection numbers, the numerical classification of smooth weak Fano threefolds with Picard number two is completed. Numerical classification is just the listing of solutions to a system of Diophantine equations formed from the relations among intersection numbers that any smooth weak Fano threefold must satisfy. The arguments used are algebraic in nature and a finite list of all possible combinations can be found in the Tables section of this thesis, [JPR05], [JPR07], and [CM10]. However, not every solution of the system of equations corresponds to an example of a weak Fano threefold. What remains open is the more delicate problem of showing that these numerical cases exhibit a geometric realization. All authors of the aforementioned papers, including myself, have shown the existence of many cases; however, open cases still remain.

Numerical classification can be summarized as follows: if a smooth weak Fano threefold exists, then it must appear on the list. Only when it is shown that there does or does not exist a weak Fano with the given numerical invariants will the geometric classification finally be complete. It is this area that I hope to include in my future research.

1.2 Definitions and Preliminaries

In this section, we recall some definitions and fix our notation which will be used throughout this thesis. All standard definitions follow from [Ha77].

By a variety X, we always mean a normal projective variety over the field of complex numbers $\mathbb{C}$. The variety X is assumed to be irreducible and nonsingular unless otherwise stated. A *prime divisor* on X is a closed subvariety of codimension one. A (Weil) *divisor* is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors. We write a divisor D as a finite formal sum $D = \sum_{i=1}^{N} n_i Y_i$, where $n_i \in \mathbb{Z}$ and the Y_i are prime divisors. The finite union $\bigcup Y_i$ is called the *support* of the divisor D. The free $\mathbb{Z}$-module generated by the irreducible curves on X is denoted by $Z_1(X)$. Then there is a *bilinear intersection form* $\mathrm{Div}(X) \times Z_1(X) \to \mathbb{Z}$, where intersection is denoted by $D.C$, such that if a prime divisor D and an irreducible curve C are in general position (i.e. C is not contained in the support of D), then $D.C$ is the number of points of intersection of D and C counted with multiplicities. Two divisors D_1 and D_2 are numerically equivalent if $D_1.C \equiv D_2.C$ for any curve C. The group of divisors modulo numerical equivalence on a variety X is the *Picard group*, denoted $\mathrm{Pic}(X)$. The rank of the Picard group is called the *Picard number* and is denoted by $\rho(X)$. The *canonical divisor* K_X of X is the Weil divisor of zeros and poles of a rational differential form of highest degree. Let $\mathcal{O}_X(D)$ be the associated sheaf of the divisor D and let $h^i(X, \mathcal{O}_X(D))$, or $h^i(D)$ for short, denote the dimension as a vector space over $\mathbb{C}$ of $H^i(X, \mathcal{O}_X(D))$. The complete linear system determined by D is denoted by $|D|$, where $|D| = \{D' \mid D' \sim D, D' \geq 0\}$, where $\sim$ denotes linear equivalence. The associated rational map is denoted $\varphi_{|D|} : X \dashrightarrow X' \subset \mathbb{P}^{h^0(D)-1}$, and is defined as $\varphi_{|D|}(P) \mapsto (f_1(P) : \ldots : f_{h^0(D)}(P))$ for some basis $f_1, \ldots, f_{h^0(D)}$ of $H^0(X, \mathcal{O}_X(D))$. The divisor D is *very ample* if $\varphi_{|D|}$ is an isomorphism and *ample* if nD is very ample for some $n > 0$. The set of numerical equivalence classes of curves is denoted $N_1(X)$. The closure of the cone spanned by effective curves in $N_1(X)$ is called the *Mori cone* and is denoted $\overline{NE}(X)$.

A morphism $\pi : X \to Y$ is called a *contraction* if the induced sheaf morphism $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is an isomorphism. A contraction is called K_X*-trivial* (resp. K_X*-negative*) if $K_X.C = 0$ (resp. $K_X.C < 0$) for all curves C contracted by π. A contraction π is called a *Mori extremal contraction* if π contracts a single K_X-negative extremal ray R of the Mori cone $\overline{NE}(X)$. That is, a curve C is contracted if and only if the numerical class of C in $N_1(X)$ lies in R.

A *flip* of a small extremal contraction $f : X \to Z$ of a threefold X is a diagram

(1.1)
$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & Z & \end{array}$$

where g is a small birational morphism and an isomorphism outside the exceptional locus of f such that K_{X^+} is positive against the finitely many curves contracted by f^+.

A related birational construction to the flip is the *flop*. A flop is a diagram as above with f and f^+ both K_X-trivial morphisms.

A variety X is said to be *Fano* if its anti-canonical divisor $-K_X$ is ample. These varieties are named after the Italian geometer Gino Fano who first studied varieties with this property. The classification of smooth Fano varieties in low dimensions is known.

In dimension one, Fano varieties are just the rational curves, i.e. those curves C with genus $g = 0$.

In dimension two, these are the well studied del Pezzo surfaces, and up to isomorphism, all are either $\mathbb{P}^2$ blown-up at at most 8 points or $\mathbb{P}^1 \times \mathbb{P}^1$.

Dimension three Fano varieties, or Fano *threefolds*, were classified in the smooth case by Fano, Iskovskikh, Shokurov, Mori, and Mukai. A complete list can be found in the appendix of [IP99]. The classification of smooth Fano threefolds was completed in stages, with Iskovskikh and Shokurov first classifying those with Picard number one and then Mori and Mukai finishing the classification for those Fano varieties with higher Picard number.

Relaxing the above definition for a variety to be Fano, one says that X is *weak Fano*, or *almost Fano*, if its anticanonical divisor $-K_X$ is both nef and big, but not necessarily ample. A divisor D is said to be *nef* if $D.C \geq 0$ for every irreducible curve $C \subset X$. This now standard terminology originated from Miles Reid and stands for "numerically eventually free." A divisor D on an irreducible projective variety X is said to be *big* if the Kodaira dimension of D, denoted by $\kappa(D)$, is equal to the dimension of X. For an n dimensional smooth complex projective variety X with a nef divisor D, this is equivalent to saying that the self-intersection number D^n is strictly positive. This is also equivalent to saying that the global sections of nD define a birational map of X for $n \gg 0$, or that $h^0(nD) = \text{const} \cdot n^{\dim X}$ for $n \gg 0$. On a weak Fano variety X, since $-K_X$ is assumed to be nef but not necessarily ample, there can be curves C such that $-K_X.C = 0$, and these curves are said to be *K-trivial*.

The classification of smooth weak Fano varieties in low dimensions is known:

In dimension one, all weak Fano varieties are Fano.

In dimension two, the Hirzebruch surface $\mathbb{F}_2$ is the unique minimal weak Fano variety with the rational section C_0 the K-trivial curve.

In dimension three, the complete classification of smooth weak Fano varieties is open and it is in this area where this thesis contributes to the collection of currently known examples of weak Fano threefolds by studying and providing examples of Sarkisov links of type II. Similar to how Fano threefolds were classified, the classification of smooth weak Fano variety is progressing by Picard number, with most of the focus currently on the case of Picard number two. Smooth weak Fano threefolds correspond to links as mentioned above and are further explained in Section 2.1.

Out of the 104 families of smooth Fano threefolds, 36 have Picard number two. In comparison, the classification of smooth weak Fano threefolds with Picard number two is still an open problem although much progress has been made. There are already over one hundred families of smooth weak Fano threefolds known to exist, with about another hundred numerical cases still open. Throughout this thesis, I consider only those weak Fano varieties with Picard number two. In my joint paper with N. Marshburn ([CM10]), the numerical classification of these weak Fano threefolds with Picard number two is completed when the Mori contraction before and after the flop is divisorial. For the non-divisorial cases, [JPR05] and [JPR07] complete the numerical classification for smooth weak Fano threefolds with Picard number two. Their contributions are further explained in the next section. This thesis presents only the cases with symmetric divisorial type contractions both before and after the flop.

1.3 Prior Results and Background Information

In Germany in 2005, Priska Jahnke, Thomas Peternell, and Ivo Radloff began the classification of smooth weak Fano threefolds with Picard number two ([JPR05]). Classification in the Fano case relied on the fact that there are two contractions of extremal rays (*Mori extremal contractions*). However, if X is only assumed to be weak Fano, the second Mori contraction is substituted by the birational contraction associated with the base point free anticanonical linear system $|-mK_X|$ for $m \gg 0$.

Jahnke, Peternell and Radloff used Mori's classification of extremal rays in dimension three to break the classification problem into subcases:

Theorem 1.3.1. *(Mori (1982)) Let X be a smooth three dimensional projective variety. Let R be an extremal ray on X, and let $\phi : X \to Y$ be the corresponding extremal contraction. Then only the following cases are possible:*

1. *Type E: R is not numerically effective. Then $\phi : X \to Y$ is a divisorial contraction of an irreducible exceptional divisor $E \subset X$ onto a curve or a point. In addition, ϕ is the blow-up of the subvariety $\phi(E)$ (with the reduced structure). All the possible types of extremal rays R which can occur in this situation are listed in the following table, where $\mu(R)$ is the length of the extremal ray R (that is, the number min$\{(-K_X) \cdot C \,|\, C \in R$ is a rational curve$\})$ and l_R is a rational curve*

such that $-K_X \cdot l_R = \mu(R)$ *and* $[l_R] = R$.

Type of R	ϕ *and* E	$\mu(R)$	l_R
E1	$\phi(E)$ *is a smooth curve, and* Y *is a smooth variety*	*1*	*a fiber of a ruled surface* E
E2	$\phi(E)$ *is a point,* Y *is a smooth variety,* $E \simeq \mathbb{P}^2$, *and* $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$	*2*	*a line on* $E \simeq \mathbb{P}^2$
E3	$\phi(E)$ *is an ordinary double point,* $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *and* $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$	*1*	$s \times \mathbb{P}^1$ *or* $\mathbb{P}^1 \times t$ *in* E
E4	$\phi(E)$ *is a double (cDV)-point,* E *is a quadric cone in* $\mathbb{P}^3$, *and* $\mathcal{O}_E(E) \simeq \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$	*1*	*a ruling of cone* E
E5	$\phi(E)$ *is a quadruple non Gorenstein point on* Y, $E \simeq \mathbb{P}^2$, *and* $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$	*1*	*a line on* $E \simeq \mathbb{P}^2$

2. *Fibered Type:* R *is numerically effective. Then* $\phi : X \to Y$ *is a relative Mori fiber space,* Y *is nonsingular, dim* $Y \leq 2$, *and all the possible situations are the following:*

   (a) *dim* $Y = 2$*: Then* $\phi : X \to Y$ *has a standard conic bundle structure, of which there are two types.*

   (b) *dim* $Y = 1$*: Then* $\phi : X \to Y$ *has a del Pezzo fibration structure, of which there are three types.*

(c) dim $Y = 0$. Then X is Fano.

See [IP99] for more details regarding the fibered cases, which are not needed in this thesis.

In [JPR05], the authors completed the numerical classification and partial geometric classification when the anti-canonical morphism $\psi_{|-mK_X|} : X \to X'$ contracts a divisor. The geometric existence of several numerical cases in their paper still remain open. In 2007, Jahnke, Peternell, and Radloff ([JPR07]) then studied the case when the anticanonical morphism $\psi_{|-mK_X|} : X \to X'$ contracts only a finite number of curves (i.e. when $\psi_{|-mK_X|}$ is *small*). They again divided their classification into subcases based on the possible Mori contractions. Concurrently in Japan, Kiyohiko Takeuchi also wrote a paper ([Tak09]) with a partial geometric and numerical classification of smooth weak Fano varieties with $\psi_{|-mK_X|} : X \to X'$ small and one of the Mori extremal contractions of del-Pezzo type. Due to a discrepancy in the literature between the use of "weak Fano" and "almost Fano", it appears that Takeuchi was unaware of the work of Jahnke, Peternell, and Radloff and vice versa.

The authors of [JPR07] left the classification of one subcase open: when both Mori contractions are divisorial (of type E) both before and after the flop. They stated in their introduction that they wished to return to finish this last case; however, they never did. This thesis, along with the thesis of N. Marshburn [Mar11], finishes the numerical classification of this last subcase as well as achieves partial results in the geometric realizability of some these cases. Our joint results can be found in our paper [CM10].

1.4 Assumptions

Throughout this thesis, complex projective threefolds X are assumed to satisfy the following conditions:

i) X is smooth;

ii) $-K_X$ is nef and big (i.e. X is a *weak Fano* variety);

iii) X has finitely many K-trivial curves (i.e. $-K_X$ is big in codimension 1);

iv) $\rho(X) = 2$;

v) The linear system $|-K_X|$ is basepoint free;

vi) The weak Fano index r_X of X is 1.

These varieties appear as smooth central objects of Sarkisov links between terminal Fano varieties with Picard number one. Classifying these links is a step toward classifying all birational maps between terminal Fano varieties with Picard number one.

Assumptions (ii) and (iv) above imply that X has two extremal contractions: a K_X-negative contraction ϕ and a K_X-trivial contraction ψ. Assumptions (iii) and (v) imply that ψ is a small nontrivial birational contraction induced by the linear system $|-K_X|$ (the *anticanonical contraction*). By [Ko89], ψ induces a flop χ, and we obtain the following diagram:

(1.2)
$$\begin{array}{ccccc} X & & \overset{\chi}{\dashrightarrow} & & X^+ \\ \downarrow{\scriptstyle \phi} & \searrow^{\psi} & & \swarrow^{\psi^+} & \downarrow{\scriptstyle \phi^+} \\ Y & & X' & & Y^+ \end{array}$$

In the above diagram, χ is a flop which is an isomorphism outside of the exceptional locus $\text{Exc}(\psi)$ and X^+ satisfies the conditions (i)-(vi) above. The morphism ϕ^+ is a K_{X^+}-negative extremal contraction and ψ^+ is the anticanonical morphism. X' is a terminal Gorenstein Fano threefold with Picard number one, but is not $\mathbb{Q}$-factorial since ψ is small. Indeed, if X' were $\mathbb{Q}$-factorial, then for a curve C contracted by ψ, we would have $0 \neq E.C = \psi_*(E).\psi_*(C) = 0$. The diagram represents a Sarkisov link of type II between the Mori fibrations $Y/\text{Spec}\ \mathbb{C}$ and $Y^+/\text{Spec}\ \mathbb{C}$.

We can further assume that $-K_X$ is generated by global sections (assumption (v)). The case when $-K_X$ is not generated by global sections is the case when X' is the deformation of the Fano threefold V_2. This is proved in Proposition 2.5 in [JPR07].

If $-K_X$ is divisible in $\text{Pic}(X)$, then $-K_X = r_X H$ for some $H \in \text{Pic}(X)$, where r_X is called the *Fano index* of X. The divisor H is called the *fundamental divisor* and the linear system $|H|$ is the *fundamental system* on X. The self intersection number H^3 is the *degree* of X. Since we are assuming X to be smooth (and in particular Gorenstein), we remark that the Fano index r_X is a positive integer. By using the smoothing of X', [Shi89] has shown that $r_X \leq 4$, with equality when $X' = \mathbb{P}^3$. In addition, [Shi89] showed that when $r_X = 3$, $X' \subset \mathbb{P}^4$ is the quadric. In both cases, $X \cong X^+$. For $r_X = 3$, both ϕ and ϕ^+ are conic $\mathbb{P}^2$-bundles over $\mathbb{P}^1$. See [JPR07] Proposition 2.12 for more details.

The case $r_X = 2$ was treated in [JP06]. Both ϕ and ϕ^+ are either $E2$ contractions, $\mathbb{P}^1$-bundles or quadric bundles. The complete list for the case when $r_X = 2$, $\rho(X) = 2$ and ψ small is given in [JPR07] Theorem 2.13. Thus we can assume that $r_X = 1$, which is assumption (vi) above.

Lastly, note the result of Remark 4.1.10 in [IP99] concerning the case when X is hyperelliptic (that is, the anticanonical map $\varphi_{|-K_X|}$ is generically a double cover). If the anticanonical morphism is generically a double cover of a $\mathbb{Q}$-factorial threefold then the flop χ is the birational involution of X induced by $\varphi_{|-K_X|}$. By generic we mean over a Zariski dense open set. In particular, X is isomorphic to X^+ and ϕ and ϕ^+ have the same type. Our diagram 1.2 then reduces to the the Stein factorization of $\varphi_{|-K_X|}$:

$$\begin{array}{ccc} X & & \\ & \searrow & \\ {\scriptstyle \varphi_{|-K_X|}} \searrow & & X' \\ & & \downarrow {\scriptstyle 2:1} \\ & & Z. \end{array}$$

When $-K_X^3 = 2$, $\varphi_{|-K_X|}$ is generically a double cover of $Z = \mathbb{P}^3$.

Chapter 2

Relation to Mori Minimal Model Program

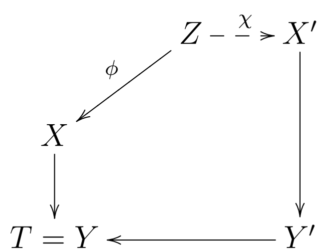
As described in the introduction, the algorithm of factoring a birational map of Mori fibrations $\alpha : Y/S \dashrightarrow Y'/S'$ is called the Sarkisov program. In dimension two, the links are known and the Sarkisov program leads to a new proof of the classical Castelnuovo-Noether Theorem, which states that the group of birational self-maps of $\mathbb{P}^2$ is generated by the projective linear transformations together with a fixed quadratic transformation. It is hoped that in higher dimensions the Sarkisov program can be used to classify birational self-maps of $\mathbb{P}^n$ or other Mori fibrations. In this chapter we give the definition and structure of the four types of Sarkisov links in dimension three. The particular interest of this thesis is when both sides of the diagram (1.1) are divisorial, corresponding to those links of type II.

2.1 Dimension Three Sarkisov Links

2.1.1 Definition

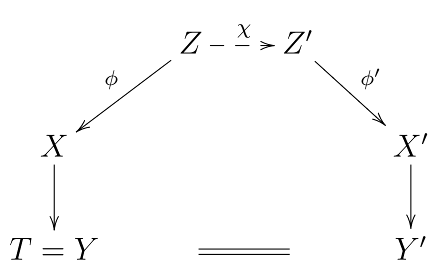
The following diagrams are called *elementary links* in dimension three of types I - IV between Mori fiber spaces Y/S and Y'/S'. In general, links occur over a variety T. The central varieties then have relative Picard number $\rho(X/T) = 2$ over T and are weak Fano relative to T. In our case, for weak Fano threefolds X, both Y and Y' are Fano threefolds and T is the point Spec $\mathbb{C}$. In this situation, when T is a point, we then say that the links below are *global links*.

(2.1.1) Type I:



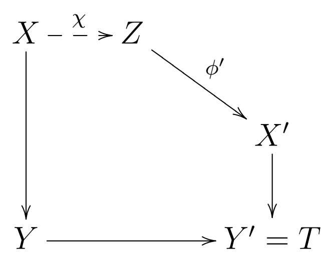
The morphism ϕ is an extremal divisorial contraction and χ is a sequence of codimension two modifications that are a composition of anti-flips, a flop, and flips, each of which may not occur in a particular link. The relative Picard number $\rho(Y'/Y) = 1$.

(2.1.2) Type II:



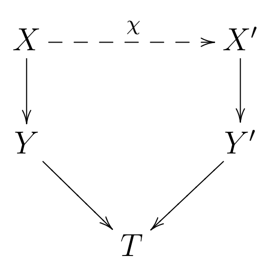
Both ϕ and ϕ' are divisorial contractions and χ is a sequence of codimension two modifications that are a composition of anti-flips, a flop, and flips, each of which may not occur in a particular link. Note that Y and Y' are isomorphic.

(2.1.3) Type III (inverse to Type I):



The contraction ϕ' is an extremal divisorial contraction and χ is a sequence of codimension two modifications that are a composition of anti-flips, a flop, and flips, each of which may not occur in a particular link. The relative Picard number $\rho(Y/Y') = 1$.

(2.1.4) Type IV:



The birational map χ indicates a sequence of codimension two modifications that are a composition of anti-flips, a flop, and flips, with each of which may not occur in a particular link. Y, Y' and T are normal projective varieties with only $\mathbb{Q}$-factorial singularities and $\rho(Y/T) = \rho(Y'/T) = 1$.

Chapter 3

E1-E1 Case

3.1 Equations and Bounds

Let us consider first the case where both extremal contractions ϕ and ϕ^+ are of type $E1$. Then Y is a smooth Fano variety of Fano index r with Picard number one, and ϕ is the blow up of a smooth curve $C \subset Y$. Let g and d denote the genus and degree of C and let E denote the exceptional divisor $\phi^{-1}(C)$. Denote by H a fundamental divisor in Y. The pullback of H to X will also be denoted by H. Since $\mathrm{Pic}(Y)$ is generated by H, $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by H and E. We will use the divisors $-K_X$ and E as generators of $\mathrm{Pic}(X)$ instead. Unless the Fano index of Y is one, these divisors do not generate $\mathrm{Pic}(X)$. However, they do generate $\mathrm{Pic}(X) \otimes \mathbb{Q}$, the Picard group of X with coefficients in $\mathbb{Q}$.

The strict transform of a divisor $D \in \mathrm{Pic}(X)$ across the flop χ is denoted by $\widetilde{D}$. Since χ is small, $\widetilde{K_X} = K_{X^+}$. We identify divisors in X and X^+ via χ and thus have an isomorphism between the Picard groups of X and X^+. In particular, for any $D \in \mathrm{Pic}(X)$, $\chi^{-1}(\chi(D)) = D$ and for any $D^+ \in \mathrm{Pic}(X^+)$, $\chi(\chi^{-1}(D^+)) = D^+$. In our notation, we can write this as $\widetilde{\widetilde{D}} = D$.

Note also that since $-K_X = \phi^*(-K_Y) - E = rH - E$, the denominators of the rational coefficients of the expression of a divisor in terms of $-K_X$ and E will divide r.

Similarly, I also define C^+, g^+, d^+, E^+, r^+, and H^+. Write

$$\widetilde{E^+} = \alpha(-K_X) + \beta E \text{ for some nonzero } \alpha, \beta \in \frac{1}{r}\mathbb{Z}. \tag{3.1}$$

and

$$\widetilde{E} = \alpha^+(-K_{X^+}) + \beta^+ E^+ \text{ for some nonzero } \alpha^+, \beta^+ \in \frac{1}{r^+}\mathbb{Z}. \tag{3.2}$$

By equating E and $\widetilde{\widetilde{E}}$, as well as E^+ and $\widetilde{\widetilde{E^+}}$, and then comparing coefficients using (3.2) and (3.1), we obtain the following relations:

$$\beta\beta^+ = 1, \;\; \alpha + \beta\alpha^+ = \alpha^+ + \beta^+\alpha = 0. \tag{3.3}$$

Since the flop χ is a small transformation, it induces an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(X^+)$. In the following lemma we enumerate cases where this isomorphism preserves intersection numbers.

Lemma 3.1.1. *For any divisors D and D' on X:*

1. $K_X.D.D' = K_{X^+}.\widetilde{D}.\widetilde{D'}$
2. *Let C be a curve in X such that C is disjoint from every flopping curve in X. Then $D.C = \widetilde{D}.\widetilde{C}$ and $\widetilde{C}$, the strict transform of C on X^+ under χ, is disjoint from any flopping curves in X^+.*

Proof. Since $-K_X$ is assumed to be basepoint free, a general anticanonical divisor is disjoint from any of the finitely many flopping curves on X. In fact, the anticanonical divisors on X are pullbacks of anticanonical divisors on X', where the flopping curves are contracted to finitely many points. The theorem then follows from the projection formula. □

The goal for numerical classification of these cases is to find solutions to the Diophantine equations that result from comparing intersection numbers on both sides of the flop. We use the preceding lemma and the following well known formulas (see [IP99]).

$$\begin{aligned} E^3 &= -rd + 2 - 2g; \\ K_X^2.E &= rd + 2 - 2g; \\ K_X.E^2 &= 2 - 2g; \\ -K_Y^3 &= -K_X^3 + 2rd + 2 - 2g. \end{aligned} \tag{3.4}$$

Define

$$\sigma := rd + 2 - 2g = K_X^2.E.$$

The analogous formulas and variables with the "+" sign adjoined hold in X^+. For example,

$$\sigma^+ := r^+d^+ + 2 - 2g^+ = K_{X^+}^2.E^+.$$

Since intersection with $-K_X$ is preserved under the flop by Lemma 3.1.1 , we have

$$\begin{aligned} -K_{X^+}.(E^+)^2 &= -K_X.\widetilde{E^+}^2; \\ -K_X.(E^2) &= -K_{X^+}.\widetilde{E}^2. \end{aligned} \tag{3.5}$$

These relations give us our Diophantine equations whose solutions give all possible numerical examples of the smooth weak Fano threefolds we are interested in. From

(3.1) and (3.2), we can rewrite this as follows:

$$\begin{aligned} 2g^+ - 2 &= -K_{X^+}(\alpha(-K_X) + \beta E)^2 \\ &= \alpha^2(-K_X)^3 + 2\alpha\beta\sigma + \beta^2(2g-2). \end{aligned} \tag{3.6}$$

$$\begin{aligned} 2g - 2 &= -K_{X^+}(\alpha^+(-K_{X^+}) + \beta^+ E^+)^2 \\ &= (\alpha^+)^2(-K_X)^3 + 2\alpha^+\beta^+\sigma^+ + (\beta^+)^2(2g^+-2). \end{aligned} \tag{3.7}$$

We now would like to run a computer program to find all solutions for these equations. Before doing this however, we have to prove effective bounds for all variables involved. We continue to follow the ideas in [JPR07]. Since $-K_X|_E$ is still nef and big, we have that $\sigma = K_X^2.E = (-K_X|_E)^2 > 0$. Similarly, $\sigma^+ > 0$. By [Shi89], since X' has only terminal singularities, X' is smoothable and the smoothing has the same Fano index as X, which is 1 by assumption. Since $|-K_X|$ is basepoint free, we have

$$2 \leq (-K_X)^3 \leq 22 \ \textit{(evens only)}.$$

As both Y and Y^+ are both smooth Fano threefolds of index r and r^+ respectively, so we have $1 \leq r, r^+ \leq 4$. By classification, we also have

$$2 \leq (-K_Y)^3 \leq \begin{cases} 22, & r=1 \\ 40, & r=2 \end{cases} \ \textit{(evens only)} \ \text{and } (-K_Y)^3 = \begin{cases} 54, & r=3 \\ 64, & r=4. \end{cases} \tag{3.8}$$

The same bounds of course hold for $(-K_{Y^+})^3$. Next we bound both d and g (and apply the same argument to bound d^+ and g^+). From $22 \geq (-K_X)^3 = (-K_Y)^3 - \sigma -$

$rd \geq 2$, we get

$$d \leq \frac{(-K_Y)^3 - 3}{r} \leq 19 \text{ and } \sigma \leq (-K_Y)^3 - rd - 2 \leq \begin{cases} 19, & r = 1 \\ 36, & r = 2 \\ 49, & r = 3 \\ 58, & r = 4. \end{cases}$$

Finally, since $0 < \sigma = rd - 2g + 2$, by looking at each value of $r = 1, \ldots, 4$ and the corresponding upper bound for d, we have

$$g \leq \frac{19r}{2} + 1.$$

Running a computer program alone at this point is impractical since we would still have to loop through all rational values of α and β (which are unbounded a priori at this point). Thus the following lemma is necessary:

Lemma 3.1.2. *Using the notation as above, if $X^+ \xrightarrow{\phi^+} Y^+$ is an $E1$ contraction, then $\beta^+ = -\frac{r}{r^+}$ and $\beta = -\frac{r^+}{r}$.*

Proof. Since ϕ^+ is of type $E1$, $-K_{X^+} = r^+H^+ - E^+$. Combining this with (3.2), we can rewrite $\widetilde{E}$ as follows:

$$\begin{aligned} \widetilde{E} &\cong \alpha^+(r^+H^+ - E^+) + \beta^+E^+ \\ &\cong \alpha^+r^+H^+ + (\beta^+ - \alpha^+)E^+. \end{aligned}$$

Note that both α^+r^+ and $\beta^+ - \alpha^+$ are integers. (This is proven directly in Section 3.2.3.)

Thus

$$
\begin{aligned}
\mathbb{Z}/r\mathbb{Z} &\cong \mathrm{Pic}(X)/\langle -K_X, E\rangle \\
&\cong \mathrm{Pic}(X^+)/\langle -K_{X^+}, \widetilde{E}\rangle \\
&\cong \mathrm{Pic}(X^+)/\langle r^+H^+ - E^+, \alpha^+ r^+ H^+ + (\beta^+ - \alpha^+)E^+\rangle.
\end{aligned}
$$

Taking the order of both sides yields:

$$
r = \begin{vmatrix} \alpha^+ r^+ & r^+ \\ \beta^+ - \alpha^+ & -1 \end{vmatrix} = -\alpha^+ r^+ - r^+\beta^+ + \alpha^+ r^+ = -r^+\beta^+
$$

and thus $\beta^+ = -\frac{r}{r^+}$ as desired.

From (3.3), it then follows that $\beta = -\frac{r^+}{r}$. □

Notice now that both α^+, and then by (3.3) α, are completely determined by replacing (3.2) into the formula for σ:

$$
\sigma = K_X^2.E = K_{X^+}^2 \widetilde{E} = \alpha^+(-K_{X^+})^3 + \beta^+ K_{X^+}^2 E^+.
$$

Thus

$$
\alpha^+ = \frac{\sigma - \beta^+\sigma^+}{(-K_X)^3}. \tag{3.9}
$$

We include the following two formulas for completeness:

$$
\widetilde{E}^3 = (\alpha^+)^3(-K_X)^3 + 3(\alpha^+)^2\beta^+\sigma^+ - 3\alpha^+(\beta^+)^2 K_X (E^+)^2 + (\beta^+)^3(E^+)^3; \tag{3.10}
$$

$$
\widetilde{E^+}^3 = \alpha^3(-K_X)^3 + 3\alpha^2\beta\sigma - 3\alpha\beta^2 K_X E^2 + \beta^3 E^3. \tag{3.11}
$$

All our variables are now bounded or completely determined via an explicit formula. With our equations finalized, we are ready to run a computer program to find all the possible numerical solutions. Programs were written in both Visual Basic and C++ to verify the results. The Visual Basic source code can be found in the Appendix 5.4 of this thesis. Here we summarize our bounds on the variables used in the program.

Variables and bounds:

$$1 \le r, r^{+} \le 4;$$

$$2 \le K_X^3 \le 22 \text{ (evens only)};$$

$$0 \le g \le \tfrac{19}{r} + 1;$$

$$0 \le g^{+} \le \tfrac{19}{r^{+}} + 1;$$

$$1 \le d, d^{+} \le 19.$$

From these variables, the following are then determined:

$$-K_Y^3 = -K_X^3 + 2rd - 2g + 2;$$

$$-K_{Y^+}^3 = -K_X^3 + 2r^{+}d^{+} - 2g^{+} + 2;$$

$$\sigma = rd + 2 - 2g;$$

$$\sigma^{+} = r^{+}d^{+} + 2 - 2g^{+};$$

$$\beta^{+} = -\tfrac{r}{r^{+}};$$

$$\beta = \tfrac{1}{\beta^{+}};$$

$$\alpha^{+} = \tfrac{\sigma - \beta^{+}\sigma^{+}}{-K_X^3};$$

$$\alpha = -\beta\alpha^{+}.$$

Equations that any weak Fano threefold with our assumptions must satisfy (3.5):

$$\begin{aligned} 2g^{+} - 2 &= \alpha^2(-K_X)^3 + 2\alpha\beta\sigma + \beta^2(2g-2); \\ 2g - 2 &= (\alpha^{+})^2(-K_X)^3 + 2\alpha^{+}\beta^{+}\sigma^{+} + (\beta^{+})^2(2g^{+}-2). \end{aligned}$$

The programs used to create tables (5.1),(5.2),(5.3), and (5.4) of this thesis are available to view in Appendix (5.4) as well as download from the author's website:

`www.math.jhu.edu\` $\sim$ `jcutrone`.

3.2 Numerical Checks

Although a computer program can now be run from the results of the prior section, there are still numerical checks that can be added to the program to eliminate some possible solutions. Some simple numerical checks to include in the program have been previously mentioned, such as (3.3) and checking that both $\sigma, \sigma^+ > 0$. From the classification of smooth Fano threefolds Y with Picard number one, we also have to check that both $(-K_Y)^3$ and $(-K_{Y^+})^3$ are even and that both these numbers satisfy the bounds from (3.8). In addition, again from classification, we have to check that if $r = 1, (-K_Y)^3 \neq 20$ and similarly if $r^+ = 1, (-K_{Y^+})^3 \neq 20$.

Another obvious check to include in the program is that r^3 divides $(-K_Y)^3$ and that $(r^+)^3$ divides $(-K_{Y^+})^3$. Using the formulas in (3.10) and (3.11), we also need to make sure both $\widetilde{E}^3$ and $\widetilde{E^+}^3 \in \mathbb{Z}$.

Define the *defect* of the flop to be

$$e = E^3 - \widetilde{E}^3. \tag{3.12}$$

Then we have the following two lemmas, which add two more checks to our growing list.

Lemma 3.2.1. *[Tak02] The correction term e in (3.12) is a strictly positive integer*

For a simpler proof then that of [Tak02] in the above Lemma, see the proof of Lemma 3.1 in [Kal09].

The integer e is not widely understood. It is related to the number of flopping curves, and in fact is equal to this number if the flop is an Atiyah flop and if for each flopping curve Γ we have $H.\Gamma = 1$. See [Mar11] for more details regarding the integer e when the flop is a simple Atiyah flop. At present, there is no known upper bound. In my tables, I still have open examples with very large values of e. If a strict upper bound could be found, then this bound could be used to eliminate the geometric realization of some open cases.

Lemma 3.2.2. *The integer r^3 divides e.*

Proof. Starting from the equality $E = rH + K_X$, take the strict transform of E under χ to get $\widetilde{E} = (\widetilde{rH + K_X}) = r\widetilde{H} + K_{X^+}$. Taking the difference of cubes in the formula for e and using the facts that $-K_X^3 = -K_{X^+}^3$ and that intersection with K_X is preserved under a flop (Lemma 3.1.1) we get the desired result. □

Since $\widetilde{E^+} = \alpha(-K_X) + \beta E$, by replacing $-K_Y$ with rH and $-K_X$ with $-K_Y - E = rH - E$, we have

$$\widetilde{E^+} = \alpha rH + (\beta - \alpha)E. \tag{3.13}$$

Since $\widetilde{E^+} \in \text{Pic}(X)$ is not divisible, we have the following:

Proposition 3.2.3. *Using all notation as above, we have the following numerical checks:*

$$\alpha r, \alpha^+ r^+, \alpha - \beta, \alpha^+ - \beta^+ \in \mathbb{Z};$$

$$GCD(\alpha r, \beta - \alpha) = 1;$$

$$GCD(\alpha^+ r^+, \beta^+ - \alpha^+) = 1.$$

Since we are assuming X is smooth, we can use the results of Batyrev and Kontsevich (see [Ba97]) which state that Hodge numbers are preserved under a flop. The only interesting Hodge number in our situation is $h^{1,2}(X)$. For smooth Fano threefolds Y with Picard number one, all these numbers are known (see [IP99]). In the $E1$ case, since we are blowing up a smooth curve of genus g, it can be shown (see [CG72]) that $h^{1,2}(X) = h^{1,2}(Y) + g$. Thus checking to see if $h^{1,2}(X) = h^{1,2}(X^+)$ is equivalent to checking the following equality:

$$h^{1,2}(Y) + g = h^{1,2}(Y^+) + g^+. \tag{3.14}$$

Using these checks to eliminate some possible cases from the output of the computer program yields Table (5.1) in the Tables section of this thesis.

Theorem 3.2.4. *For a geometrically realizable link (1.2), if both ϕ and ϕ^+ are of type $E1$, then the numerical invariants associate to the link are found in Table (5.1). Thirteen families of these links exist are known to exist.*

3.3 Elimination of Cases

In this section, we will eliminate some of the numerical cases listed in Table (5.1).

Proposition 3.3.1. *The following E1-E1 numerical cases in Table (5.1) are not geometrically realizable:*

$$Nos. 27, 32, 36, 53, 62, 66, 73, 82, 85, 91, 95.$$

Proof. We will show that case *No. 27* can not exist. A similar argument can be applied to the remaining cases. From the data in case *No. 27*, we have that

$$\begin{aligned}\widetilde{E^+} &\sim -4K_X - E \\ &\sim 4(-K_Y - E) - E \\ &\sim 4(2H - E) - E \\ &\sim 8H - 5E.\end{aligned}$$

Since $\widetilde{E^+}$ is the strict transform of an exceptional divisor, $\widetilde{E^+}$ must be the unique member of the linear system $|\widetilde{E^+}|$. However, since

$$|\widetilde{E^+}| = |8H - 5E| \supset 5|H - E| + 3|H|$$

and since dim $|H| > 0$, the linear system $|H - E|$ must be empty. We will show that this is in fact not the case.

To show that $|H - E| \neq \emptyset$, it is equivalent to show that $h^0(X, H - E) > 0$. Since we are in the $E1$ case, this is in turn equivalent to showing that $h^0(Y, H - C) > 0$, where C is the smooth curve of genus 0 and degree 1 being blown up. The inequality $h^0(Y, H - C) > 0$ is equivalent to $h^0(Y, \mathcal{I}_C(H)) > 0$, which is what we will show. Here $\mathcal{I}_C$ is the ideal sheaf of the curve C on Y.

Start with the short exact sequence:

$$0 \to \mathcal{I}_C \to \mathcal{O}_Y \to \mathcal{O}_C \to 0.$$

Twist by H to get the short exact sequence:

$$0 \to \mathcal{I}_C(H) \to \mathcal{O}_Y(H) \to \mathcal{O}_C(H) \to 0.$$

The corresponding long exact sequence of cohomology then gives us the exact sequence:

$$0 \to H^0(Y, \mathcal{I}_C(H)) \to H^0(Y, \mathcal{O}_Y(H)) \to H^0(Y, \mathcal{O}_C(H)) \to \dots$$

To compute the dimensions of each of the vector spaces, we will use the Riemann-Roch formula. From the well known formula for smooth (weak) Fano threefolds ([Isk78]), we have:

$$h^0(X, \mathcal{O}_X(jH)) = \frac{j(r+j)(r+2j)}{12} H^3 + \frac{2j}{r} + 1.$$

Plugging in our values of $j = 1$, $r = 2$ and $H^3 = -K_Y^3/r^3 = 8/2^3 = 1$, we get that $h^0(Y, \mathcal{O}_Y(H)) = 3$.

To compute $h^0(Y, \mathcal{O}_C(H))$ we will restrict to the curve C and compute $h^0(C, \mathcal{O}_C(H))$ using the Riemann-Roch formula for nonsingular curves. We then have

$$h^0(C, \mathcal{O}_C(H)) = \deg H|_C + 1 - g = 1 + 1 - 0 = 2.$$

Notice here that the $h^1(C, \mathcal{O}_C(H)) = 0$ by Serre Duality, as the degree of $-H|_C + K_C$ is -3. Since $h^0(Y, \mathcal{O}_Y(H)) = 3$ and $H^0(Y, \mathcal{O}_C(H)) = 2$, from the long exact sequence above we have $h^0(Y, \mathcal{I}_C(H)) \geq 1$ and thus the linear system $|H - E|$ is non-empty. □

Proposition 3.3.2. *Case No. 9 on the $E1 - E1$ table (5.1) does not exist.*

Proof. By classification, Y is embedded in $\mathbb{P}^6$ via $|-K_Y|$ and is the complete intersection of three quadrics. The curve C is of genus 1 and degree 3, so the linear span $\langle C \rangle$ of C is just $\mathbb{P}^2$ by Riemann-Roch. Since $-K_X = \phi^*(-K_Y) - E$ is basepoint free by assumption, the curve C must be the base locus of the linear system $|-K_Y - C|$. Then $C = \text{Bs}|-K_Y - C| = \text{Bs}|-K_Y - \langle C \rangle| = \langle C \rangle \cap Y$. Since C is a cubic and Y is an intersection of quadrics, this is impossible. Thus this case does not exist. □

3.4 Geometric Realization of Cases

In this section we discuss the geometric realization of some of the cases found in Table 5.1.

Remark 3.4.1. Some of the cases listed in Table 5.1 were previously shown to exist as weak Fano varieties by Takeuchi. In other instances, the methods used by Iskovskikh and others to show the existence of the smooth Fano threefold with the same numerical invariants directly apply. These cases are mentioned with their appropriate references in the corresponding tables.

Proposition 3.4.2. *Case No. 111 on the $E1 - E1$ table (5.1) exists.*

Proof. Let $S \subset \mathbb{P}^3$ be a nonsingular cubic surface, the blow-up of $\mathbb{P}^2$ at six general points (no three on a line, no six on a conic). Let the Picard group of S be generated by $l, e_1, \ldots, e_6$, where l is the pullback of a line in $\mathbb{P}^2$ and $e_1, \ldots, e_6$ are the exceptional divisors. The intersection numbers between the generators of $\mathrm{Pic}(S)$ are $l^2 = 1$, $l.e_i = 0$ for any $i = 1, \ldots, 6$ and $e_i.e_j = -\delta_{ij}$, where δ_{ij} is the Kronecker delta. In S, consider the divisor

$$C \sim 7l - 4e_1 - 3e_2 - 3e_3 - 2e_4 - 2e_5 - 2e_6.$$

Since for any of the 27 lines L on S, $C.L \geq 0$ and $C^2 > 0$, by the theory of cubic surfaces ([Ha77]), the linear system $|C|$ contains an irreducible nonsingular member which we will also denote by C.

The degree of any effective divisor $D \sim al - \sum b_i e_i$ on S as a curve in $\mathbb{P}^3$ is $3a - \sum b_i$, and the genus of D is $\frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum(b_i^2 - b_i)$. Thus the degree and genus of C are 5 and 0 respectively, in accordance with the given data. Let X be the blowup of C in $\mathbb{P}^3$. Let $\widetilde{S}$ in X be the strict transform of the cubic surface S. Then $\widetilde{S} \in |3H - E| = |-K_X - H|$.

Suppose $\Gamma \subset X$ is a curve such that $-K_X.\Gamma \leq 0$. Then $\widetilde{S}.\Gamma < 0$ since $\widetilde{S} \sim -K_X - H$ and thus Γ is contained in $\widetilde{S}$. Now $\phi|_{\widetilde{S}}$ is an isomorphism of $\widetilde{S}$ with S, so write $\Gamma \sim al - \sum b_i e_i$. We now claim that Γ must be a line in the cubic surface $\widetilde{S}$. To show this, it suffices to show that $\Gamma^2 = a^2 - \sum b_i^2 < 0$, since the lines on a cubic surface are the only divisors with negative self-intersection. Starting with $-K_X.\Gamma \leq 0$, we rewrite this inequality as $(4H - E).\Gamma \leq 0$ or equivalently $4H.\Gamma \leq E.\Gamma$. Viewing the intersection on the right in S, we have $4\deg_{\mathbb{P}^3}\Gamma \leq C.\Gamma$. This is equivalent to

$$4(al - \sum b_i e_i).(3l - \sum e_i) \leq (7l - 4e_1 - 3e_2 - 3e_3 - 2e_4 - 2e_5 - 2e_6).(al - \sum b_i e_i)$$

which is equivalent to

$$5a \leq b_2 + b_3 + 2b_4 + 2b_5 + 2b_6.$$

If Γ is e_1, we have $-K_X.\Gamma = 0$. If $\Gamma = e_i$ for $i = 2, \ldots, 6$ we get that $-K_X.\Gamma > 0$. Now we can assume Γ is not one of the e_i's. Since Γ is effective, it follows that $5a \leq 2\sum b_i$. Squaring both sides gives $25a^2 \leq 4(\sum b_i)^2$. Using the Cauchy-Schwartz inequality, we have

$$25a^2 \leq 4(\sum b_i)^2 \leq 4 \cdot 6 \sum b_i^2 < 25 \sum b_i^2$$

Canceling the 25 yields the desired

$$a^2 < \sum b_i^2.$$

Since we now know that Γ is one of the possible 27 lines on a cubic surface, a simple check shows that the only curve which is $-K_X$-trivial is the exceptional divisor e_1, with all other lines being $-K_X$-positive. Therefore X is a weak Fano threefold with only 1 flopping curve. Since this example is the only case on all tables with genus 0 and degree 5, this must be case 111 on our table. $\square$

Remark 3.4.3. *Case No. 76:* This case is shown to exist in [JPR05], page 40, in their example No. 19.

Chapter 4

Non E1-E1 Cases

4.1 Numerical Classification

In this section we assume both $\phi : X \to Y$ and $\phi^+ : X^+ \to Y^+$ are both either $E2, E3/E4$ or $E5$ contractions. The cases $E3$ and $E4$ are numerically equivalent, and thus we do not distinguish between them in what follows. We start the classification of these remaining divisorial type contractions with the following observation:

Proposition 4.1.1. *If $\phi : X \to Y$ is an $E2, E3/E4$ or $E5$ type contraction, then α and β as in (3.1) are integers.*

Proof. Case 1: The contraction ϕ *is E2*: If the Fano index of Y, r_Y, is two, then the Fano index of X, would be two, contradicting our assumption that $r_X = 1$. By classification of smooth Fano threefolds with Picard number one, if $r_Y = 3$ or 4, then Y is either the smooth quadric $Q \subset \mathbb{P}^4$ or $\mathbb{P}^3$ respectively. In both of these situations, the blow up of a point would be a Fano variety. Thus we only need to consider the case when Y has Fano index one. Let l be a line on Y which we know to exist by the classic theorem of Shokurov.

Then $-K_X.\phi^*(l) = -K_Y.l = 1$ and $E.\phi^*(l) = 0$. Therefore $\mathbb{Z} \ni \widetilde{E^+}.\phi^*(l) =$

$(\alpha(-K_X)+\beta E).\phi^*(l)=\alpha$. Therefore $\widetilde{E^+}+\alpha K_X=\beta E$ is Cartier, so $\beta\in\mathbb{Z}$.

Case 2: ϕ is E3/E4 or E5: Let ψ be the flopping contraction. Then ψ restricted to E is a finite birational morphism. The linear system corresponding to the morphism $\psi|_E$ is the subsystem of $|-K_X|_E|$ corresponding to the image of $H^0(X,-K_X)\to H^0(X,-K_X|_E)$. By Mori's theorem classifying extremal rays (1.3.1), we know that in the $E3/E4$ case, $\mathcal{O}_X(-K_X)|_E\cong\mathcal{O}_Q(1)$ on a quadric $Q\subset\mathbb{P}^3$ and for the $E5$ case, $\mathcal{O}_X(-K_X)|_E\cong\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. No proper subsystem of either of the complete linear systems associated to these sheaves gives a finite birational morphism, so $H^0(X,-K_X)\to H^0(X,-K_X|_E)$ is surjective. Therefore the linear system of $\psi|_E$ is $|-K_X|_E|$. Thus $\psi|_E$ is an embedding since $-K_X|_E=\mathcal{O}(1)$ is very ample. This implies the intersection of E with any flopping curve Γ must be transversal at a single point, ie $E.\Gamma=1$. Then $\mathbb{Z}\ni\widetilde{E^+}.\Gamma=(\alpha(-K_X)+\beta E).\Gamma=\beta$. So $\widetilde{E^+}-\beta E=\alpha(-K_X)$ is Cartier and since the index of X is 1 by assumption, $\alpha\in\mathbb{Z}$. □

By the above proposition, we immediately obtain:

Lemma 4.1.2. *If $X\xrightarrow{\phi}Y$ and $X^+\xrightarrow{\phi^+}Y^+$ are both contractions of type either $E2,E3/E4$ or $E5$, then $\alpha=\alpha^+$ and $\beta=\beta^+=-1$.*

Proof. Since β and β^+ are both negative integers and $\beta\beta^+=1$ by (3.3), we must have $\beta=\beta^+=-1$. Since $\alpha+\beta\alpha^+=0$, we get $\alpha=\alpha^+$. □

We now proceed to show which combinations of symmetric $E2-E5$ contractions can exist. Note first that by Theorem 1.3.1, a simple calculation shows that $K_X.E^2=2$

for any contraction of type $E2, E3/E4$, or $E5$. Then

$$\begin{aligned} 2 &= K_{X^+}.{E^+}^2 \\ &= K_X.\widetilde{E^+}^2 \\ &= K_X.(-\alpha K_X - E)^2 \\ &= \alpha^2 K_X^3 + 2\alpha K_X^2.E + K_X.E^2 \\ &= \alpha(\alpha K_X^3 + 2K_X^2.E) + 2. \end{aligned}$$

Since $\alpha \neq 0$ by (3.1), we obtain:

$$-K_X^3 = \frac{2K_X^2.E}{\alpha}.$$

By symmetry, this formula also holds for ϕ^+. That is,

$$-K_{X^+}^3 = \frac{2K_{X^+}^2.E^+}{\alpha^+}.$$

Recalling that $-K_X^3 = -K_{X^+}^3$ and $\alpha = \alpha^+$, we have $K_X^2.E = K_{X^+}^2.E^+$. Also, $K_X^2.E = 4, 2$ or 1 for contractions of type $E2, E3/E4$ and $E5$ respectively. Using Theorem 1.3.1, this proves the following:

Theorem 4.1.3. *If $X \xrightarrow{\phi} Y$ is of type $E2 - E5$, then $X^+ \xrightarrow{\phi^+} Y^+$ is of the same type. The possible values of $(-K_X^3, \alpha)$ are found in Tables 5.2, 5.3, and 5.4.*

To calculate the remaining the numbers on our tables (5.2),(5.3),(5.4), we note the following formulas for $-K_Y^3$, again using Theorem 1.3.1:

$$\begin{aligned} -K_Y^3 &= -K_X^3 + 8, \quad \phi : X \to Y \text{ an } E2 \text{ contraction;} \\ -K_Y^3 &= -K_X^3 + 2, \quad \phi : X \to Y \text{ an } E3/E4 \text{ contraction;} \\ -K_Y^3 &= -K_X^3 + \tfrac{1}{2}, \quad \phi : X \to Y \text{ an } E5 \text{ contraction.} \end{aligned}$$

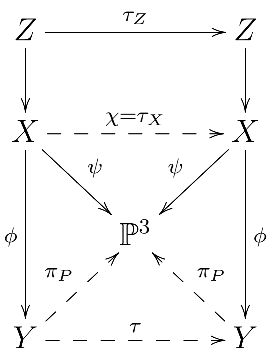
4.2 Existence of Cases

In Table 5.2, when both contractions are of type $E2$, there are three numerical cases. One case has been shown to exist by Takeuchi, one shown not to exist by A. Kaloghiros, and one case remains open.

In Table 5.3, when both contractions are of type $E3/E4$, both cases have been previously shown to exist. I make one remark regarding the second numerical case studied by Pukhlikov ([Puk88]):

Remark 4.2.1. *Case 2*: Consider a quartic threefold $Y \subset \mathbb{P}^4$ with a single ordinary double point $P \in Y$. Pukhlikov found the group of birational involutions of a general Y. We focus on the birational involution τ corresponding to projection from P: a general line in $\mathbb{P}^4$ through P intersects Y at two additional points and τ interchanges those two points. Following [Puk89], we show τ induces a Sarkisov link of type II.

The indeterminacy of the projection $\pi_P : Y \to \mathbb{P}^3$ from P is resolved by blowing up P. We obtain a smooth threefold X and double cover $\psi : X \to \mathbb{P}^3$ which coincides with the anticanonical morphism of X. A general Y contains 24 lines through P, so X is weak Fano with 24 flopping curves. The flop χ is the extension τ_X of τ to X. Pukhlikov shows that if we blow up the flopping curves, we obtain a variety Z on which the extension τ_Z of τ is regular. Thus χ is an Atiyah flop with the corresponding commutative diagram.



The existence of the one numerical case $E5 - E5$ in Table 5.4 remains open.

Chapter 5

Tables

5.1 E1 - E1

The following table is a list of all the numerical possibilities for the $E1-E1$ case, where r and r^+ are the Fano indices of Y and Y^+ in (1.2) respectively. All other notations can be found in (3.1) and (3.2). Due to space constraints, missing from the table are the values of α^+ and β^+. Those can be determined from the given values of α and β using (3.3). There are 111 entries on the table, with 12 known not to exist, 13 known to exist, and the remaining 86 unknown. These cases are denoted by the standard scientific notations "x", ":)", and "?" respectively.

Table 5.1: E1-E1

No.	$-K_X^3$	$-K_Y^3$	$-K_{Y^+}^3$	α	β	r	d	g	r^+	d^+	g^+	e/r^3	Exist?	Ref
1.	2	6	6	3	-1	1	1	0	1	1	0	47	:)	[Isk78]
2.	2	8	8	4	-1	1	2	0	1	2	0	88	?	
3.	2	10	10	5	-1	1	3	0	1	3	0	153	?	
4.	2	12	12	6	-1	1	4	0	1	4	0	248	?	
5.	2	14	14	7	-1	1	5	0	1	5	0	379	?	
6.	2	16	16	8	-1	1	6	0	1	6	0	552	?	
7.	2	18	18	9	-1	1	7	0	1	7	0	773	?	
8.	2	22	22	11	-1	1	9	0	1	9	0	1383	?	
9.	2	8	8	3	-1	1	3	1	1	3	1	21	x	(3.3.2)
10.	2	10	10	4	-1	1	4	1	1	4	1	56	?	
11.	2	12	12	5	-1	1	5	1	1	5	1	115	?	
12.	2	14	14	6	-1	1	6	1	1	6	1	204	?	
13.	2	16	16	7	-1	1	7	1	1	7	1	329	?	
14.	2	18	18	8	-1	1	8	1	1	8	1	496	?	
15.	2	22	22	10	-1	1	10	1	1	10	1	980	?	
16.	2	12	12	4	-1	1	6	2	1	6	2	24	?	
17.	2	14	14	5	-1	1	7	2	1	7	2	77	?	
18.	2	16	16	6	-1	1	8	2	1	8	2	160	?	
19.	2	18	18	7	-1	1	9	2	1	9	2	279	?	
20.	2	22	22	9	-1	1	11	2	1	11	2	649	?	
21.	2	16	16	5	-1	1	9	3	1	9	3	39	?	
22.	2	18	18	6	-1	1	10	3	1	10	3	116	?	
23.	2	22	22	8	-1	1	12	3	1	12	3	384	?	
24.	2	18	18	5	-1	1	11	4	1	11	4	1	?	
25.	2	22	22	7	-1	1	13	4	1	13	4	179	?	
26.	2	22	22	6	-1	1	14	5	1	14	5	28	?	
27.	2	8	8	4	-1	2	1	0	2	1	0	11	x	(3.3.1)
28.	2	16	16	8	-1	2	3	0	2	3	0	69	?	
29.	2	24	24	12	-1	2	5	0	2	5	0	223	?	
30.	2	32	32	16	-1	2	7	0	2	7	0	521	?	
31.	2	40	40	20	-1	2	9	0	2	9	0	1011	?	
32.	2	16	16	6	-1	2	4	2	2	4	2	20	x	(3.3.1)
33.	2	24	24	10	-1	2	6	2	2	6	2	114	?	
34.	2	32	32	14	-1	2	8	2	2	8	2	328	?	
35.	2	40	40	18	-1	2	10	2	2	10	2	710	?	
36.	2	24	24	8	-1	2	7	4	2	7	4	41	x	(3.3.1)
37.	2	32	32	12	-1	2	9	4	2	9	4	183	?	

Table 5.2: E1-E1 (continued)

No.	$-K_X^3$	$-K_Y^3$	$-K_{Y^+}^3$	α	β	r	d	g	r^+	d^+	g^+	e/r^3	Exist?	Ref
38.	2	40	40	16	-1	2	11	4	2	11	4	469	?	
39.	2	32	32	10	-1	2	10	6	2	10	6	80	?	
40.	2	40	40	14	-1	2	12	6	2	12	6	282	?	
41.	2	32	32	8	-1	2	11	8	2	11	8	13	?	
42.	2	40	40	12	-1	2	13	8	2	13	8	143	?	
43.	2	40	40	10	-1	2	14	10	2	14	10	46	?	
44.	2	54	54	25	-1	3	9	2	3	9	2	571	?	
45.	2	54	54	22	-1	3	10	5	3	10	5	372	?	
46.	2	54	54	19	-1	3	11	8	3	11	8	221	?	
47.	2	54	54	16	-1	3	12	11	3	12	11	112	?	
48.	2	54	54	13	-1	3	13	14	3	13	14	39	?	
49.	2	64	64	30	-1	4	8	2	4	8	2	418	?	
50.	2	64	64	26	-1	4	9	6	4	9	6	261	?	
51.	2	64	64	22	-1	4	10	10	4	10	10	146	?	
52.	2	64	64	18	-1	4	11	14	4	11	14	67	?	
53.	2	64	64	14	-1	4	12	18	4	12	18	18	x	(3.3.1)
54.	4	10	10	2	-1	1	2	0	1	2	0	28	:)	[Tak89]
55.	4	14	14	3	-1	1	4	0	1	4	0	68	?	
56.	4	18	18	4	-1	1	6	0	1	6	0	144	?	
57.	4	22	22	5	-1	1	8	0	1	8	0	268	?	
58.	4	12	12	2	-1	1	4	1	1	4	1	8	?	
59.	4	16	16	3	-1	1	6	1	1	6	1	42	?	
60.	4	18	18	3	-1	1	8	2	1	8	2	16	?	
61.	4	22	22	4	-1	1	10	2	1	10	2	80	?	
62.	4	16	10	1.5	-0.5	2	3	1	1	3	1	3	x	(3.3.1)
63.	4	24	14	2.5	-0.5	2	5	1	1	5	1	25	:)	[Isk78]
64.	4	32	18	3.5	-0.5	2	7	1	1	7	1	77	?	
65.	4	40	22	4.5	-0.5	2	9	1	1	9	1	171	?	
66.	4	24	24	4	-1	2	6	3	2	6	3	6	x	(3.3.1)
67.	4	32	32	6	-1	2	8	3	2	8	3	40	?	
68.	4	40	40	8	-1	2	10	3	2	10	3	110	?	
69.	4	40	40	6	-1	2	12	7	2	12	7	18	?	
70.	4	54	16	11/3	-1/3	3	9	3	1	5	0	103	?	
71.	4	54	54	13	-1	3	8	0	3	8	0	164	?	
72.	4	54	54	10	-1	3	10	6	3	10	6	60	?	

Table 5.3: E1-E1 (continued)

No.	$-K_X^3$	$-K_Y^3$	$-K_{Y+}^3$	α	β	r	d	g	r^+	d^+	g^+	e/r^3	Exist?	Ref
73.	4	54	54	7	-1	3	12	12	3	12	12	4	x	(3.3.1)
74.	4	64	12	2.75	-0.25	4	9	7	1	3	0	45	:)	[Tak89]
75.	4	64	64	14	-1	4	8	3	4	8	3	82	?	
76.	4	64	64	10	-1	4	10	11	4	10	11	20	:)	[JPR05]
77.	6	10	10	1	-1	1	1	0	1	1	0	11	:)	[Isk78]
78.	6	16	16	2	-1	1	4	0	1	4	0	32	?	
79.	6	22	22	3	-1	1	7	0	1	7	0	89	?	
80.	6	18	18	2	-1	1	6	1	1	6	1	12	?	
81.	6	40	18	2.5	-0.5	2	9	2	1	5	0	47	?	
82.	6	16	16	2	-1	2	2	0	2	2	0	4	x	(3.3.1)
83.	6	40	40	6	-1	2	8	0	2	8	0	82	?	
84.	6	32	32	4	-1	2	7	2	2	7	2	17	?	
85.	6	40	40	4	-1	2	11	6	2	11	6	1	x	(3.3.1)
86.	6	54	22	8/3	-1/3	3	8	1	1	8	1	48	?	
87.	6	54	12	5/3	-1/3	3	10	7	1	2	0	14	:)	[Tak89]
88.	6	54	54	7	-1	3	9	4	3	9	4	31	?	
89.	6	64	40	4.5	-0.5	4	8	4	2	10	4	24	?	
90.	6	64	64	10	-1	4	7	0	4	7	0	47	?	
91.	6	64	64	6	-1	4	10	12	4	10	12	2	x	(3.3.1)
92.	8	14	14	1	-1	1	2	0	1	2	0	10	:)	[Tak89]
93.	8	22	22	2	-1	1	6	0	1	6	0	36	?	
94.	8	40	16	1.5	-0.5	2	9	3	1	3	0	12	?	
95.	8	24	24	2	-1	2	4	1	2	4	1	2	x	(3.3.1)
96.	8	40	40	4	-1	2	8	1	2	8	1	28	?	
97.	8	54	18	5/3	-1/3	3	8	2	1	4	0	20	?	
98.	8	64	22	1.75	-0.25	4	7	1	1	7	1	14	?	
99.	8	64	64	6	-1	4	8	5	4	8	5	10	?	
100.	10	18	18	1	-1	1	3	0	1	3	0	9	?	
101.	10	40	22	1.5	-0.5	2	7	0	1	5	0	18	?	
102.	10	54	54	4	-1	3	8	3	3	8	3	8	?	
103.	10	64	32	2.5	-0.5	4	7	2	2	5	0	9	?	
104.	12	22	22	1	-1	1	4	0	1	4	0	8	?	
105.	12	54	40	7/3	-2/3	3	7	1	2	7	1	7	?	
106.	12	64	16	0.75	-0.25	4	7	3	1	1	0	5	:)	[Isk78]
107.	14	40	40	2	-1	2	6	0	2	6	0	6	?	
108.	14	54	18	2/3	-1/3	3	7	2	1	1	0	4	:)	[Isk78]
109.	16	54	22	2/3	-1/3	3	6	0	1	2	0	4	:)	[Tak89]
110.	18	40	22	0.5	-0.5	2	5	0	1	1	0	3	:)	[Isk78]
111.	22	64	64	2	-1	4	5	0	4	5	0	1	:)	(3.4.2)

5.2 E2 - E2

The following table is a list of all the numerical possibilities for the $E2 - E2$ case. The Fano indices of Y^+ and Y can always be assumed to equal 1 as shown in the proof of Proposition 4.1.1. All other notations can be found in (3.1) and (3.2). Due to space constraints, missing from the table are the values of α^+ and β^+ which are equal to α and β respectively using (3.3). There are 3 entries on the table, 1 known not to exist, 1 which is geometrically realizable, and 1 unknown.

Table 5.4: E2-E2

No.	$-K_X^3$	$-K_Y^3$	α	β	e	Exist?	Ref
1.	8	16	1	-1	12	:)	[Tak89]
2.	4	12	2	-1	30	x	[Kal09]
3.	2	10	4	-1	90	?	

5.3 E3/4 - E3/4

The following table is a list of all the numerical possibilities for the $E3/4 - E3/4$ case. All notation can be found in (3.1) and (3.2). Due to space constraints, missing from the table are the values of α^+ and β^+, which are equal to α and β respectively using (3.3). There are 2 entries on the table, both known to exist.

Table 5.5: E3/4-E3/4

No.	$-K_X^3$	$-K_Y^3$	α	β	e	Exist?	Ref
1.	4	6	1	-1	12	:)	[Kal09]
2.	2	4	2	-1	24	:)	[Puk88]

5.4 E5 - E5

The following table is a list of the only numerical possibility for the $E5 - E5$ case. All notation can be found in (3.1) and (3.2). Due to space constraints, missing from the table are the values of α^+ and β^+ which are equal to α and β respectively using (3.3).

Table 5.6: E5-E5

No.	$-K_X^3$	$-K_Y^3$	α	β	e	Exist?	Ref
1.	2	2.5	1	-1	15	?	

Algorithm Source Code

This program finds all numerical solutions for the $E1 - E1$ case and was written in Visual Basic and compiled as a macro in Microsoft Excel. The solutions for the other symmetric divisorial cases were determined by hand as shown in the corresponding section in this paper.

```
Sub E1E1()
 Dim rowcounter, r, rplus, minusK_X3, minusK_Y3, g, d As Integer
 Dim sigma, sigmaplus, maxsigma, maxsigmaplus, maxr, maxminusK_X3 As Int
 Dim minminusK_Y3, minminusK_Y3plus, maxminusK_Y3, maxminusK_Y3plus As Int
 Dim maxg, maxgplus, maxd, maxdplus As Integer
 Dim K_X2E, K_XE2, K_XE2plus, K_X2Eplus As Long
 Dim y, yplus, k, kplus, x, xplus As Double
 Dim e, eplus As Long
 Dim alpha, beta, alphaplus, betaplus As Double
 Dim K_ESquiggle2, K_EplusSquiggle2, EPlusSquiggle3, Eplus3 As Double
 Dim ESquiggle3, E3 As Double
 Dim SheetName As String

 Application.ScreenUpdating = False

 maxr = 4
 maxminusK_X3 = 22
 rowcounter = 2
 SheetName = "E1 - E1"

 For r = 1 To maxr
    'calculate max values:
```

```
    Select Case r
      Case 1
        maxminusK_Y3 = 22
        minminusK_Y3 = 2
        maxsigma = 17
      Case 2
        maxminusK_Y3 = 40
        minminusK_Y3 = 8
        maxsigma = 34
      Case 3
        maxminusK_Y3 = 54
        minminusK_Y3 = 54
        maxsigma = 47
      Case 4
        maxminusK_Y3 = 64
        minminusK_Y3 = 64
        maxsigma = 56
    End Select

  For rplus = 1 To r
    Select Case rplus
      Case 1
        maxminusK_Y3plus = 22
        minminusK_Y3plus = 2
        maxsigmaplus = 17
      Case 2
        maxminusK_Y3plus = 40
        minminusK_Y3plus = 8
```

```
            maxsigmaplus = 34
        Case 3
            maxminusK_Y3plus = 54
            minminusK_Y3plus = 54
            maxsigmaplus = 47
        Case 4
            maxminusK_Y3plus = 64
            minminusK_Y3plus = 64
            maxsigmaplus = 56
    End Select

maxg = Round(19 * r/2 + 1, 0)
maxgplus = Round(19 * rplus / 2 + 1, 0)
betaplus = -r / rplus
beta = 1 / betaplus
yplus = -1 / betaplus
y = 1 / yplus

If (maxminusK_Y3 - 3) / r >= 17 Then maxd = 19
    Else maxd = Round((maxminusK_Y3 - 3) / r, 0)
If (maxminusK_Y3plus - 3) / rplus >= 17 Then maxdplus = 17
    Else maxdplus = Round((maxminusK_Y3plus - 3) / rplus, 0)

For minusK_X3 = 2 To maxminusK_X3 Step 1
    For g = 0 To maxg
    For gplus = 0 To maxgplus
        For d = 1 To maxd
        For dplus = 1 To maxdplus
```

```
minusK_Y3 = minusK_X3 + 2 * r * d - 2 * g + 2
minusK_Y3plus = minusK_X3 + 2 * rplus * dplus - 2 * gplus + 2

If minusK_Y3 > maxminusK_Y3 Or minusK_Y3 < minminusK_Y3
   Then GoTo NextdPlusLine
If minusK_Y3plus > maxminusK_Y3plus
   Or minusK_Y3plus < minminusK_Y3plus
      Then GoTo NextdPlusLine
If minusK_Y3 Mod r ^ 3 <> 0 Then GoTo NextdPlusLine
If minusK_Y3plus Mod rplus ^ 3 <> 0 Then GoTo NextdPlusLine
If minusK_Y3 Mod 2 <> 0 Then GoTo NextdPlusLine
If minusK_Y3plus Mod 2 <> 0 Then GoTo NextdPlusLine

K_XE2 = 2 - 2 * g
K_X2E = r * d + 2 - 2 * g
K_XE2plus = 2 - 2 * gplus
K_X2Eplus = rplus * dplus + 2 - 2 * gplus
sigma = K_X2E
sigmaplus = K_X2Eplus

If sigma < 0 Or sigma > maxsigma
   Then GoTo NextdPlusLine
If sigmaplus < 0 Or sigmaplus > maxsigmaplus
   Then GoTo NextdPlusLine

E3 = -1 * r * d + 2 - 2 * g
Eplus3 = -1 * rplus * dplus + 2 - 2 * gplus
alphaplus = (sigma - betaplus * sigmaplus) / minusK_X3
```

```
alpha = -1 * alphaplus / betaplus
K_EplusSquiggle2 = (alpha - 1) ^ 2 * alpha * minusK_X3
   + (3 * alpha ^ 2 - 4 * alpha + 1) * beta * K_X2E
   + (-3 * alpha + 2) * beta ^ 2 * K_XE2
   + beta ^ 3 * E3

If sigmaplus < 0 Or sigmaplus > maxsigmaplus
   Then GoTo NextdPlusLine

E3 = -1 * r * d + 2 - 2 * g
Eplus3 = -1 * rplus * dplus + 2 - 2 * gplus
alphaplus = (sigma - betaplus * sigmaplus) / minusK_X3
alpha = -1 * alphaplus * betaplus
x = y * alpha
xplus = yplus * alphaplus

If Int((x + 1) / y) <> (x + 1) / y Then GoTo NextdPlusLine
If Int((xplus + 1) / yplus) <> (xplus + 1) / yplus
   Then GoTo NextdPlusLine

k = (x + 1) / y
kplus = (xplus + 1) / yplus
K_EplusSquiggle2 = (alpha - 1) ^ 2 * alpha * minusK_X3
   + (3 * alpha ^ 2 - 4 * alpha + 1) * beta * K_X2E
   + (-3 * alpha + 2) * beta ^ 2 * K_XE2 + beta ^ 3 * E3
K_ESquiggle2 = (alphaplus - 1) ^ 2 * alphaplus * minusK_X3
   + (3 * alphaplus ^ 2 - 4 * alphaplus + 1)
   * betaplus * K_X2Eplus
```

```
   + (-3 * alphaplus + 2) * betaplus ^ 2 * K_XE2plus
   + betaplus ^ 3 * Eplus3
EPlusSquiggle3 = alpha ^ 3 * minusK_X3
   + 3 * alpha ^ 2 * beta * sigma
   - 3 * alpha * beta ^ 2 * K_XE2 + beta ^ 3 * E3
ESquiggle3 = alphaplus ^ 3 * minusK_X3
   + 3 * alphaplus ^ 2 * betaplus
   * sigmaplus - 3 * alphaplus * betaplus ^ 2 * K_XE2plus
   + betaplus ^ 3 * Eplus3

e = E3 - ESquiggle3
eplus = Eplus3 - EPlusSquiggle3

If Int(alphaplus * r / betaplus) <> alphaplus * r / betaplus
   Then GoTo NextdPlusLine
If Int(alpha * rplus / beta) <> alpha * rplus / beta
   Then GoTo NextdPlusLine
If Int((alphaplus + 1) / betaplus) <> (alphaplus + 1) / betaplus
   Then GoTo NextdPlusLine
If Int((alpha + 1) / beta) <> (alpha + 1) / beta
   Then GoTo NextdPlusLine
If alpha + beta * alphaplus <> 0
   Or alphaplus + betaplus * alpha <> 0
      Then GoTo NextdPlusLine
If Int(K_EplusSquiggle2) <> K_EplusSquiggle2
   Then GoTo NextdPlusLine
If Int(K_ESquiggle2) <> K_ESquiggle2 Then GoTo NextdPlusLine
If Int(EPlusSquiggle3) <> EPlusSquiggle3 Then GoTo NextdPlusLine
```

```
            If Int(ESquiggle3) <> ESquiggle3 Then GoTo NextdPlusLine
            If Int(alpha - beta) <> (alpha - beta) Then GoTo NextdPlusLine
            If Int(alphaplus - betaplus) <> (alphaplus - betaplus)
               Then GoTo NextdPlusLine
            If Int(r * alpha) <> (r * alpha) Then GoTo NextdPlusLine
            If Int(rplus * alphaplus) <> (r * alpha)
               Or Int(rplus * betaplus) <> rplus * betaplus
                  Then GoTo NextdPlusLine
            If e <= 0 Or eplus <= 0 Then GoTo NextdPlusLine

'Tests C1-C4:
   'C1:
      If minusK_Y3 <> y ^ 2 * (minusK_X3 * k ^ 2
         - 2 * k * sigmaplus + 2 * gplus - 2)
            Then GoTo NextdPlusLine
   'C1+:
      If minusK_Y3plus <> yplus ^ 2 * (minusK_X3 * kplus ^ 2
         - 2 * kplus * sigma + 2 * g - 2)
            Then GoTo NextdPlusLine
   'C2:
      If 0 <> minusK_X3 * k ^ 2 * x + sigmaplus * (2 * k - 3 * k ^ 2 * y)
         + (2 * gplus - 2) * (3 * k * y - 1) + (rplus * dplus - 2
         + 2 * gplus + eplus) * y
            Then GoTo NextdPlusLine
   'C2+:
      If 0 <> minusK_X3 * kplus ^ 2 * xplus + sigma * (2 * kplus
         - 3 * kplus ^ 2 * yplus) + (2 * g - 2) * (3 * kplus * yplus - 1)
         + (r * d - 2 + 2 * g + e) * yplus
```

```
         Then GoTo NextdPlusLine
'C3:
   If minusK_X3 * k * x - sigmaplus * (2 * y * k - 1)
      + (2 * gplus - 2) * y <> r * d / y
         Then GoTo NextdPlusLine
'C3+
   If minusK_X3 * kplus * xplus - sigma * (2 * yplus * kplus - 1)
      + (2 * g - 2) * yplus <> rplus * dplus / yplus
         Then GoTo NextdPlusLine
'C4
   If minusK_X3 * x ^ 2 - 2 * sigmaplus * y * x
      + (2 * gplus - 2) * y ^ 2
      - 2 * g + 2 <> 0
         Then GoTo NextdPlusLine
'C4+
   If minusK_X3 * xplus ^ 2 - 2 * sigma * yplus * xplus +
      (2 * g - 2) * yplus ^ 2 - 2 * gplus + 2 <> 0
         Then GoTo NextdPlusLine


Sheets(SheetName).Cells(rowcounter, 1) = rowcounter - 1
Sheets(SheetName).Cells(rowcounter, 2) = r
Sheets(SheetName).Cells(rowcounter, 3) = rplus
Sheets(SheetName).Cells(rowcounter, 4) = minusK_X3
Sheets(SheetName).Cells(rowcounter, 5) = alpha
Sheets(SheetName).Cells(rowcounter, 6) = beta
Sheets(SheetName).Cells(rowcounter, 7) = alphaplus
Sheets(SheetName).Cells(rowcounter, 8) = betaplus
Sheets(SheetName).Cells(rowcounter, 9) = g
```

```
Sheets(SheetName).Cells(rowcounter, 10) = d
Sheets(SheetName).Cells(rowcounter, 11) = gplus
Sheets(SheetName).Cells(rowcounter, 12) = dplus
Sheets(SheetName).Cells(rowcounter, 13) = minusK_Y3
Sheets(SheetName).Cells(rowcounter, 14) = minusK_Y3plus
Sheets(SheetName).Cells(rowcounter, 15) = K_EplusSquiggle2
Sheets(SheetName).Cells(rowcounter, 16) = K_ESquiggle2
Sheets(SheetName).Cells(rowcounter, 17) = EPlusSquiggle3
Sheets(SheetName).Cells(rowcounter, 18) = Eplus3
Sheets(SheetName).Cells(rowcounter, 19) = ESquiggle3
Sheets(SheetName).Cells(rowcounter, 20) = E3
Sheets(SheetName).Cells(rowcounter, 21) = e
Sheets(SheetName).Cells(rowcounter, 22) = eplus
Sheets(SheetName).Cells(rowcounter, 23) = sigma
Sheets(SheetName).Cells(rowcounter, 24) = sigmaplus

rowcounter = rowcounter + 1
NextdPlusLine:
   Next dplus
   Next d
      Next gplus
      Next g
         Next minusK_X3
            Next rplus
            Next r
Application.ScreenUpdating = True
End Sub
```

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Vitae

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