SCALED CORRELATIONS OF CRITICAL POINTS OF RANDOM SECTIONS ON RIEMANN SURFACES

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Abstract

In this thesis we prove that as $N \to \infty$, the scaling limit of the correlation between critical points ζ_1 and ζ_2 of random holomorphic sections of the N-th power of a positive line bundle over a compact Riemann surface tends to $\frac{2}{3\pi^2}$ for small $\sqrt{N}|\zeta_1 - \zeta_2|$. The scaling limit is directly calculated using a general form of the Kac-Rice formula and formulas and theorems of Pavel Bleher, Bernard Shiffman, and Steve Zelditch.

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Introduction

This dissertation studies the behavior of the critical points of gaußian random holomorphic sections of the N-th power of a holomorphic line bundle L on a Riemann surface M as $N \to \infty$, as is studied in [DSZ04], [DSZ06a], and [DSZ06b]. In the particular case where $L = \mathcal{O}(1)$, the so-called hyperplane section bundle over $M = \mathbb{CP}^1$, sections of L^N correspond to homogeneous polynomials of degree N, the SU_2 polynomials, so the results in this dissertation apply to the critical points of random polynomials $\sum \sqrt{\binom{n}{k}} c_k z^k$ with c_k identically distributed gaußian random variables. In this way, this dissertation examines one small facet of the theory of random polynomials and random holomorphic functions.

Since what may have been the first study of critical points of random curves in [Ric39], this area of research has led to results of interest in mathematics, probability theory, and physics. For instance, the classical result of Hammersley in [Ham56] that for $f(z) := \sum_{j=0}^{N} c_j z^j$ with c_j independent standard gaußian random variables, as $N \to \infty$, the complex zeroes tend toward the unit circle in $\mathbb{C}$ and its generalization by Bloom and Shiffman in [BS07] (also discussed in [Blo05]), namely that as $N \to \infty$, the common zeroes of m random polynomials $f_k(z) := \sum_{|J| \le N} c_J^k z_1^{j_1} \cdots z_m^{j_m}$ in $\mathbb{C}^m$ are concentrated near the "distinguished boundary" of the m-dimensional polydisc. Since the zeroes of a collection of m polynomials in m variables is almost surely discrete, for random f_i, the set $\{f_1(z) = f_2(z) = \ldots = f_m(z) = 0\}$ is a random point process on $\mathbb{C}^m$ of interest in probability theory.

How much should zeroes and critical points of random polynomials or random holomorphic functions $\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ be expected to vary from their expected behavior? This type of question is addressed in [Sod05], [ST04], [ST06], [ST05]. This dissertation examines how pairs of critical points are correlated by examining the 2-point correlation function, $K_2(z, w)$.

The main theorem of this dissertation says that the scaling limit of the correlation between critical points of random holomorphic sections of the N-th power of a positive line bundle over a compact Riemann surface tends to $\dfrac{2}{3\pi^2}$ as $N \to \infty$ for small $r := \sqrt{N}|\zeta_1 - \zeta_2|$. i.e.

Theorem 1.1. *For any positive hermitian line bundle L over any compact Riemann surface M*

$$\lim_{N\to\infty} \frac{1}{N^2} K^N_{211}\left(\frac{\zeta_1}{\sqrt{N}}, \frac{\zeta_2}{\sqrt{N}}\right) = \frac{2}{3\pi^2} + \mathcal{O}\big(r^2\big) \tag{1.1}$$

where $r := \operatorname{dist}(\zeta_1, \zeta_2)$ i.e. as the distance between critical points gets smaller, their scaled limit correlation approaches $\frac{2}{3\pi^2}$ uniformly in ζ_1, ζ_2.

$K^N_{211}(z, w)$ is calculated via the generalized form of the Kac-Rice formula of [Kac49],[Ric44]

$$K(t) = \int |\xi| \operatorname{JPD}(0, \xi;\, t)\, d\xi$$

where $\operatorname{JPD}(x, \xi;\, t)$ denotes the joint probability distribution of $x = f(t)$ and $\xi = f'(t)$.

Though we know no immediate interpretation of the constant $\frac{2}{3\pi^2}$, the fact that it is not 0 is interesting. This contrasts with the fact that the scaling limit correlation of *zeroes* of random sections on a compact Riemann surface is $r^2 + \mathcal{O}\big(r^6\big)$ as was proved in general in [BSZ00a] and [BSZ00b] and specfically for $\mathbb{CP}^1$ in [Han96].

The introductions of [DSZ04], [DSZ06a], and [BSZ00b] give a description of the basic objects of study and the physical motivation for them. The next few sections summarize the more thorough descriptions given there.

Metrics, connections, and curvature

Throughout these definitions, M will denote a complex manifold with $\dim_{\mathbb{C}} M = n$ with complex coordinates $(z_1, \ldots, z_n)$. M will also be thought of as a $2n$-dimensional real manifold with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ where $z_j =: x_j + iy_j$. In general L will be a holomorphic line bundle over M. For standard results and definitions about line bundles, see Chapter 1 of [GH94] for instance.

Definition 1.2.

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{1.2}$$

Using the Cauchy-Riemann equations, one can check that this definition means

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \iff \quad f \text{ is holomorphic}$$

Definition 1.3. T_M denotes the set of smooth complex-valued vectors on M. i.e. $T_{M,p}$ is the space of $\mathbb{C}$-linear derivations in the ring of complex-valued $\mathcal{C}^\infty$ functions on M near p.

Definition 1.4. Since

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)$$

is a basis for T_M,

$$\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right)$$

is also. So if we define the spaces spanned by the $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ alone

$$T'_M := \operatorname{span}\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right)$$
$$T''_M := \operatorname{span}\left(\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right)$$

we get a natural division of T_M into "holomorphic" and "antiholomorphic" parts:

$$T_M = T'_M \oplus T''_M$$

Definition 1.5. T^*_M denotes the dual space of T_M, i.e. the set of smooth complex-valued covectors or 1-forms on M.

Definition 1.6. The dual basis

$$(dz_1, d\bar{z}_1, \dots, dz_n, d\bar{z}_n) := \left(\frac{\partial}{\partial z_1}^*, \frac{\partial}{\partial \bar{z}_1}^*, \dots, \frac{\partial}{\partial z_n}^*, \frac{\partial}{\partial \bar{z}_n}^* \right)$$

for T_M^* gives a natural division of T_M^* into "holomorphic" and "antiholomorphic" parts

$$T_M^* = T_M^{*\prime} \oplus T_M^{*\prime\prime}$$

where

$$T_M^{*\prime} := \operatorname{span}(dz_1, \dots, dz_n)$$

$$T_M^{*\prime\prime} := \operatorname{span}(d\bar{z}_1, \dots, d\bar{z}_n)$$

Note that the dual basis' definition and (1.2) imply

$$dz_j = dx_j + idy_j \quad d\bar{z}_j = dx_j - idy_j$$

so

$$\overline{f\,dz} = \bar{f}\,d\bar{z}$$

Definition 1.7. $\bigwedge^p T_M^*$ denotes the set of smooth complex-valued p-forms on M.

Definition 1.8.

$$\bigwedge^p T_M^{*\prime} := \left\{ \sum_{|J|=p} f_J\,dz_J \;\middle|\; f_J \in \mathcal{C}^\infty(M) \right\}$$

where

$$\text{“}\,|J| = p\,\text{”} := \text{“}\, J = (j_1, \dots, j_p) \quad \text{with} \quad 1 < j_1 < \cdots < j_p < n\,\text{”}$$

$$dz_J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$$

Definition 1.9.

$$\bigwedge^q T_M^{*\prime\prime} := \left\{ \sum_{|J|=q} f_J\,d\bar{z}_J \;\middle|\; f_J \in \mathcal{C}^\infty(M) \right\}$$

where

$$\text{“}\,|J| = q\,\text{”} := \text{“}\, J = (j_1, \dots, j_q) \quad \text{with} \quad 1 < j_1 < \cdots < j_q < n\,\text{”}$$

$$d\bar{z}_J := d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

Definition 1.10.

$$T_M^{*(p,q)} := \bigwedge^p T_M^{*\prime} \otimes \bigwedge^q T_M^{*\prime\prime}$$

Definition 1.11. d can be split into two new operators ∂ and $\bar{\partial}$ with respect to this decomposition i.e.

$$df = \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j}_{\partial f} + \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j}_{\bar{\partial} f}$$

Notice

$$d = \partial + \bar{\partial}$$

$$\partial^2 = \bar{\partial}^2 = 0$$

Definition 1.12. $\mathcal{A}^p(L)$ denotes the sheaf of smooth L-valued p-forms. i.e. for any open $U \subset M$, $\mathcal{A}^p(L)(U) := \{\omega\big|_x \otimes e_U(x) \mid \omega \in \bigwedge^p T_M^*\big|_U$ and e_U a local frame above $U\}$

Definition 1.13. $\mathcal{A}^{p,q}(L)$ denotes the sheaf of smooth L-valued (p,q)-forms. i.e. for any open $U \subset M$, $\mathcal{A}^{p,q}(L)(U) := \{\omega\big|_x \otimes e_U(x) \mid \omega \in T_M^{*(p,q)}\Big|_U$ and e_U a local frame above $U\}$

Definition 1.14. The $\bar{\partial}$ operator can be extended to act on sections via

$$\begin{array}{rccc} \bar{\partial}: & \mathcal{A}^{p,q}(L) & \longrightarrow & \mathcal{A}^{p,q+1}(L) \\ & \omega \otimes e & \longmapsto & \bar{\partial}\omega \otimes e \end{array}$$

where the $\bar{\partial}\omega$ on the right side is the previously defined $\bar{\partial}$ acting on the (p,q)-form ω.

Definition 1.15. Given a holomorphic line bundle $L \to M$, a *hermitian metric* on L is an assignment of a hermitian inner product $\langle\ ,\ \rangle_x$ to each fiber L_x varying smoothly with $x \in M$. i.e. for any local frame $e_U(x)$ of L,

$$h_U(x) := \langle e_U(x), e_U(x)\rangle_x \in \mathcal{C}^\infty(M)$$

since there is a one to one relationship between $\langle\ ,\ \rangle$'s and h_U's both are referred to as "the metric". For the sake of readability, the U will often be suppressed.

Definition 1.16. A holomorphic line bundle with a hermitian metric is said to be a *hermitian line bundle.*

Definition 1.17. A *connection* on a line bundle $L \to M$ is a map

$$\nabla : \mathcal{A}^p(L) \to \mathcal{A}^{p+1}(L)$$

It is defined for each U by a choice of 1-form $\theta \in T_M^*|_U$ and a frame $e_U(x)$ over U by

$$\nabla e_U(x) := \theta_x \otimes e_U(x)$$

and requiring ∇ to satisfy Lebnitz's rule. i.e.

For any $\sigma = g \cdot e_U \in \mathcal{A}^0(L)(U)$ and $f \in \mathcal{C}^\infty(U)$

$$\begin{aligned}\nabla(f \otimes \sigma) &= df \otimes \sigma + f \cdot \nabla(\sigma) \\ &= df \otimes (ge_U) + f \cdot \nabla(ge_U) \\ &= g\, df \otimes e_U + f(dg \otimes e_U + g \cdot \nabla e_U) \\ &= g\, df \otimes e_U + f(dg \otimes e_U + g \cdot (\theta \otimes e_U)) \\ &= (g\, df + f\, dg + g\, \theta) \otimes e_U\end{aligned}$$

and for any $\sigma = g \cdot e_U \in \mathcal{A}^0(L)(U)$ and $\omega = \sum_{|J|+|K|=p} f_{JK}\, dz_J \wedge d\bar{z}_K \in \mathcal{A}^p(U)$

$$\begin{aligned}\nabla(\omega \otimes \sigma) &= d\omega \otimes \sigma + (-1)^p \omega \wedge \nabla\sigma \\ &= d\left(\sum f_{JK}\, dz_J d\bar{z}_K\right) \otimes (g \cdot e_U) + (-1)^p \left(\sum f_{JK}\, dz_J d\bar{z}_K\right) \wedge [(dg + g\theta) \otimes e_U] \\ &= \left(\sum g\, df_{JK} dz_J d\bar{z}_K\right) \otimes e_U + (-1)^p \sum (f_{JK}\, dz_J d\bar{z}_K dg + f_{JK} g\, dz_J d\bar{z}_K \theta) \otimes e_U \\ &= \left[\sum \left(g\, df_{JK} dz_J d\bar{z}_K + (-1)^p f_{JK}\, dz_J d\bar{z}_K dg + f_{JK} g\, dz_J d\bar{z}_K \theta\right)\right] \otimes e_U\end{aligned}$$

Definition 1.18. Using the $T^*_M = T^{*\prime}_M \oplus T^{*\prime\prime}_M$ decomposition, we can write $\nabla = \nabla' + \nabla''$ where

$$\nabla' : \mathcal{A}^0(L) \to \mathcal{A}^{1,0}(L)$$

$$\nabla'' : \mathcal{A}^0(L) \to \mathcal{A}^{0,1}(L)$$

Definition 1.19. There is a unique connection on any hermitian line bundle $(L, \langle\ ,\ \rangle) \to M$ such that

- $\nabla'' = \bar{\partial}$ ("compatibility with the complex structure")
- $d\left(\langle \alpha(z), \beta(z) \rangle_z\right) = \langle (\nabla\alpha)(z), \beta(z) \rangle_z + \langle \alpha(z), (\nabla\beta)(z) \rangle_z$ ("compatibility with the metric")

This connection is called *the Chern connection* associated to $\langle\ ,\ \rangle$. It is usually denoted ∇_h, though when the h is understood, it is often written simply ∇. Compatibility with the complex structure implies the 1-form θ associated to ∇_h must be a $(1,0)$-form:

$$\theta \otimes e := \nabla_h e = \nabla'_h e + \nabla''_h e = \nabla'_h e + \bar{\partial} e = \nabla'_h e + \bar{\partial}(1 \otimes e) = \nabla'_h e + (\bar{\partial} 1 \otimes e) = \nabla'_h e \in \mathcal{A}^{1,0}(L)$$

Definition 1.20. The properties of the Chern connection on a hermitian line bundle yield a unique $(1,1)$-form

$$\begin{aligned}
\nabla_h \nabla_h e &= \nabla_h(\theta \otimes e) \\
&= d\theta \otimes e + (-1)^1 \theta \wedge \nabla_h e \\
&= d\theta \otimes e - \theta \wedge (\theta \otimes e) \\
&= d\theta \otimes e \\
&= (\partial\theta + \bar{\partial}\theta) \otimes e \\
&= \bar{\partial}\theta \otimes e \in \mathcal{A}^{1,1}(L) \quad \text{since } \theta \text{ is a } (1,0)\text{-form}
\end{aligned}$$

This form $\bar{\partial}\theta$ is called the *curvature* of ∇_h. It is denoted by Θ_h. The Θ is upper-case because in the more general theory where L is a vector bundle of rank higher than one, Θ_h is a matrix whose entries are $(1,1)$-forms. In the line bundle case, Θ_h is just a 1×1 matrix whose entry is our $\bar{\partial}\theta$.

Furthermore

$$\begin{aligned}
\left.\sum_{j=1}^{n} \frac{\partial h}{\partial z_j}\, dz_j\right|_z + \left.\sum_{j=1}^{n} \frac{\partial h}{\partial \bar{z}_j}\, d\bar{z}_j\right|_z &= dh|_z = d\langle e(z), e(z)\rangle_z = \langle \nabla_h e(z), e(z)\rangle_z + \langle e(z), \nabla_h e(z)\rangle_z \\
&= \langle \theta_z \otimes e(z), e(z)\rangle_z + \langle e(z), \theta_z \otimes e(z)\rangle_z \\
&= \theta_z \langle e(z), e(z)\rangle_z + \bar{\theta}\langle e(z), e(z)\rangle_z \\
&= \underbrace{h(z)\,\theta_z}_{1,0} + \underbrace{h(z)\,\bar{\theta}_z}_{0,1}
\end{aligned}$$

so

$$\sum_{j=1}^{n} \frac{\partial h}{\partial z_j}\, dz_j = h\,\theta$$

so

$$\frac{1}{h}\sum_{j=1}^{n} \frac{\partial h}{\partial z_j} = \theta \Rightarrow \partial \log h = \theta$$

so the curvature $(1,1)$-form of ∇_h can be written

$$\Theta_h = \bar{\partial}\partial \log h$$

Definition 1.21. If there exists a metric h on L such that $\frac{i}{2\pi}\Theta_h$ is a *positive* $(1,1)$*-form*, L is called a *positive line bundle*. In local coordinates, $\frac{i}{2\pi}\Theta_h$ being a positive $(1,1)$-form means

$$\forall z \in M \quad \forall v \in (T'_M)_z \quad \Theta_h(z)(v, \bar{v}) < 0$$

Jointly Gaußian Random Variables

Throughout these definitions, $\mathcal{L}$ denotes Lebesgue measure on $\mathbb{C}$ and $\mathcal{B}$ denotes the borel subsets of $\mathbb{C}$.

Definition 1.22. On a probability space (Ω, Σ, P), a *random variable* is a map

$$X : (\Omega, \Sigma, P) \to (Y, \Sigma', \mu)$$

with Σ' the σ-algebra of μ-measurable sets in Y such that

$$B \in \Sigma' \Rightarrow X^{-1}(B) \in \Sigma$$

For "expected value" and "probability density function" to have their usual meanings, usually $(Y, \Sigma', \mathcal{L})$ is in fact $(\mathbb{C}, \mathcal{B}, \mathcal{L})$ or $(\mathbb{R}, \{\text{borel sets in } \mathbb{R}\}, \text{Lebesgue measure on } \mathbb{R})$.

Definition 1.23. The *probability distribution* for a random variable

$$X : (\Omega, \Sigma, P) \to (Y, \Sigma', \mu)$$

is

$$\begin{aligned} X_*P : &\Sigma' \longrightarrow [0,1] \\ &B \longmapsto \mathrm{P}\big[X^{-1}(B)\big] \end{aligned}$$

The next three definitions consider $(Y, \Sigma', \mathcal{L}) = (\mathbb{R}, \text{BorelSets}, d\text{Lebesgue})$.

Definition 1.24. In this restricted case, the probability distribution is entirely determined by $\mathrm{P}\big[X^{-1}((-\infty, t])\big]$ since any borel set can be built out of $(-\infty, t]$'s. This function

$$F_X(t) := \mathrm{P}\big[X^{-1}((-\infty, t])\big]$$

is called the *cumulative distribution function* of X.

Definition 1.25. If

$$\mathrm{P}\big[X^{-1}(B)\big] = 0 \quad \forall B \in \Sigma' \text{ such that } \mu(B) = 0$$

then there exists a function f_X such that

$$F_X(t) = \int_{-\infty}^{t} f_X(r)\, dr$$

f_X is called the *probability density function* for X. A general probability density function for $X : (\Omega, \Sigma, P) \to (\mathbb{R}, \mathrm{BorelSets}, d\mathrm{Lebesgue})$ is any f_X such that

$$\mathrm{P}\big[X^{-1}(B)\big] = \int_{X^{-1}(B)} dP = \int_B f_X \, d\mu \qquad \forall B \in \Sigma'$$

Definition 1.26. If $X \in L^1(P)$ then the *expected value* of X is said to exist and is defined to be

$$\mathrm{E}\,[X] := \int_\Omega X \, dP$$

If X happens to have a probability density function,

$$\mathrm{E}\,[X] = \int_{\mathbb{R}} t f_X(t) \, dt$$

Definition 1.27. $X_1, \ldots, X_n$ are called *independent* random variables when

$$\mathrm{P}\big[X_1^{-1}(B) \cap \cdots \cap X_n^{-1}(B)\big] = \prod_{j=1}^{n} \mathrm{P}\big[X_j^{-1}(B)\big] \qquad \forall B \in \Sigma'$$

Definition 1.28. A *centered complex gaußian* random variable is a random variable

$$X : (\Omega, \Sigma, P) \to (\mathbb{C}, \mathcal{B}, \mathcal{L})$$

whose distribution is

$$(X_* P)(B) := \mathrm{P}\big[X^{-1}(B)\big] = \int_B \frac{1}{\pi\sigma^2} e^{-\frac{1}{\sigma^2}|z|^2} \, d\mathcal{L}(z)$$

When $\sigma = 1$ we say X is a *standard complex gaußian.*

Note any centered gaußian has expected value 0:

$$\int_\Omega X \, dP = \int_{\mathbb{C}} \frac{z}{\pi\sigma^2} e^{-\frac{1}{\sigma^2}|z|^2} \, dz = 0$$

where the "dz" above is shorthand for $d\mathcal{L}(z)$. In fact this integral is taken over both real dimensions (in this case "dz" $= \frac{i}{2} dz \wedge d\bar{z}$), unlike what the notation seems to imply.

Definition 1.29. More generally a collection of random variables $X_j : \Omega \to \mathbb{C}$ is said to be *jointly gaußian* if the complex valued random variable

$$a_1 X_1 + \cdots + a_n X_n$$

is a centered complex gaußian for any $a_j \in \mathbb{C}$.

Definition 1.30. The $n \times n$ symmetric positive semi-definite matrix

$$\Delta := [\,\mathrm{E}\,[X_i X_j]\,]_{i,j=1\ldots n}$$

is called the *covariance matrix* of $\vec{X}$. When the X_i are linearly independent, as in our calculation, Δ is non-singular, i.e. *positive definite.*

When the X_i are not linearly dependent, Definition 1.29 is equivalent to a more probability density style description. Specifically, Definition 1.29 in this case is equivalent to demanding that the random vector

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} : \Omega^n \to \mathbb{C}^n$$

has distribution

$$(\vec{X}_* P)(B) = \mathrm{P}\Big[\vec{X}^{-1}(B)\Big] = \int_B \frac{1}{\pi^n \det \Delta} e^{-\langle \Delta^{-1} z, z\rangle}\, dz$$

where

$$\text{“}dz\text{”} = d\mathcal{L}(z_1) \wedge \ldots \wedge d\mathcal{L}(z_n)$$

Definition 1.31. The distribution $\vec{X}_* P$ for any P as above is called the *joint probability distribution* of the X_j.

Lemma 1.32. *If $X_1, \ldots, X_n$ are jointly gaußian then the entries of $L(\vec{X})$ are, too, for any linear surjection $L : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^m$*

Proof:

If $[\,L\,] = [\,\ell_{ij}\,]$ and

$$L\left(\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right) = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}$$

then

$$\begin{aligned} a_1 Y_1 + \cdots + a_m Y_m &= a_1(\ell_{11} X_1 + \cdots + \ell_{1n} X_n) + \cdots + a_m(\ell_{m1} X_1 + \cdots + \ell_{mn} X_n) \\ &= (a_1 \ell_{11} + \cdots + a_m \ell_{m1}) X_1 + \cdots + (a_1 \ell_{1n} + \cdots + a_m \ell_{mn}) X_n \end{aligned}$$

a centered complex gaußian for any $a_j \in \mathbb{C}$. *Q.E.D.*

Random Sections and the Two Point Kernel

Here we define what we mean by "random sections" of the bundle $L^N \to M$.

Definition 1.33. The metric h induces hermitian metrics h^N on L^N given by $h^N(z) := h(z)^N$ i.e.

$$\langle s_1 \otimes \cdots \otimes s_N, t_1 \otimes \cdots \otimes t_N \rangle_{L^N} = \langle f_1 e_U \otimes \cdots \otimes f_N e_U, g_1 e_U \otimes \cdots \otimes g_N e_U \rangle_{L^N}$$

$$= f_1 \cdots f_N \bar{g}_1 \cdots \bar{g}_N \langle e_U \otimes \cdots \otimes e_U, e_U \otimes \cdots \otimes e_U \rangle_{L^N}$$

$$:= f_1 \cdots f_N \bar{g}_1 \cdots \bar{g}_N \langle e_U, e_U \rangle_L^N = f_1 \cdots f_N \bar{g}_1 \cdots \bar{g}_N h(z)^N$$

Definition 1.34. Using h^N we can create a new hermitian inner-product on $H^0(M, L^N)$ by

$$\langle s, t \rangle = \int_M \langle s, t \rangle_{L^N} \, d\mathrm{Vol}_M \qquad s, t \in H^0(M, L^N)$$

Throughout the rest of this section, (s_j^N) will denote an orthonormal basis for $H^0(M, L^N)$ with respect to $\langle\ ,\ \rangle$.

Definition 1.35. We can define a gaußian probability measure P on $H^0(M, L^N)$. Given

$$H^0(M, L^N) \ni s = \sum_{j=1}^{\ell} c_j(s)\, s_j^N$$

for any borel collection of sections $\mathcal{S}$,

$$\mathrm{P}[\mathcal{S}] := \int_{\mathcal{S}} \frac{1}{\pi^n} e^{-\langle c_j(s), c_j(s) \rangle} \, dc(s)$$

where $dc(s)$ is 2ℓ-dimensional Lebesgue measure.

P is characterized by the property that the 2ℓ real variables $\mathrm{Re}(c_j)$ and $\mathrm{Im}(c_j)$ are independent random variables with mean 0 and variance $1/2$. Specifically

$$\mathrm{E}[c_j] = 0 \qquad \mathrm{E}[c_j c_k] = 0 \qquad \mathrm{E}[c_j \bar{c}_k] = \delta_{jk}$$

For $c_1, \dots, c_\ell$ jointly gaußian, consider the random holomorphic section

$$s(z) := \sum_{j=1}^{\ell} c_j s_j^N(z)$$

and the map

$$\begin{bmatrix} c_1 \\ \vdots \\ c_\ell \end{bmatrix} \overset{\lambda}{\longmapsto} \begin{bmatrix} (\nabla'_z s)(z) \\ (\nabla'_z \nabla'_z s)(z) \\ (\nabla''_z \nabla'_z s)(z) \\ (\nabla'_w s)(w) \\ (\nabla'_w \nabla'_w s)(w) \\ (\nabla''_w \nabla'_w s)(w) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{\ell} c_j (\nabla'_z s_j^N)(z) \\ \sum_{j=1}^{\ell} c_j (\nabla'_z \nabla'_z s_j^N)(z) \\ \sum_{j=1}^{\ell} c_j (\nabla''_z \nabla'_z s_j^N)(z) \\ \sum_{j=1}^{\ell} c_j (\nabla'_w s_j^N)(w) \\ \sum_{j=1}^{\ell} c_j (\nabla'_w \nabla'_w s_j^N)(w) \\ \sum_{j=1}^{\ell} c_j (\nabla''_w \nabla'_w s_j^N)(w) \end{bmatrix}$$

For fixed z and w, λ is a linear map so Lemma 1.32 says the entries of $\lambda(\vec{c})$ are jointly gaußian. Their covariance matrix Δ has entries

$$
\begin{array}{lll}
\Delta_{11}=\mathrm{E}\big[(\nabla'_z s)(z)\otimes\overline{(\nabla'_z s)(z)}\big] & \Delta_{12}=\mathrm{E}\big[(\nabla'_z s)(z)\otimes\overline{(\nabla'_z\nabla'_z s)(z)}\big] & \Delta_{13}=\mathrm{E}\big[(\nabla'_z s)(z)\otimes\overline{(\nabla''_z\nabla'_z s)(z)}\big] \\
\Delta_{21}=\mathrm{E}\big[(\nabla'_z\nabla'_z s)(z)\otimes\overline{(\nabla'_z s)(z)}\big] & \Delta_{22}=\mathrm{E}\big[(\nabla'_z\nabla'_z s)(z)\otimes\overline{(\nabla'_z\nabla'_z s)(z)}\big] & \Delta_{23}=\mathrm{E}\big[(\nabla'_z\nabla'_z s)(z)\otimes\overline{(\nabla''_z\nabla'_z s)(z)}\big] \\
\Delta_{31}=\mathrm{E}\big[(\nabla''_z\nabla'_z s)(z)\otimes\overline{(\nabla'_z s)(z)}\big] & \Delta_{32}=\mathrm{E}\big[(\nabla''_z\nabla'_z s)(z)\otimes\overline{(\nabla'_z\nabla'_z s)(z)}\big] & \Delta_{33}=\mathrm{E}\big[(\nabla''_z\nabla'_z s)(z)\otimes\overline{(\nabla''_z\nabla'_z s)(z)}\big] \\
\Delta_{41}=\mathrm{E}\big[({\nabla'}_w s)(w)\otimes\overline{(\nabla'_z s)(z)}\big] & \Delta_{42}=\mathrm{E}\big[({\nabla'}_w s)(w)\otimes\overline{(\nabla'_z\nabla'_z s)(z)}\big] & \Delta_{43}=\mathrm{E}\big[({\nabla'}_w s)(w)\otimes\overline{(\nabla''_z\nabla'_z s)(z)}\big] \\
\Delta_{51}=\mathrm{E}\big[({\nabla'}_w{\nabla'}_w s)(w)\otimes\overline{(\nabla'_z s)(z)}\big] & \Delta_{52}=\mathrm{E}\big[({\nabla'}_w{\nabla'}_w s)(w)\otimes\overline{(\nabla'_z\nabla'_z s)(z)}\big] & \Delta_{53}=\mathrm{E}\big[({\nabla'}_w{\nabla'}_w s)(w)\otimes\overline{(\nabla''_z\nabla'_z s)(z)}\big] \\
\Delta_{61}=\mathrm{E}\big[(\nabla''_w{\nabla'}_w s)(w)\otimes\overline{(\nabla'_z s)(z)}\big] & \Delta_{62}=\mathrm{E}\big[(\nabla''_w{\nabla'}_w s)(w)\otimes\overline{(\nabla'_z\nabla'_z s)(z)}\big] & \Delta_{63}=\mathrm{E}\big[(\nabla''_w{\nabla'}_w s)(w)\otimes\overline{(\nabla''_z\nabla'_z s)(z)}\big] \\
\Delta_{14}=\mathrm{E}\big[(\nabla'_z s)(z)\otimes\overline{({\nabla'}_w s)(w)}\big] & \Delta_{15}=\mathrm{E}\big[(\nabla'_z s)(z)\otimes\overline{({\nabla'}_w{\nabla'}_w s)(w)}\big] & \Delta_{16}=\mathrm{E}\big[(\nabla'_z s)(z)\otimes\overline{(\nabla''_w{\nabla'}_w s)(w)}\big] \\
\Delta_{24}=\mathrm{E}\big[(\nabla'_z\nabla'_z s)(z)\otimes\overline{({\nabla'}_w s)(w)}\big] & \Delta_{25}=\mathrm{E}\big[(\nabla'_z\nabla'_z s)(z)\otimes\overline{({\nabla'}_w{\nabla'}_w s)(w)}\big] & \Delta_{26}=\mathrm{E}\big[(\nabla'_z\nabla'_z s)(z)\otimes\overline{(\nabla''_w{\nabla'}_w s)(w)}\big] \\
\Delta_{34}=\mathrm{E}\big[(\nabla''_z\nabla'_z s)(z)\otimes\overline{({\nabla'}_w s)(w)}\big] & \Delta_{35}=\mathrm{E}\big[(\nabla''_z\nabla'_z s)(z)\otimes\overline{({\nabla'}_w{\nabla'}_w s)(w)}\big] & \Delta_{36}=\mathrm{E}\big[(\nabla''_z\nabla'_z s)(z)\otimes\overline{(\nabla''_w{\nabla'}_w s)(w)}\big] \\
\Delta_{44}=\mathrm{E}\big[({\nabla'}_w s)(w)\otimes\overline{({\nabla'}_w s)(w)}\big] & \Delta_{45}=\mathrm{E}\big[({\nabla'}_w s)(w)\otimes\overline{({\nabla'}_w{\nabla'}_w s)(w)}\big] & \Delta_{46}=\mathrm{E}\big[({\nabla'}_w s)(w)\otimes\overline{(\nabla''_w{\nabla'}_w s)(w)}\big] \\
\Delta_{54}=\mathrm{E}\big[({\nabla'}_w{\nabla'}_w s)(w)\otimes\overline{({\nabla'}_w s)(w)}\big] & \Delta_{55}=\mathrm{E}\big[({\nabla'}_w{\nabla'}_w s)(w)\otimes\overline{({\nabla'}_w{\nabla'}_w s)(w)}\big] & \Delta_{56}=\mathrm{E}\big[({\nabla'}_w{\nabla'}_w s)(w)\otimes\overline{(\nabla''_w{\nabla'}_w s)(w)}\big] \\
\Delta_{64}=\mathrm{E}\big[(\nabla''_w{\nabla'}_w s)(w)\otimes\overline{({\nabla'}_w s)(w)}\big] & \Delta_{65}=\mathrm{E}\big[(\nabla''_w{\nabla'}_w s)(w)\otimes\overline{({\nabla'}_w{\nabla'}_w s)(w)}\big] & \Delta_{66}=\mathrm{E}\big[(\nabla''_w{\nabla'}_w s)(w)\otimes\overline{(\nabla''_w{\nabla'}_w s)(w)}\big]
\end{array}
\tag{1.3}
$$

by abuse of notation. Each entry should be replaced by its coefficient when written in a local frame about z and w.

In fact all of the entries of Δ used in our calculation can be rewritten in terms of derivatives of an important invariant of P called the "two point kernel".

Definition 1.36. The *two-point kernel* (or *covariance kernel*) for $H^0(M, L^N)$ is defined by

$$\Pi_N(z,w) := \sum_{j=1}^{\ell} s_j^N(z) \otimes \overline{s_j^N(w)} \in L_z^N \otimes \overline{L_w^N} \quad (z,w) \in M$$

Since L^N is hermitian, Π_N is the Szegő kernel of (L^N, h^N), i.e. the orthogonal projection

$$\Pi_{N,h^N,\mathrm{Vol}_M} : L^2(M, L^N) \to H^0(M, L^N)$$

with respect to $\langle\ ,\ \rangle$.

Π_N and the entries of Δ are related because

$$\begin{aligned}
\mathrm{E}\left[s(z)\otimes\overline{s(w)}\right] &= \mathrm{E}\left[\sum_{j=1}^{\ell} c_j s_j^N(z)\otimes\overline{\sum_{k=1}^{\ell} c_k s_k^N(w)}\right] = \sum_{j,k=1}^{\ell}\mathrm{E}\left[c_j\bar{c}_k\right] s_j^N(z)\otimes\overline{s_k^N(w)} \\
&= \sum_{j,k=1}^{\ell}\delta_{jk} s_j^N(z)\otimes\overline{s_k^N(w)} = \sum_{j=1}^{\ell} s_j^N(z)\otimes\overline{s_j^N(w)} \\
&=: \Pi_N(z,w)
\end{aligned}$$

so differentiating both sides yields, for instance,

$$\begin{aligned}
\nabla_z'\nabla_w''\Pi_N(z,w) &= \nabla_z'\nabla_w''\mathrm{E}\left[s(z)\otimes\overline{s(w)}\right] = \nabla_z'\mathrm{E}\left[s(z)\otimes\nabla_w''\overline{s(w)}\right] = \nabla_z'\mathrm{E}\left[s(z)\otimes\overline{\nabla_w' s(w)}\right] \\
&= \mathrm{E}\left[\nabla_z' s(z)\otimes\overline{\nabla_w' s(w)}\right] = \Delta_{14}
\end{aligned} \tag{1.4}$$

The Kac-Rice Theorem

Various generalizations of Rice's original theorem [Ric39](3) are referred to as "The Kac-Rice Theorem" in current literature. What is meant by "using the Kac-Rice theorem" is that the expected density of zeroes of a random linear combination of functions

$$f_{\vec{a}}(x) := \sum_{j=0}^{\ell} a_j f_j(x)$$

is found by integrating the joint distribution of $f_{\vec{a}}$ and $f'_{\vec{a}}$ with $f_{\vec{a}}$ replaced by 0 against $\|f'_{\vec{a}}\|$. For instance, the single-variable real Kac-Rice theorem says the following:

Take P a probability measure on $\mathbb{R}^\ell$ and $f_1, \dots, f_\ell$ a collection of analytic functions. For fixed t,

$$x_t := f_{\vec{a}}(t) : \mathbb{R}^\ell \to \mathbb{R}$$

and

$$\xi_t := f'_{\vec{a}}(t) : \mathbb{R}^\ell \to \mathbb{R}$$

are random variables so they have a joint probability distribution, namely the distribution for the random variable

$$X_t := \begin{bmatrix} x_t \\ \xi_t \end{bmatrix} : \mathbb{R}^\ell \to \mathbb{R}^2$$

specifically

$$(X_t)_* P =: D_t(x, \xi)$$

Definition 1.37. For the function $f_{\vec{a}}$ define the measure

$$Z_{f_{\vec{a}}} := \sum_{f_{\vec{a}}(t_j)=0} \delta_{t_j}$$

The demand that the f_j be analytic ensures $f_{\vec{a}}$ has only finitely many zeroes on bounded intervals, so $Z_{f_{\vec{a}}}$ is a sum of point masses. $Z_{f_{\vec{a}}}$ can be generalized to mean the current of integration along the regular points of the variety $\{f_{\vec{a}} = 0\}$ even when not discrete, but that is not necessary for our computation.

The Kac-Rice theorem says

$$\mathrm{E}\left[Z_{f_{\vec{a}}}\right] = K(t)\, dt$$

where

$$K(t) := \int_{\mathbb{R}} D_t(0,\xi)|\xi|\,d\xi$$

Definition 1.38. The $K(t)\,dt$ above is called the *one-point correlation* or *one-point density* of $Z_{f_{\vec{a}}}$.

Definition 1.39. If we define the measure of n simultaneous zeroes

$$Z^n_{f_{\vec{a}}} := \sum_{\{\vec{t}\in M\times\cdots\times M | f_{\vec{a}}(t_1)=\cdots=f_{\vec{a}}(t_n)=0\}} \delta_{\vec{t}}$$

the Kac-Rice theorem says the same for a measure $K_n(\vec{t})\,d\vec{t}$ called the *n-point correlation* or *n-point density* of $Z^n_{f_{\vec{a}}}$.

This dissertation is, in fact, concerned with the 2-point correlation of the simultaneous zeroes of two random *sections* $\nabla s^N(z), \nabla s^N(w) \in T^*_M \otimes L^N$ for $s^N \in H^0(M, L^N)$ which are called *critical points* of s^N. Here "random" means that the s^N are chosen with respect to the gaußian probability measure on $H^0(M, L^N)$ given in Definition 1.35 and $\dim_{\mathbb{C}} M = 1$. In this particular case, the Kac-Rice theorem is

$$\mathrm{E}\left[Z_{\nabla s(z),\nabla s(w)}\right] = K_2(z,w)\,dz\,dw$$

where

$$\begin{aligned} K_2(z,w) &= \int_W D_{z,w}(\vec{0},\vec{\xi})\det(\xi^1(\xi^1)^*)^{\frac{1}{2}}\det(\xi^2(\xi^2)^*)^{\frac{1}{2}}\,d\xi \\ &= \int_W D_{z,w}(\vec{0},\vec{\xi})\cdot|\det\xi^1|\cdot|\det(\xi^2)|\,d\xi \qquad \text{due to the makeup of } \xi^1 \text{ and } \xi^2 \end{aligned}$$

where $D_{z,w}(\vec{x},\vec{\xi})$ is the joint probability distribution of the random sections

$$x_1 = \nabla s(z) = \nabla'_z s(z) \qquad \text{since } s \text{ is holomorphic}$$

$$x_2 = \nabla s(w) = \nabla'_w s(w) \qquad \text{since } s \text{ is holomorphic}$$

$$\xi_1 = \nabla'\nabla s(z)$$

$$\xi_2 = \nabla''\nabla s(z)$$

$$\xi_3 = \nabla'\nabla s(w)$$

$$\xi_4 = \nabla''\nabla s(w)$$

and

$$\begin{aligned} \xi^1 &= \text{“}\nabla\nabla s(z)\text{”} = \begin{bmatrix} \nabla'\nabla s(z) & \overline{\nabla''\nabla s(z)} \\ \nabla''\nabla s(z) & \overline{\nabla'\nabla s(z)} \end{bmatrix} = \begin{bmatrix} \nabla'_z\nabla'_z s(z) & \overline{\nabla''_z\nabla'_z s(z)} \\ \nabla''_z\nabla'_z s(z) & \overline{\nabla'_z\nabla'_z s(z)} \end{bmatrix} \\ \xi^2 &= \text{“}\nabla\nabla s(w)\text{”} = \begin{bmatrix} \nabla'\nabla s(w) & \overline{\nabla''\nabla s(w)} \\ \nabla''\nabla s(w) & \overline{\nabla'\nabla s(w)} \end{bmatrix} = \begin{bmatrix} \nabla'_w\nabla'_w s(w) & \overline{\nabla''_w\nabla'_w s(w)} \\ \nabla''_w\nabla'_w s(w) & \overline{\nabla'_w\nabla'_w s(w)} \end{bmatrix} \end{aligned}$$

as in [BSZ01](33). Here

$$W = \nabla'\big((T_M^* \otimes L^N\big)_z\big) \times \nabla''\big((T_M^* \otimes L^N\big)_z\big) \times \nabla'\big((T_M^* \otimes L^N\big)_w\big) \times \nabla''\big((T_M^* \otimes L^N\big)_w\big)$$

and $d\xi$ means Lebesgue measure with respect to the hermitian metric on W.

As in the previous section, $x_1, x_2, \xi_1, \xi_2, \xi_3$, and ξ_4 are jointly gaußian with covariance matrix $\Delta =$(1.3) so

$$D_{z,w}(x_1, x_2, \xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{\pi^6 \det \Delta(z,w)} \exp\left[- \left\langle \Delta^{-1}(z,w) \begin{bmatrix} \vec{x} \\ \vec{\xi} \end{bmatrix}, \begin{bmatrix} \vec{x} \\ \vec{\xi} \end{bmatrix} \right\rangle \right]$$

so

$$D_{z,w}(0, 0, \xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{\pi^6 \det \Delta(z,w)} \exp\left[- \left\langle \Delta^{-1}(z,w) \begin{bmatrix} \vec{0} \\ \vec{\xi} \end{bmatrix}, \begin{bmatrix} \vec{0} \\ \vec{\xi} \end{bmatrix} \right\rangle \right]$$

Dividing Δ into blocks

$$\Delta = \begin{bmatrix} [\,A\,]_{2\times 2} & [\,B\,]_{2\times 4} \\ [\,B^*\,]_{4\times 2} & [\,C\,]_{4\times 4} \end{bmatrix}_{6\times 6} \tag{1.5}$$

and using the formula for inverting matrices presented in blocks

$$\begin{aligned} \Delta^{-1} &= \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \underbrace{(C - B^*A^{-1}B)^{-1}}_{\Lambda^{-1}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^*A^{-1} & I \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} A^{-1} + A^{-1}B\Lambda^{-1}B^*A^{-1} \end{bmatrix}_{2\times 2} & \begin{bmatrix} -A^{-1}B\Lambda^{-1} \end{bmatrix}_{2\times 4} \\ \begin{bmatrix} -\Lambda^{-1}B^*A^{-1} \end{bmatrix}_{4\times 2} & \begin{bmatrix} \Lambda^{-1} \end{bmatrix}_{4\times 4} \end{bmatrix} \end{aligned}$$

meaning

$$\Delta^{-1} \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1} \\ \begin{bmatrix} \vec{\xi} \end{bmatrix}_{4\times 1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A^{-1} + A^{-1}B\Lambda^{-1}B^*A^{-1} \end{bmatrix} & \begin{bmatrix} -A^{-1}B\Lambda^{-1} \end{bmatrix} \\ \begin{bmatrix} -\Lambda^{-1}B^*A^{-1} \end{bmatrix} & \begin{bmatrix} \Lambda^{-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \vec{\xi} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -A^{-1}B\Lambda^{-1}\vec{\xi} \\ \Lambda^{-1}\vec{\xi} \end{bmatrix}$$

so

$$\left[\left\langle \tilde{\Delta}^{-1} \begin{bmatrix} \vec{0} \\ \vec{\xi} \end{bmatrix}, \begin{bmatrix} \vec{0} \\ \vec{\xi} \end{bmatrix} \right\rangle \right] = \left[(-A^{-1}B\Lambda^{-1}\vec{\xi})^* \quad (\Lambda^{-1}\vec{\xi})^* \right] \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \vec{\xi} \end{bmatrix} \end{bmatrix} = \left[(\Lambda^{-1}\vec{\xi})^* \vec{\xi} \right] = \left[\langle \Lambda^{-1}\vec{\xi}, \vec{\xi} \rangle \right]$$

Using the formula for determinants of matrices presented as blocks says

$$\det \Delta = (\det A)(\det \Lambda) \tag{1.6}$$

so the integrand in $K_2(z, w)$ is

$$\frac{e^{-\langle \Lambda^{-1}(z,w)\vec{\xi},\vec{\xi}\rangle}}{\pi^6 \det A(z,w) \det \Lambda(z,w)} \det(\xi^1(\xi^1)^*)^{\frac{1}{2}} \det(\xi^2(\xi^2)^*)^{\frac{1}{2}}\, d\xi$$

Now, as mentioned above, the particular entries of the ξ^j allow the $\det(\xi^j(\xi^j)^*)^{\frac{1}{2}}$ to be simplified as in [DSZ04](34).

$$\begin{aligned}
\det(\xi^1(\xi^1)^*)^{\frac{1}{2}} \det(\xi^2(\xi^2)^*)^{\frac{1}{2}} &= \det\left(\begin{bmatrix} \xi_1 & \bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{bmatrix}\begin{bmatrix} \xi_1 & \bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{bmatrix}^*\right)^{\frac{1}{2}} \det\left(\begin{bmatrix} \xi_1 & \bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{bmatrix}\begin{bmatrix} \xi_1 & \bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{bmatrix}^*\right)^{\frac{1}{2}} \\
&= \left(\left(|\xi_1|^2 - |\xi_2|^2\right)^2\right)^{\frac{1}{2}} \left(\left(|\xi_3|^2 - |\xi_4|^2\right)^2\right)^{\frac{1}{2}} \\
&= \left||\xi_1|^2 - |\xi_2|^2\right| \cdot \left||\xi_3|^2 - |\xi_4|^2\right|
\end{aligned}$$

Identifying W with $\mathbb{C}^4$, the two-point correlation of critical points on a Riemann surface is

$$K_2(z, w) = \int_{\mathbb{C}^4} \frac{e^{-\langle \Lambda^{-1}\vec{\xi},\vec{\xi}\rangle}}{\pi^6 \det A \det \Lambda} \left||\xi_1|^2 - |\xi_2|^2\right| \cdot \left||\xi_3|^2 - |\xi_4|^2\right| d\vec{\xi} \tag{1.7}$$

The Scaling Limit

Although the critical point equation $\nabla s(z) = 0$ is not holomorphic, it is still smooth, so the results of [BSZ00b] about the zeroes of random smooth sections apply. The main theorem (3.6) of [BSZ00b] actually says that as $N \to \infty$, the "scaling limit" of the correlation of zeroes is independent of choice of M, L, and h. Specifically

$$\frac{1}{N^{nk}} K^N_{nk}\left(\frac{z_1}{\sqrt{N}}, \dots, \frac{z_n}{\sqrt{N}}\right) = K^\infty_{nkm}(z_1, \dots, z_n) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

where K^∞_{nkm} depends only on n, k, and m.

If we write the 2-point correlation (1.7) as K^N_2 to reflect the N dependency in our case, this theorem says

$$\frac{1}{N^2} K^N_2\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) = K^\infty_2(z, w) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

so proving Theorem 1.1 only requires that we calculate the N limit of the left hand side for z close to w for a particularly nice choice of M, L, and h.

Theorem 3.1 of [BSZ00b] says roughly that in the N limit, the $\frac{1}{N}\Pi_N(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}})$ entries in K^N_2 can be replaced by $\Pi^{\mathbf{H}}_1(\tilde{z}, \tilde{w})$, the Szegő kernel of the *reduced Heisenberg Group* $\mathbf{H}_{\text{red}}$ which we define below. For a more thorough geometric discussion of $\mathbf{H}_{\text{red}}$'s construction and properties, see [BSZ00b]§1.3.2.

Definition 1.40. Take the trivial bundle $L := \mathbb{C} \times \mathbb{C}$ over $\mathbb{C}$ with curvature $h(z) := e^{-|z|^2}$. Then $h^{-1}(z) = e^{|z|^2}$ gives a metric on the dual bundle $L^* \to \mathbb{C}$. Form the "circle bundle", X, of elements $v \in L^*$ such that $h^{-1}(v) = 1$. i.e. $X = \{(z, \zeta) \in \mathbb{C} \times \mathbb{C} \mid |\zeta| = e^{-\frac{|z|^2}{2}}\}$. This bundle $X \to \mathbb{C}$ is the *reduced Heisenberg Group*, written $\mathbf{H}_{\text{red}}$. When necessary, since $X \cong \mathbb{C} \times S^1$, we will write elements as (z, θ). Because $L \to \mathbb{C}$ here is the trivial bundle, we may use the frame $e_U = e_{\mathbb{C}} = 1$, the constant function 1.

The Szegő kernel for $\mathbf{H}_{\text{red}}$, $\Pi^{\mathbf{H}}_1$ is by definition the kernel of orthogonal projection from $\mathcal{L}^2(\mathbf{H}_{\text{red}})$ to the Hardy space for $\mathbf{H}_{\text{red}}$, $\mathcal{H}^2_1$. These spaces can be viewed as

$$\mathcal{L}^2(\mathbf{H}_{\text{red}}) = \left\{\tilde{f}(z, \theta) = f(z)e^{i\theta}e^{\frac{-|z|^2}{2}} \;\middle|\; f \in \mathcal{C}^\infty, \int_{\mathbb{C}} f(z)e^{-|z|^2}\, dz\, d\bar{z} < \infty\right\}$$

and

$$\mathcal{H}_1^2 = \left\{ \tilde{f}(z,\theta) = f(z)e^{i\theta}e^{\frac{-|z|^2}{2}} \;\middle|\; f \text{ holomorphic}, \int_{\mathbb{C}} f(z)e^{-|z|^2}\, dz\, d\bar{z} < \infty \right\}$$

In fact, $\varphi_k(z) := \frac{1}{\sqrt{\pi k!}} z^k e^{i\theta} e^{\frac{-|z|^2}{2}}$ are an orthonormal basis for $\mathcal{L}^2(\mathbf{H}_{\text{red}})$ since

$$\frac{1}{\pi\sqrt{j!k!}} \int_{\mathbb{C}^2} z^j \bar{z}^k\, e^{i(\theta-\varphi)} e^{-|z|^2} \left(\frac{i}{2}\, dz\, d\bar{z}\right) = \delta_{jk}$$

so the kernel for the orthogonal projection $\mathcal{L}^2(\mathbf{H}_{\text{red}}) \to \mathcal{H}_1^2$ is

$$\tilde{\Pi}_1^{\mathbf{H}}((z,\theta),(w,\varphi)) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(w)} = \frac{1}{\pi} e^{i(\theta-\varphi)} e^{z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2}$$

The extra factor $e^{\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2}$ appears because we have chosen a non-trivial metric h for the trivial bundle. As long as the connection is computed correctly, we could work in any frame, so the formula becomes

$$\Pi_1^{\mathbf{H}}(z,w) = \frac{1}{\pi} e^{z\bar{w}}$$

based on the frame $e_{\mathbb{C}} = 1$ (see Definition 1.36). Since (1.7) doesn't change when $\Pi_1^{\mathbf{H}}(z,w)$ is multiplied by a non-zero scalar (hence likewise if Δ is multiplied by a non-zero scalar), we will use

$$\Pi_1^{\mathbf{H}}(z,w) = e^{z\bar{w}}$$

for ease of calculation.

The Chern connection is defined by its action on a frame

$$\nabla_z^{\mathbf{H}} e_{\mathbb{C}}(z) = \left(\nabla_z^{\mathbf{H}}\right)' e_{\mathbb{C}}(z) = -\bar{z}\, dz \otimes e_{\mathbb{C}}(z)$$

Often, we will write $\Pi_1^{\mathbf{H}}((z,\theta),(w,\varphi))$ and mean only the function coefficient $e^{z\bar{w}}$.

For our case, we form the Szegő kernel for the Nth power of an arbitrary positive line bundle over a Riemann surface $(L^N, h^N) \to M$ similarly, defining the circle bundle

$$X_M := \{ s \in L^* \mid \langle s(z), s(z) \rangle = 1 \}$$

and calling the Szegő kernel

$$\tilde{\Pi}_N : \tilde{X} \times \bar{X} \longrightarrow \mathbb{C}$$

This Szegő kernel is related to our earlier two-point function by

$$\tilde{\Pi}_N\left(\left(\frac{z}{\sqrt{N}},0\right),\left(\frac{w}{\sqrt{N}},0\right)\right)=\left(\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)\ ,\ \ h(z)^{\frac{N}{2}}(e_U^*)^N(z)\otimes h(w)^{\frac{N}{2}}(e_U^*)^N(w)\right)$$

With those definitions, we are in the position to state Theorem 3.1 of [BSZ00b] precisely in our case:

Choose $z_0\in M$, local coordinate map z, and a local frame e_L over a neighborhood of z_0 so that

$$\Theta_h(z_0)=(\partial\bar{\partial}\log h)(z_0)=dz\wedge d\bar{z}|_{z_0}$$

and

$$\frac{\partial h}{\partial z}(z_0)=\frac{\partial^2 h}{\partial z^2}(z_0)=0$$

then

$$\frac{1}{N}\tilde{\Pi}_N\left(\left(z_0+\frac{u}{\sqrt{N}},\frac{\theta}{N}\right),\left(z_0+\frac{v}{\sqrt{N}},\frac{\varphi}{N}\right)\right)=\tilde{\Pi}_1^{\mathbf{H}}((u,\theta),(v,\varphi))+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

where $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ means a function whose $\mathcal{C}^k$-norm is $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ in the standard sense for all k.

For

$$\begin{aligned}f_w(z)&:=\tilde{\Pi}_N\Big((z_0+z,0)\ ,\ (z_0+w,0)\Big)\\ g_w(z)&:=f_w\left(\frac{z}{\sqrt{N}}\right)=\tilde{\Pi}_N\Big((z_0+\frac{z}{\sqrt{N}},0)\ ,\ (z_0+w,0)\Big)\end{aligned}\tag{1.8}$$

[BSZ00b]'s theorem says

$$\frac{1}{N}f_{\frac{w}{\sqrt{N}}}\left(\frac{z}{\sqrt{N}}\right)=\frac{1}{N}g_{\frac{w}{\sqrt{N}}}(z)=\Pi_1^{\mathbf{H}}((z,0),(w,0))+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

Taking derivatives on both sides of (1.8)

$$g_w'(z)=\frac{1}{\sqrt{N}}f_w'\left(\frac{z}{\sqrt{N}}\right)$$

so

$$\begin{aligned}\frac{1}{N^{\frac{3}{2}}}\left(\frac{\partial\tilde{\Pi}_N}{\partial z}\right)\left((z_0+\frac{z}{\sqrt{N}},0),(z_0+\frac{w}{\sqrt{N}},0)\right)&=\frac{1}{N^{\frac{3}{2}}}f'_{\frac{w}{\sqrt{N}}}\left(\frac{z}{\sqrt{N}}\right)=\frac{1}{N}g'_{\frac{w}{\sqrt{N}}}(z)\\ &=\frac{1}{N}\frac{\partial}{\partial z}\left(f_{\frac{w}{\sqrt{N}}}\left(\frac{z}{\sqrt{N}}\right)\right)=\frac{\partial}{\partial z}\left(\frac{1}{N}f_{\frac{w}{\sqrt{N}}}\left(\frac{z}{\sqrt{N}}\right)\right)\\ &=\frac{\partial}{\partial z}\left(\Pi_1^{\mathbf{H}}((z,0),(w,0))+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right)\end{aligned}$$

Choose a local frame e_U^N for L^N over $U \subset M$ such that $h(z) = 1-|z|^2+\mathcal{O}(|z|^3)$ by taking an arbitrary frame and multiplying by a smooth function with appropriate first and second order terms. Then

$$\nabla_{h^N} = d + N\partial \log h$$

and

$$\begin{aligned}
\frac{\partial}{\partial z} \log h &= \frac{\partial}{\partial z} \log(1 - |z|^2 + \mathcal{O}(|z|^3)) \\
&= \frac{1}{1 - |z|^2 + \mathcal{O}(|z|^3)} (-\bar{z} + \mathcal{O}(|z|^2)) \\
&= (1 + \mathcal{O}(|z|^2))(-\bar{z} + \mathcal{O}(|z|^2)) \\
&= -\bar{z} + \mathcal{O}(|z|^2)
\end{aligned}$$

meaning

$$N\left(\frac{\partial}{\partial z} \log h\right)\left(\frac{z}{\sqrt{N}}\right) = N\left(-\bar{z} + \mathcal{O}(|z|^2)\right)\Big|_{\frac{z}{\sqrt{N}}} = N\left(-\frac{\bar{z}}{\sqrt{N}} + \mathcal{O}\left(\frac{|z|^2}{N}\right)\right) = -\sqrt{N}\bar{z} + \mathcal{O}(|z|^2)$$

so

$$\begin{aligned}
\left(\frac{1}{N^{\frac{3}{2}}}\nabla_z^N \Pi_N\right)\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) &= \frac{1}{N^{\frac{3}{2}}}\left[d + N\partial \log h\right] \Pi_N\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) \\
&= \frac{1}{N^{\frac{3}{2}}}\frac{\partial \Pi_N}{\partial z}\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) + \frac{1}{N^{\frac{3}{2}}} N\partial \log h\left(\frac{z}{\sqrt{N}}\right) \\
&= \frac{1}{N^{\frac{3}{2}}}\frac{\partial \tilde{\Pi}_N}{\partial z}\left((\frac{z}{\sqrt{N}}, 0), (\frac{w}{\sqrt{N}}, 0)\right) + \frac{1}{N^{\frac{3}{2}}} N\partial \log h\left(\frac{z}{\sqrt{N}}\right) \\
&= \frac{\partial}{\partial z}\left(\Pi_1^{\mathbf{H}}((z,0),(w,0)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right) + \frac{1}{N^{\frac{3}{2}}}(-\sqrt{N}\bar{z} + \mathcal{O}(|z|^2)) \\
&= \frac{\partial}{\partial z}\left(\Pi_1^{\mathbf{H}}((z,0),(w,0)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right) - \frac{1}{N}\bar{z} + \frac{1}{N^{\frac{3}{2}}}\mathcal{O}(|z|^2) \\
&= \nabla_z^{\mathbf{H}} \Pi_1^{\mathbf{H}}((z,0),(w,0)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)
\end{aligned}$$

so

$$\frac{1}{N^2}A^p_{p'} := \frac{1}{N^2}\nabla'_z\nabla''_w\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})}$$
$$= \left(\nabla^{\mathbf{H}}_z\right)'\left(\nabla^{\mathbf{H}}_w\right)''\Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
$$\frac{1}{N^{\frac{5}{2}}}B^p_{p'1} := \frac{1}{N^{\frac{5}{2}}}\nabla'_z\nabla''_w\nabla''_w\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})}$$
$$= \left(\nabla^{\mathbf{H}}_z\right)'\left(\nabla^{\mathbf{H}}_w\right)''\left(\nabla^{\mathbf{H}}_w\right)''\Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
$$\frac{1}{N^{\frac{5}{2}}}B^p_{p'2} := \frac{1}{N^{\frac{5}{2}}}\nabla'_z\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})} = \left(\nabla^{\mathbf{H}}_z\right)'\Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
$$\frac{1}{N^3}C^{p\,1}_{p'1} := \frac{1}{N^3}\nabla'_z\nabla'_z\nabla''_w\nabla''_w\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})}$$
$$= \left(\nabla^{\mathbf{H}}_z\right)'\left(\nabla^{\mathbf{H}}_z\right)'\left(\nabla^{\mathbf{H}}_w\right)''\left(\nabla^{\mathbf{H}}_w\right)''\Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
$$\frac{1}{N^3}C^{p\,1}_{p'2} := \frac{1}{N^3}\nabla'_z\nabla'_z\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})} = \left(\nabla^{\mathbf{H}}_z\right)'\left(\nabla^{\mathbf{H}}_z\right)'\Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
$$\frac{1}{N^3}C^{p\,2}_{p'1} := \frac{1}{N^3}\nabla''_w\nabla''_w\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})}$$
$$= \left(\nabla^{\mathbf{H}}_w\right)''\left(\nabla^{\mathbf{H}}_w\right)''\Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
$$\frac{1}{N^3}C^{p\,2}_{p'2} := \frac{1}{N^3}\Pi_N\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right)|_{(\zeta_p,\zeta_{p'})} = \Pi^{\mathbf{H}}_1((z,0),(w,0))|_{(\zeta_p,\zeta_{p'})}+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

and

$$\frac{1}{N^3}\Lambda = \Lambda^{\mathbf{H}} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$

where $\Lambda^{\mathbf{H}}$ is Λ with all of the Π_N terms replaced by $\Pi^{\mathbf{H}}_1$ terms and

$$B = [\,[\,B^p_{p'1}\,]\,[\,B^p_{p'2}\,]\,]$$
$$C = \begin{bmatrix} [\,C^{p\,1}_{p'1}\,] & [\,C^{p\,1}_{p'2}\,] \\ [\,C^{p\,2}_{p'1}\,] & [\,C^{p\,2}_{p'2}\,] \end{bmatrix}$$

So finally,

$$\frac{1}{N^2}K^N_2\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right) = \frac{N^{-2}}{\pi^6 N^{16}\det\Delta^{\mathbf{H}}}\int_{\mathbb{C}^4} e^{-N^{-3}\langle(\Lambda^{\mathbf{H}})^{-1}\vec{\xi},\vec{\xi}\rangle}\|\det\vec{\xi}\|\,d\vec{\xi}$$

where $\Delta^{\mathbf{H}}$ is Δ with all of the Π_N terms replaced by $\Pi^{\mathbf{H}}_1$ terms and $\|\det\vec{\xi}\|$ is shorthand for $|\det\xi^1||\det\xi^2|$.

Now perform the change of variables

$$\vec{v} := \begin{bmatrix} h_1 \\ x_1 \\ h_2 \\ x_2 \end{bmatrix} := N^{-\frac{3}{2}}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = N^{-\frac{3}{2}}\vec{\xi}$$

Now $\|\det\vec{\xi}\| = N^6\|\det\vec{v}\|$ and $d\vec{\xi} = N^{12}d\vec{v}$ so

$$\frac{1}{N^2}K^N_2\left(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}\right) = \frac{1}{\pi^6\det\Delta^{\mathbf{H}}}\int_{\mathbb{C}^4} e^{-\langle(\Lambda^{\mathbf{H}})^{-1}\vec{v},\vec{v}\rangle}\|\det\vec{v}\|\,d\vec{v}$$

Additional definitions and notes

Definition 1.41. A *critical point of* $s \in H^0(M, L)$ *with respect to* ∇_h is any $z \in M$ such that $\nabla_h s(z) = 0$. For almost any $s \in H^0(M, L)$, the set

$$\mathrm{Crit}^{\nabla_h}(s) := \{z \in M \mid \nabla_h s(z) = 0\}$$

is discrete, so the following definition makes sense.

Definition 1.42. The measure associated to $\mathrm{Crit}^{\nabla_h}(s)$ is

$$C_s^{\nabla_h} := \sum_{z \in \mathrm{Crit}^{\nabla_h}(s)} \delta_z$$

where δ_z is the point-mass at z.

Definition 1.43. The volume form associated to h, dV_h, is given by

$$dV_h := \frac{1}{m!}\left(-\frac{i}{2}\partial\bar{\partial}\log h\right)^m$$

Definition 1.44. The *two-point correlation* was not originally defined as the necessary integrand in the Kac-Rice formula. It can be defined directly as

$$K_2(z, w) = \lim_{\varepsilon \to 0} \frac{\mathrm{E}\left[\#[\mathrm{Crit}^{\nabla_h}(s) \cap B_\varepsilon(z)] \cdot \#[\mathrm{Crit}^{\nabla_h}(s) \cap B_\varepsilon(w)]\right]}{\mathrm{Vol}\left[B_\varepsilon(z) \times B_\varepsilon(w)\right]}$$

where

$$B_\varepsilon(z) := \text{the ball of radius } \varepsilon \text{ about } z$$

$$\#A := \text{the cardinality of } A$$

$K_2(z, w)$ also comes from the distribution equation

$$\mathrm{E}\left[C_s^{\nabla_h} \boxtimes C_s^{\nabla_h}\right] = K_2(z, w)\, dV_h(z) \boxtimes dV_h(w)$$

where $\boxtimes$ is the product on currents defined in [SZ08] by

$$S \boxtimes T = \pi_1^* S \wedge \pi_2^* T \in \mathcal{D}'^{p+q}(M \times M)$$

for $S \in \mathcal{D}'^p(M)$ and $T \in \mathcal{D}'^q(M)$ where $\pi_1, \pi_2 : M \times M \to M$ are the projections to the first and second factors, respectively.

Definition 1.45. Throughout the calculation $\mathcal{O}(t^n)$, for $n \in [1, \infty)$, will mean a function f such that

$$\exists \delta, M > 0 \quad \text{such that} \quad \Big(t \in (0, \delta) \Rightarrow |f(t)| \leq M t^n \Big)$$

In fact, every time it is used in this dissertation, it is sufficient to think of $\mathcal{O}(t^n)$ as a function real analytic at 0 whose first non-zero Taylor term when expanded there is a multiple of t^n. i.e.

$$f(t) = a_n t^n + a_{n+1} t^{n+1} + a_{n+2} t^{n+2} + \cdots$$

with $a_n \neq 0$. Technically, any time $\mathcal{O}(t^n)$ is mentioned, it would be necessary to mention the radius of convergence, and often manipulation of a term involving $\mathcal{O}(t^n)$ will result in a new term involving $\mathcal{O}(t^n)$, where the radius of convergence has shrunk. This calculation only requires that the radius of convergence stays positive. As long as this is the case, the reader should not pay attention to this technicality.

Proof of the main result

To prove Theorem 1.1, we want to find

$$K_{211}^{\infty}(\zeta_1,\zeta_2) = \lim_{N\to\infty} \frac{1}{N^2} K_{21}^N\left(\frac{\zeta_1}{\sqrt{N}}, \frac{\zeta_2}{\sqrt{N}}\right)$$

for any $\zeta_1, \zeta_2 \in M$ where $\dim_{\mathbb{C}} M = 1$. As above, Kac-Rice says we need only calculate

$$\frac{1}{N^2} K_2^N\left(\frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}}\right) = \frac{1}{\pi^6 \det \Delta^{\mathbf{H}}} \int_{\mathbb{C}^4} e^{-\langle (\Lambda^{\mathbf{H}})^{-1}\vec{v}, \vec{v}\rangle} \| \det \vec{v} \| \, d\vec{v}$$

As in (1.6), $\det \Delta^{\mathbf{H}} = (\det A^{\mathbf{H}})(\det \Lambda^{\mathbf{H}})$ and since $K_{211}^{\infty}(z,w)$ depends only on the distance between z and w, we can choose $z = 0$ and $w = r > 0$. So

$$K_{211}^{\infty}(z,w) = J(r) := \frac{1}{\pi^6 \det(A(0,r)) \det(\Lambda(0,r))} \int_{\mathbb{C}^4} \left| |h_1|^2 - |x_1|^2 \right| \cdot \left| |h_2|^2 - |x_2|^2 \right| e^{-\langle \Lambda^{-1}(0,r)v, v\rangle} \, dv$$

where

$$v = \begin{bmatrix} h_1 \\ x_1 \\ h_2 \\ x_2 \end{bmatrix}$$

The absolute value bars simplification using Wick's formula as in [BSZ00b], the fact that ∇s is not a holomorphic section bars using the Poincaré-Lelong formula as in [BSZ00a], and, unfortunately, we are unable to use the ingenious method used in the proof of Lemma 3.1 of [DSZ06a] where the authors were able to rewrite J using Fourier transforms. In [DSZ06a], the authors noticed that they could replace each $\left| |h_j|^2 - |x_j|^2 \right|$ by

$$\lim_{\varepsilon_j, \varepsilon_j' \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} |p| e^{-\varepsilon_j |\xi|^2 - \varepsilon_j' |p|^2} \cdot e^{i\xi(p - |h_j|^2 + |x_j|^2)} \, d\xi \, dp$$

because it can be simplified to

$$\begin{aligned}
&\lim_{\varepsilon_j' \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} |p| e^{-\varepsilon_j' |p|^2} \left(\int_{\mathbb{R}} \lim_{\varepsilon_j \to 0} e^{i\xi(p - |h_j|^2 + |x_j|^2)} e^{\varepsilon_j |\xi|^2} \, d\xi \right) dp \\
&= \lim_{\varepsilon_j' \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} |p| e^{-\varepsilon_j' |p|^2} \left(\int_{\mathbb{R}} \lim_{\varepsilon_j \to 0} e^{i\xi(p - |h_j|^2 + |x_j|^2)} \, d\xi \right) dp \\
&= \lim_{\varepsilon_j' \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} |p| e^{-\varepsilon_j' |p|^2} \left(2\pi \delta_0(p - |h_j|^2 + |x_j|^2) \right) dp \\
&= \lim_{\varepsilon_j' \to 0} \int_{\mathbb{R}} |p| e^{-\varepsilon_j' |p|^2} \delta_{|h_j|^2 - |x_j|^2}(p) \, dp \\
&= \lim_{\varepsilon_j' \to 0} \left| |h_j|^2 - |x_j|^2 \right| e^{-\varepsilon' \left| |h_j|^2 - |x_j|^2 \right|^2} \\
&= \left| |h_j|^2 - |x_j|^2 \right|
\end{aligned}$$

With that substitution and some work,

$$J = \lim_{\vec{\varepsilon}\to\vec{0}} J_{\vec{\varepsilon}} = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{|p_1 p_2| e^{i(\xi_1 p_1)} e^{i(\xi_2 p_2)}}{\det(i\Lambda D - I)}\, d\xi_1\, d\xi_2\, dp_1\, dp_2$$

with

$$D = \begin{bmatrix} -\xi_1 & & & \\ & \xi_1 & & \\ & & -\xi_2 & \\ & & & \xi_2 \end{bmatrix}$$

In the case of [DSZ06a], the authors took advantage of the fact that $\det(i\Lambda D - I)$ was the product of many linear factors and the integral could be done using residues. In our case $\det(i\Lambda D - I)$ is an extremely complicated rational function of r and e^{r^2}. So in this dissertation we will carefully expand J as a function of r.

Calculate Λ as in [BSZ00b] and [DSZ04]:

$$\begin{aligned}
&\Lambda := C - B^* A^{-1} B \\
&A := [\, A^p_{p'} \,] = \left[\mathrm{E}\left[x^p_j \bar{x}^{p'}_{j'} \right] \right] = \begin{bmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{bmatrix} \\
&B := [\, B^p_{p'q'} \,] = \left[\mathrm{E}\left[x^p_j \bar{\xi}^{p'}_{j'q'} \right] \right] = \begin{bmatrix} B^1_{11} & B^1_{12} & B^1_{21} & B^1_{22} \\ B^2_{11} & B^2_{12} & B^2_{21} & B^2_{22} \end{bmatrix} \\
&C := [\, C^{p\,q}_{p'q'} \,] = \left[\mathrm{E}\left[\xi^p_{jq} \bar{\xi}^{p'}_{j'q'} \right] \right] = \begin{bmatrix} C^{1\,1}_{11} & C^{1\,1}_{12} & C^{1\,1}_{21} & C^{1\,1}_{22} \\ C^{1\,2}_{11} & C^{1\,2}_{12} & C^{1\,2}_{21} & C^{1\,2}_{22} \\ C^{2\,1}_{11} & C^{2\,1}_{12} & C^{2\,1}_{21} & C^{2\,1}_{22} \\ C^{2\,2}_{11} & C^{2\,2}_{12} & C^{2\,2}_{21} & C^{2\,2}_{22} \end{bmatrix}
\end{aligned}$$

where $p, p' \in \{1,2\}$ $\quad q, q' \in \{1,2\}$ and the p, q index the rows, and the p', q' index the columns.

$$\begin{aligned}
A^p_{p'} &:= \nabla'_z \nabla''_w \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})} \\
B^p_{p'1} &:= \nabla'_z \nabla''_w \nabla''_w \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})} \\
B^p_{p'2} &:= \nabla'_z \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})} \\
C^{p\,1}_{p'1} &:= \nabla'_z \nabla'_z \nabla''_w \nabla''_w \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})} \\
C^{p\,1}_{p'2} &:= \nabla'_z \nabla'_z \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})} \\
C^{p\,2}_{p'1} &:= \nabla''_w \nabla''_w \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})} \\
C^{p\,2}_{p'2} &:= \Pi^H_1(z,w)|_{(\zeta_p,\zeta_{p'})}
\end{aligned}$$

Notice p, q index rows and p', q' index columns.

For *any* function $\Pi(z, w)$ holomorphic in z and antiholomorphic in w where

$$\nabla_z e(z) = \nabla_z' e(z) = g(z)\, dz \otimes e(z) \quad \nabla_z \bar{e}(z) = \nabla_z'' \bar{e}(z) = \overline{g(z)}\, d\bar{z} \otimes \bar{e}(z)$$

$$\nabla_z'' e(z) = 0 \qquad\qquad \nabla_z' \bar{e}(z) = 0$$

we have

$$\begin{aligned}
\nabla_z'(\Pi(z,w) \otimes e(z) \otimes \bar{e}(w)) &= \left[\frac{\partial \Pi}{\partial z} \otimes dz\right] \otimes e(z) \otimes \bar{e}(w) + \Pi(z,w) \otimes \left[g(z)\, dz \otimes e(z)\right] \otimes \bar{e}(w) \\
&= \left(\frac{\partial \Pi}{\partial z} + g(z)\Pi(z,w)\right) \otimes dz \otimes e(z) \otimes \bar{e}(w) \\
\nabla_z' \nabla_z'(\Pi(z,w) \otimes e(z) \otimes \bar{e}(w)) &= \left[\left(\frac{\partial^2 \Pi}{\partial z^2} + \frac{\partial g}{\partial z}\Pi(z,w) + g(z)\frac{\partial \Pi}{\partial z}\right) \otimes dz\right] \otimes dz \otimes e(z) \otimes \bar{e}(w) \\
&\quad + \left(\frac{\partial \Pi}{\partial z} + g(z)\Pi(z,w)\right) \otimes dz \otimes \left[g(z)\, dz \otimes e(z)\right] \otimes \bar{e}(w) \\
&= \left(\frac{\partial^2 \Pi}{\partial z^2} + \frac{\partial g}{\partial z}\Pi(z,w) + g(z)\frac{\partial \Pi}{\partial z} + g(z)\frac{\partial \Pi}{\partial z} + g(z)^2\Pi(z,w)\right) \\
&\qquad \otimes dz \otimes dz \otimes e(z) \otimes \bar{e}(w) \\
&= \left(\frac{\partial^2 \Pi}{\partial z^2} + \frac{\partial g}{\partial z}\Pi(z,w) + 2g(z)\frac{\partial \Pi}{\partial z} + g(z)^2\Pi(z,w)\right) \\
&\qquad \otimes dz \otimes dz \otimes e(z) \otimes \bar{e}(w) \\
\nabla_w''(\Pi(z,w) \otimes e(z) \otimes \bar{e}(w)) &= \left[\frac{\partial \Pi}{\partial \bar{w}} \otimes d\bar{w}\right] \otimes e(z) \otimes \bar{e}(w) + \Pi(z,w) \otimes e(z) \otimes \left[\overline{g(w)}\, d\bar{w} \otimes \bar{e}(w)\right] \\
&= \left(\frac{\partial \Pi}{\partial \bar{w}} + \overline{g(w)}\Pi(z,w)\right) \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w) \\
\nabla_w'' \nabla_w''(\Pi(z,w) \otimes e(z) \otimes \bar{e}(w)) &= \left[\left(\frac{\partial^2 \Pi}{\partial \bar{w}^2} + \frac{\partial \bar{g}}{\partial \bar{w}}\Pi(z,w) + \overline{g(w)}\frac{\partial \Pi}{\partial \bar{w}}\right) \otimes d\bar{w}\right] \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w) \\
&= \left(\frac{\partial \Pi}{\partial \bar{w}} + \overline{g(w)}\Pi(z,w)\right) \otimes d\bar{w} \otimes e(z) \otimes \left[\overline{g(w)}\, d\bar{w} \otimes \bar{e}(w)\right] \\
&= \left(\frac{\partial^2 \Pi}{\partial \bar{w}^2} + \frac{\partial \bar{g}}{\partial \bar{w}}\Pi(z,w) + 2\overline{g(w)}\frac{\partial \Pi}{\partial \bar{w}} + \overline{g(w)}^2\Pi(z,w)\right) \\
&\qquad \otimes d\bar{w} \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w)
\end{aligned}$$

$$
\begin{aligned}
\nabla'_z \nabla''_w (\Pi(z,w) \otimes e(z) \otimes \bar{e}(w)) &= \left[\left(\frac{\partial^2 \Pi}{\partial z \partial \bar{w}} + \overline{g(w)} \frac{\partial \Pi}{\partial z} \right) \otimes dz \right] \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w) \\
&\quad + \left(\frac{\partial \Pi}{\partial \bar{w}} + \overline{g(w)} \Pi(z,w) \right) \otimes d\bar{w} \otimes \left[g(z)\, dz \otimes e(z) \right] \otimes \bar{e}(w) \\
&= \left(\frac{\partial^2 \Pi}{\partial z \partial \bar{w}} + g(z) \frac{\partial \Pi}{\partial \bar{w}} + \overline{g(w)} \frac{\partial \Pi}{\partial z} + g(z) \overline{g(w)} \Pi(z,w) \right) \\
&\qquad \otimes dz \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w) \\
\nabla'_z \nabla''_w \nabla''_w (\Pi(z,w) \otimes e(z) \otimes \bar{e}(w)) &= \left[\left(\frac{\partial^3 \Pi}{\partial z \partial \bar{w}^2} + \frac{\partial \bar{g}}{\partial \bar{w}} \frac{\partial \Pi}{\partial z} + 2\overline{g(w)} \frac{\partial^2 \Pi}{\partial z \partial \bar{w}} + \overline{g(w)}^2 \frac{\partial \Pi}{\partial z} \right) \otimes dz \right] \\
&\qquad \otimes d\bar{w} \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w) \\
&\quad + \left(\frac{\partial^2 \Pi}{\partial \bar{w}^2} + \frac{\partial \bar{g}}{\partial \bar{w}} \Pi(z,w) + 2\overline{g(w)} \frac{\partial \Pi}{\partial \bar{w}} + \overline{g(w)}^2 \Pi(z,w) \right) \\
&\qquad \otimes d\bar{w} \otimes d\bar{w} \otimes \left[g(z)\, dz \otimes e(z) \right] \bar{e}(w) \\
&= \begin{pmatrix} \dfrac{\partial^3 \Pi}{\partial z \partial \bar{w}^2} + g(z) \dfrac{\partial^2 \Pi}{\partial \bar{w}^2} + 2\overline{g(w)} \dfrac{\partial^2 \Pi}{\partial z \partial \bar{w}} + \left(\dfrac{\partial \bar{g}}{\partial \bar{w}} + \overline{g(w)}^2 \right) \dfrac{\partial \Pi}{\partial z} \\ +2g(z)\overline{g(w)} \dfrac{\partial \Pi}{\partial \bar{w}} + g(z) \left(\dfrac{\partial \bar{g}}{\partial \bar{w}} + \overline{g(w)}^2 \right) \Pi(z,w) \end{pmatrix} \\
&\qquad \otimes dz \otimes d\bar{w} \otimes d\bar{w} \otimes e(z) \otimes \bar{e}(w)
\end{aligned}
$$

In this particular case where $g(z) = -\bar{z}$ because $h(z) := e^{-z\bar{z}}$, $\Pi(z,w) = e^{z\bar{w}}$, $\zeta_1 := 0$, and $\zeta_2 := r$, we have

$$
\begin{aligned}
A^{p}_{p'} &= e^{z\bar{w}} \left(1 + z\bar{w} - \bar{z}z - w\bar{w} + \bar{z}w\right) \\
B^{p}_{p'1} &= e^{z\bar{w}} \left(z - w\right) \left(z\bar{w} - \bar{z}z + 2 + \bar{z}w - w\bar{w}\right) \\
B^{p}_{p'2} &= e^{z\bar{w}} \left(\bar{w} - \bar{z}\right) \\
C^{p\,1}_{p'1} &= e^{z\bar{w}} \begin{pmatrix} 2 - 4\,\bar{z}z - 4\,w\bar{w} + 4\,\bar{z}w - 2\,\bar{w}\bar{z}z^2 - 2\,\bar{w}^2 zw - 2\,\bar{w}\bar{z}w^2 \\ -2\,\bar{z}^2 zw + 4\,z\bar{w} + \bar{w}^2 z^2 + \bar{w}^2 w^2 + \bar{z}^2 z^2 + \bar{z}^2 w^2 + 4\,\bar{w}z\bar{z}w \end{pmatrix} \\
C^{p\,1}_{p'2} &= e^{z\bar{w}} \left(\bar{w} - \bar{z}\right)^2 \\
C^{p\,2}_{p'1} &= e^{z\bar{w}} \left(z - w\right)^2 \\
C^{p\,2}_{p'2} &= e^{z\bar{w}}
\end{aligned}
$$

so

$$A = \begin{bmatrix} 1 & 1-r^2 \\ 1-r^2 & e^{r^2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -r\left(2-r^2\right) & r \\ r\left(2-r^2\right) & -r & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & -4\,r^2+r^4+2 & r^2 \\ 0 & 1 & r^2 & 1 \\ -4\,r^2+r^4+2 & r^2 & 2\,e^{r^2} & 0 \\ r^2 & 1 & 0 & e^{r^2} \end{bmatrix}$$

so

$$\Lambda(r) = \frac{1}{-e^{r^2}+1-2\,r^2+r^4} \cdot {}^1M(t)$$

where

$${}^1M_{11}(r) = -2\,e^{r^2}+2-2\,r^4+r^6 \qquad {}^1M_{12}(r) = r^2\left(-2+r^2\right)$$

$${}^1M_{21}(r) = r^2\left(-2+r^2\right) \qquad {}^1M_{22}(r) = -e^{r^2}+1-r^2+r^4$$

$${}^1M_{31}(r) = 4\,r^2e^{r^2}-4\,r^2+3\,r^4-r^6-r^4e^{r^2}-2\,e^{r^2}+2 \qquad {}^1M_{32}(r) = -r^2\left(e^{r^2}+1-r^2\right)$$

$${}^1M_{41}(r) = -r^2\left(e^{r^2}+1-r^2\right) \qquad {}^1M_{42}(r) = -e^{r^2}+1-r^2$$

$${}^1M_{13}(r) = 4\,r^2e^{r^2}-4\,r^2+3\,r^4-r^6-r^4e^{r^2}-2\,e^{r^2}+2 \qquad {}^1M_{14}(r) = -r^2\left(e^{r^2}+1-r^2\right)$$

$${}^1M_{23}(r) = -r^2\left(e^{r^2}+1-r^2\right) \qquad {}^1M_{24}(r) = -e^{r^2}+1-r^2$$

$${}^1M_{33}(r) = -e^{r^2}\left(2\,e^{r^2}-2+2\,r^4-r^6\right) \qquad {}^1M_{34}(r) = r^2\left(-2+r^2\right)e^{r^2}$$

$${}^1M_{43}(r) = r^2\left(-2+r^2\right)e^{r^2} \qquad {}^1M_{44}(r) = -e^{r^2}\left(e^{r^2}-1+r^2-r^4\right)$$

which is actually a function of $t := r^2 > 0$.

i.e.

$$\Lambda(t) = \frac{1}{-e^{t}+1-2\,t+t^2} \cdot {}^2M(t)$$

where

$$
\begin{aligned}
{}^2M_{11}(t) &= -2\,e^t + 2 - 2\,t^2 + t^3 & {}^2M_{12}(t) &= t\,(-2+t) \\
{}^2M_{21}(t) &= t\,(-2+t) & {}^2M_{22}(t) &= -e^t + 1 - t + t^2 \\
{}^2M_{31}(t) &= 4\,t\,e^t - 4\,t + 3\,t^2 - t^3 - t^2e^t - 2\,e^t + 2 & {}^2M_{32}(t) &= t\,\left(-e^t - 1 + t\right) \\
{}^2M_{41}(t) &= t\,\left(-e^t - 1 + t\right) & {}^2M_{42}(t) &= -e^t + 1 - t \\
{}^2M_{13}(t) &= 4\,t\,e^t - 4\,t + 3\,t^2 - t^3 - t^2e^t - 2\,e^t + 2 & {}^2M_{14}(t) &= t\,\left(-e^t - 1 + t\right) \\
{}^2M_{23}(t) &= t\,\left(-e^t - 1 + t\right) & {}^2M_{24}(t) &= -e^t + 1 - t \\
{}^2M_{33}(t) &= e^t\left(-2\,e^t + 2 - 2\,t^2 + t^3\right) & {}^2M_{34}(t) &= t\,(-2+t)\,e^t \\
{}^2M_{43}(t) &= t\,(-2+t)\,e^t & {}^2M_{44}(t) &= e^t\left(-e^t + 1 - t + t^2\right)
\end{aligned}
$$

so

$$
\begin{aligned}
\det\Lambda(t) &= \frac{\left(e^t\right)^2 t^4 - e^tt^4 - 4\,t^3e^t - 4\,t^3\left(e^t\right)^2 + 12\,t^2\left(e^t\right)^2 - 12\,e^tt^2 - 12\,e^t + 12\,\left(e^t\right)^2 - 4\,\left(e^t\right)^3 + 4}{-e^t + 1 - 2\,t + t^2} \\
&= \frac{1}{6480}t^8 + \frac{1}{3888}t^9 + \frac{869}{4082400}t^{10} + \frac{37}{326592}t^{11} + \frac{1213}{29393280}t^{12} + \mathcal{O}\left(t^{13}\right) \qquad (1.9) \\
\det A(t) &= e^t - t^2 + 2t - 1 \\
&= 3t - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \mathcal{O}\left(t^5\right) \qquad (1.10)
\end{aligned}
$$

Now

$$
\Lambda^{-1}(t) = \frac{1}{-t^4e^t + t^4e^{2\,t} - 4\,t^3e^t - 4\,t^3e^{2\,t} - 12\,t^2e^t + 12\,t^2e^{2\,t} - 4\,e^{3\,t} - 12\,e^t + 12\,e^{2\,t} + 4}\cdot {}^3M(t)
$$

where

$$^{3}M_{11}(t) = t^3 \left(e^t\right)^2 - t^2 \left(e^t\right)^2 - e^t t^2 + 4\,t \,\left(e^t\right)^2 - 4\,t\,e^t - 2\,\left(e^t\right)^3 + 4\,\left(e^t\right)^2 - 2\,e^t$$

$$^{3}M_{21}(t) = -t^3 e^t - t^3 \left(e^t\right)^2 + 2\,t^2 \left(e^t\right)^2 - 2\,e^t t^2$$

$$^{3}M_{31}(t) = t^3 e^t + t^2 \left(e^t\right)^2 + e^t t^2 + 4\,t\,e^t - 4\,t\,\left(e^t\right)^2 + 2\,\left(e^t\right)^2 - 4\,e^t + 2$$

$$^{3}M_{41}(t) = -2\,t^3 e^t + 2\,t\,\left(e^t\right)^2 - 4\,t\,e^t + 2\,t$$

$$^{3}M_{12}(t) = -t^3 e^t - t^3 \left(e^t\right)^2 + 2\,t^2 \left(e^t\right)^2 - 2\,e^t t^2$$

$$^{3}M_{22}(t) = \left(e^t\right)^2 t^4 - e^t t^4 - 4\,t^3 \left(e^t\right)^2 - 2\,e^t t^2 + 10\,t^2 \left(e^t\right)^2 - 4\,t\,\left(e^t\right)^2 + 4\,t\,e^t - 4\,\left(e^t\right)^3 + 8\,\left(e^t\right)^2 - 4\,e^t$$

$$^{3}M_{32}(t) = -2\,t^3 e^t + 2\,t\,\left(e^t\right)^2 - 4\,t\,e^t + 2\,t$$

$$^{3}M_{42}(t) = 4\,t^3 e^t + 2\,t^2 - 10\,e^t t^2 + 4\,t\,e^t - 4\,t + 4\,\left(e^t\right)^2 - 8\,e^t + 4$$

$$^{3}M_{13}(t) = t^3 e^t + t^2 \left(e^t\right)^2 + e^t t^2 + 4\,t\,e^t - 4\,t\,\left(e^t\right)^2 + 2\,\left(e^t\right)^2 - 4\,e^t + 2$$

$$^{3}M_{23}(t) = -2\,t^3 e^t + 2\,t\,\left(e^t\right)^2 - 4\,t\,e^t + 2\,t$$

$$^{3}M_{33}(t) = t^3 e^t - e^t t^2 - t^2 + 4\,t\,e^t - 4\,t - 2\,\left(e^t\right)^2 + 4\,e^t - 2$$

$$^{3}M_{43}(t) = -t^3 - t^3 e^t + 2\,e^t t^2 - 2\,t^2$$

$$^{3}M_{14}(t) = -2\,t^3 e^t + 2\,t\,\left(e^t\right)^2 - 4\,t\,e^t + 2\,t$$

$$^{3}M_{24}(t) = 4\,t^3 e^t + 2\,t^2 - 10\,e^t t^2 + 4\,t\,e^t - 4\,t + 4\,\left(e^t\right)^2 - 8\,e^t + 4$$

$$^{3}M_{34}(t) = -t^3 - t^3 e^t + 2\,e^t t^2 - 2\,t^2$$

$$^{3}M_{44}(t) = e^t t^4 - t^4 - 4\,t^3 e^t - 2\,t^2 + 10\,e^t t^2 - 4\,t\,e^t + 4\,t - 4\,\left(e^t\right)^2 + 8\,e^t - 4$$

so

$$\Lambda^{-1} = t^{-5} \cdot Y(t)$$

where

$$Y_{11}(t) = 30t^2 + 9t^3 + \mathcal{O}(t^4) \qquad Y_{12}(t) = 360t - \frac{18}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{21}(t) = 360t - \frac{18}{7}t^3 + \mathcal{O}(t^4) \qquad Y_{22}(t) = 4320 - 1080t + \frac{960}{7}t^2 - \frac{72}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{31}(t) = -30t^2 + 9t^3 + \mathcal{O}(t^4) \qquad Y_{32}(t) = -360t + 180t^2 - \frac{318}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{41}(t) = -360t + 180t^2 - \frac{318}{7}t^3 + \mathcal{O}(t^4) \qquad Y_{42}(t) = -4320 + 3240t - \frac{8520}{7}t^2 + \frac{2148}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{13}(t) = -30t^2 + 9t^3 + \mathcal{O}(t^4) \qquad Y_{14}(t) = -360t + 180t^2 - \frac{318}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{23}(t) = -360t + 180t^2 - \frac{318}{7}t^3 + \mathcal{O}(t^4) \qquad Y_{24}(t) = -4320 + 3240t - \frac{8520}{7}t^2 + \frac{2148}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{33}(t) = 30t^2 - 21t^3 + \mathcal{O}(t^4) \qquad Y_{34}(t) = 360t - 360t^2 + \frac{1242}{7}t^3 + \mathcal{O}(t^4)$$

$$Y_{43}(t) = 360t - 360t^2 + \frac{1242}{7}t^3 + \mathcal{O}(t^4) \qquad Y_{44}(t) = 4320 - 5400t + \frac{23640}{7}t^2 - \frac{9852}{7}t^3 + \mathcal{O}(t^4)$$

Note

$$\lim_{t\to 0^+} Y(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4320 & 0 & -4320 \\ 0 & 0 & 0 & 0 \\ 0 & -4320 & 0 & 4320 \end{bmatrix}$$

Now the original integral can be estimated by estimating the diagonalization of $Y(t)$. Though the proof does not depend on knowing the origin of the $U(t)$ and $D(t)$ used to approximate diagonalizing $Y(t)$, their construction is given in the appendix.

$$\begin{aligned} J(t) :&= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||h_1|^2 - |x_1|^2\right| \cdot \left||h_2|^2 - |x_2|^2\right| e^{-\langle \Lambda^{-1}(t)v,v\rangle}\, dv \\ &= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||h_1|^2 - |x_1|^2\right| \cdot \left||h_2|^2 - |x_2|^2\right| e^{-t^{-5}\langle Y(t)v,v\rangle}\, dv \\ &= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||h_1|^2 - |x_1|^2\right| \cdot \left||h_2|^2 - |x_2|^2\right| e^{-\left\langle Y(t)(t^{-\frac{5}{2}}v),(t^{-\frac{5}{2}}v)\right\rangle}\, dv \end{aligned}$$

Making the substitution $w = t^{-\frac{5}{2}}v$ is actually saying

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = t^{-\frac{5}{2}} \begin{bmatrix} h_1 \\ x_1 \\ h_2 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \\ dw_4 \end{bmatrix} = t^{-\frac{5}{2}} \begin{bmatrix} dh_1 \\ dx_1 \\ dh_2 \\ dx_2 \end{bmatrix}$$

$$\Rightarrow dw = \frac{i}{2} dw_1 d\bar{w}_1 \ldots \frac{i}{2} dw_4 d\bar{w}_4 = t^{-20} \frac{i}{2} dh_1 d\bar{h}_1 \ldots \frac{i}{2} dx_4 d\bar{x}_4 = t^{-20} dv$$

making

$$
\begin{aligned}
J(t) &= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||h_1|^2 - |x_1|^2\right| \cdot \left||h_2|^2 - |x_2|^2\right| e^{-\langle Y(t)(t^{-\frac{5}{2}}v),(t^{-\frac{5}{2}}v)\rangle}\, dv \\
&= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} t^5 \left||w_1|^2 - |w_2|^2\right| \cdot t^5 \left||w_3|^2 - |w_4|^2\right| e^{-\langle Y(t)w,w\rangle}\, t^{20} dw \\
&= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||w_1|^2 - |w_2|^2\right| \cdot \left||w_3|^2 - |w_4|^2\right| e^{-\langle Y(t)w,w\rangle}\, dw
\end{aligned}
$$

Now

$$
\begin{aligned}
Y(t) = U(t)^* D(t) U(t) &= \Big(\tilde{U}(t)^* + \big[\mathcal{O}(t^3)\big]^*_{4\times 4}\Big)\Big(\tilde{D}(t) + \underset{\text{(diag)}}{\big[\mathcal{O}(t^{12})\big]_{4\times 4}}\Big)\Big(\tilde{U}(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big) \\
&= \tilde{U}(t)^* \tilde{D}(t) \tilde{U}(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4}
\end{aligned}
$$

where

$$
D(t) := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{\lambda}_1 & 0 & 0 & 0 \\ 0 & \tilde{\lambda}_2 & 0 & 0 \\ 0 & 0 & \tilde{\lambda}_3 & 0 \\ 0 & 0 & 0 & \tilde{\lambda}_4 \end{bmatrix}}_{\tilde{D}(t)} + \underbrace{\begin{bmatrix} \mathcal{O}(t^{12}) & 0 & 0 & 0 \\ 0 & \mathcal{O}(t^{12}) & 0 & 0 \\ 0 & 0 & \mathcal{O}(t^{12}) & 0 \\ 0 & 0 & 0 & \mathcal{O}(t^{12}) \end{bmatrix}}_{\underset{\text{(diag)}}{[\mathcal{O}(t^{12})]_{4\times 4}}}
$$

where

$$
\begin{aligned}
\tilde{\lambda}_1 :=& 8640 - 6480\,t + \frac{25020}{7}\,t^2 - \frac{10050}{7}\,t^3 + \frac{66380}{147}\,t^4 - \frac{261767}{2352}\,t^5 + \frac{48960935}{2173248}\,t^6 - \frac{29628553}{8149680}\,t^7 \\
&+ \frac{208429618963}{427173626880}\,t^8 - \frac{560822276587}{8543472537600}\,t^9 + \frac{46335059891}{6133775155200}\,t^{10} + \frac{518190034231}{1794129232896000}\,t^{11} \\
\tilde{\lambda}_2 :=& 6\,t^3 - 3\,t^4 + \frac{111}{80}\,t^5 - \frac{161}{960}\,t^6 - \frac{20561}{1209600}\,t^7 + \frac{561019}{21772800}\,t^8 + \frac{3916753}{15676416000}\,t^9 \\
&- \frac{827998967}{282175488000}\,t^{10} + \frac{5185091420987}{15643809054720000}\,t^{11} \\
\tilde{\lambda}_3 :=& \frac{1}{3}\,t^4 - \frac{1}{12}\,t^5 + \frac{1}{72}\,t^6 + \frac{1}{32}\,t^7 + \frac{6223}{207360}\,t^8 + \frac{256685}{8957952}\,t^9 + \frac{588107563}{22574039040}\,t^{10} + \frac{6399891227}{325066162176}\,t^{11} \\
\tilde{\lambda}_4 :=& \frac{3}{8}\,t^5 - \frac{1}{16}\,t^6 - \frac{65}{768}\,t^7 - \frac{101}{3072}\,t^8 - \frac{877}{40960}\,t^9 - \frac{37303}{1474560}\,t^{10} - \frac{2563021}{123863040}\,t^{11}
\end{aligned}
$$

and

$$
U(t) = \tilde{U}(t) + \underbrace{\begin{bmatrix} \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) \\ \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) \\ \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) \\ \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) & \mathcal{O}(t^3) \end{bmatrix}}_{[\mathcal{O}(t^3)]_{4\times 4}}
$$

where

$$
\begin{array}{llll}
\tilde{u}_{11} = -\frac{\sqrt{2}}{24}t - \frac{\sqrt{2}}{48}t^2 & \tilde{u}_{12} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}t + \frac{5\sqrt{2}}{288}t^2 & \tilde{u}_{13} = \dfrac{\sqrt{2}}{24}t + 0t^2 & \tilde{u}_{14} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}t - \frac{5\sqrt{2}}{288}t^2 \\
\tilde{u}_{21} = -\frac{1}{2} - \frac{3}{16}t - \frac{65}{2304}t^2 & \tilde{u}_{22} = \dfrac{1}{2} - \dfrac{3}{16}t + \dfrac{5}{256}t^2 & \tilde{u}_{23} = -\frac{1}{2} + \frac{1}{16}t + \frac{167}{2304}t^2 & \tilde{u}_{24} = \frac{1}{2} + \frac{1}{16}t - \frac{41}{768}t^2 \\
\tilde{u}_{31} = \frac{\sqrt{2}}{2} + 0t + \frac{43\sqrt{2}}{576}t^2 & \tilde{u}_{32} = \dfrac{\sqrt{2}}{12} + 0t + \dfrac{61\sqrt{2}}{576}t^2 & \tilde{u}_{33} = -\frac{\sqrt{2}}{2} + 0t + \frac{7\sqrt{2}}{64}t^2 & \tilde{u}_{34} = \frac{\sqrt{2}}{6}t + \frac{109\sqrt{2}}{576}t^2 \\
\tilde{u}_{41} = \frac{1}{2} - \frac{3}{16}t - \frac{193}{768}t^2 & \tilde{u}_{42} = \dfrac{1}{2} - \dfrac{1}{16}t - \dfrac{53}{768}t^2 & \tilde{u}_{43} = \frac{1}{2} + \frac{1}{16}t + \frac{215}{768}t^2 & \tilde{u}_{44} = \frac{1}{2} + \frac{3}{16}t - \frac{29}{768}t^2
\end{array}
$$

and $\big[\mathcal{O}(t^3)\big]_{4\times 4}$ taken so that $U(t)$ is invertible.

The following lemmas say that $U(t)$ is approximately orthogonal.

Lemma 1.46. $U(t)U(t)^* = I + \big[\mathcal{O}(t^3)\big]_{4\times 4}$

Proof:

$$\begin{aligned} U(t)U(t)^* &= \Big(\tilde{U}(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big)\Big(\tilde{U}(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big)^* \\ &= \Big(\tilde{U}(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big)\Big(\tilde{U}(t)^* + \big[\mathcal{O}(t^3)\big]^*_{4\times 4}\Big) \\ &= \tilde{U}(t)\tilde{U}(t)^* + \big[\mathcal{O}(t^3)\big]_{4\times 4}\tilde{U}(t)^* + \tilde{U}(t)\big[\mathcal{O}(t^3)\big]_{4\times 4} + \big[\mathcal{O}(t^3)\big]_{4\times 4}\big[\mathcal{O}(t^3)\big]_{4\times 4} \\ &= \tilde{U}(t)\tilde{U}(t)^* + \big[\mathcal{O}(t^3)\big]_{4\times 4} + \big[\mathcal{O}(t^3)\big]_{4\times 4} + \big[\mathcal{O}(t^6)\big]_{4\times 4} \\ &= \tilde{U}(t)\tilde{U}(t)^* + \big[\mathcal{O}(t^3)\big]_{4\times 4} \\ &= {}^4M(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4} \end{aligned}$$

where

$$\begin{aligned} {}^4M_{11}(t) &= 1 + \frac{1}{288}\,t^3 + \frac{43}{20736}\,t^4 & {}^4M_{12}(t) &= \frac{221}{27648}\,\sqrt{2}t^3 + \frac{205}{110592}\,\sqrt{2}t^4 \\ {}^4M_{21}(t) &= \frac{221}{27648}\,\sqrt{2}t^3 + \frac{205}{110592}\,\sqrt{2}t^4 & {}^4M_{22}(t) &= 1 + \frac{13}{2304}\,t^3 + \frac{12317}{1327104}\,t^4 \\ {}^4M_{31}(t) &= -\frac{85}{1152}\,t^3 - \frac{83}{13824}\,t^4 & {}^4M_{32}(t) &= -\frac{23}{1024}\,\sqrt{2}t^3 - \frac{367}{165888}\,\sqrt{2}t^4 \\ {}^4M_{41}(t) &= \frac{323}{9216}\,\sqrt{2}t^3 + \frac{173}{36864}\,\sqrt{2}t^4 & {}^4M_{42}(t) &= \frac{85}{1152}\,t^3 + \frac{517}{18432}\,t^4 \end{aligned}$$

$$\begin{aligned} {}^4M_{13}(t) &= -\frac{85}{1152}\,t^3 - \frac{83}{13824}\,t^4 & {}^4M_{14}(t) &= \frac{323}{9216}\,\sqrt{2}t^3 + \frac{173}{36864}\,\sqrt{2}t^4 \\ {}^4M_{23}(t) &= -\frac{23}{1024}\,\sqrt{2}t^3 - \frac{367}{165888}\,\sqrt{2}t^4 & {}^4M_{24}(t) &= \frac{85}{1152}\,t^3 + \frac{517}{18432}\,t^4 \\ {}^4M_{33}(t) &= 1 + \frac{31}{192}\,t^3 + \frac{595}{4608}\,t^4 & {}^4M_{34}(t) &= \frac{89}{9216}\,\sqrt{2}t^3 - \frac{287}{110592}\,\sqrt{2}t^4 \\ {}^4M_{43}(t) &= \frac{89}{9216}\,\sqrt{2}t^3 - \frac{287}{110592}\,\sqrt{2}t^4 & {}^4M_{44}(t) &= 1 + \frac{95}{768}\,t^3 + \frac{21781}{147456}\,t^4 \end{aligned}$$

so

$$U(t)U(t)^* = {}^4M(t) + \big[\mathcal{O}(t^3)\big]_{4\times 4} = I + \big[\mathcal{O}(t^3)\big]_{4\times 4}$$

Q.E.D.

Lemma 1.47. $U(t)^{-1} = U(t)^* + \big[\mathcal{O}(t^3)\big]_{4\times 4}$

Proof:

$$\begin{aligned} & U(t)U(t)^* = I + \big[\mathcal{O}(t^3)\big]_{4\times 4} \\ \Rightarrow \quad & U(t)^* = U(t)^{-1}\Big(I + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big) \\ \Rightarrow \quad & U(t)^{-1} = U(t)^*\Big(I + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big)^{-1} \end{aligned}$$

since $\big[\mathcal{O}(t^3)\big]^n_{4\times 4} \xrightarrow{n\to\infty} [\,0\,]_{4\times 4}$ for small t,

$$\Big(I + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big)^{-1} = I - \big[\mathcal{O}(t^3)\big]_{4\times 4} + \big[\mathcal{O}(t^3)\big]^2_{4\times 4} - \big[\mathcal{O}(t^3)\big]^3_{4\times 4} \pm \ldots = I + \big[\mathcal{O}(t^3)\big]_{4\times 4}$$

so

$$\begin{aligned} & U(t)^{-1} = U(t)^*\Big(I + \big[\mathcal{O}(t^3)\big]_{4\times 4}\Big) \\ \Rightarrow \quad & U(t)^{-1} = U(t)^* + U(t)^*\big[\mathcal{O}(t^3)\big]_{4\times 4} \\ \Rightarrow \quad & U(t)^{-1} = U(t)^* + \big[\mathcal{O}(t^3)\big]_{4\times 4} \end{aligned}$$

Q.E.D.

Now

$$\begin{aligned} J(t) &= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \big||w_1|^2 - |w_2|^2\big| \cdot \big||w_3|^2 - |w_4|^2\big| \, e^{-\langle U(t)^* D(t) U(t) w, w\rangle} \, dw \\ &= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \big||w_1|^2 - |w_2|^2\big| \cdot \big||w_3|^2 - |w_4|^2\big| \, e^{-\langle D(t) U(t) w, U(t) w\rangle} \, dw \end{aligned}$$

Make the substitution

$$\begin{aligned} z &:= U(t)w \\ z_i &= \sum_j u_{ij} w_j \\ w_i &= \sum_j u^{ij} z_j = \sum_j \big(u_{ji} + \mathcal{O}(t^3)\,\big) z_j \\ dw &= \det(U(t)) \cdot dz = \big(1 + \mathcal{O}(t^3)\,\big)\, dz = \Big[dz + \mathcal{O}(t^3)\ dz\Big] \end{aligned}$$

In the following, only the properties of the t^0, t^1, and t^2 terms of the u_{ji} are used so, for the sake of readability, "u_{ji}" will always be written in place of "$u_{ji} + \mathcal{O}(t^3)$".

$$J(t) = \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \big||w_1|^2 - |w_2|^2\big| \cdot \big||w_3|^2 - |w_4|^2\big| \, e^{-\langle D(t) U(t) w, U(t) w\rangle} \, dw \tag{1.11}$$

$$= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} z_j \right|^2 - \left| \sum_j u_{j2} z_j \right|^2 \right| \cdot \left| \left| \sum_j u_{j3} z_j \right|^2 - \left| \sum_j u_{j4} z_j \right|^2 \right| e^{-\langle D(t) z, z\rangle} \left(dz + \mathcal{O}\big(t^3\big)\, dz\right)$$

$$= \frac{t^{30} + \mathcal{O}\big(t^{33}\big)}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} z_j \right|^2 - \left| \sum_j u_{j2} z_j \right|^2 \right| \cdot \left| \left| \sum_j u_{j3} z_j \right|^2 - \left| \sum_j u_{j4} z_j \right|^2 \right| e^{-\sum_{j=1}^4 \lambda_j z_j \bar{z}_j}\, dz$$

$$= \frac{t^{30} + \mathcal{O}\big(t^{33}\big)}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} z_j \right|^2 - \left| \sum_j u_{j2} z_j \right|^2 \right| \cdot \left| \left| \sum_j u_{j3} z_j \right|^2 - \left| \sum_j u_{j4} z_j \right|^2 \right| e^{-\sum_{j=1}^4 |\sqrt{\lambda_j} z_j|^2}\, dz$$

Make the substitution

$$w_j := \sqrt{\lambda_j} z_j \quad \Longrightarrow \quad z_j = \frac{w_j}{\sqrt{\lambda_j}} \quad \text{and} \quad dw_j = \sqrt{\lambda_j}\, dz_j$$

so

$$\begin{aligned} dz &= \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge \frac{i}{2} dz_4 \wedge d\bar{z}_4 = \frac{i}{2} \frac{dw_1}{\sqrt{\lambda_1}} \wedge \frac{d\bar{w}_1}{\sqrt{\lambda_1}} \wedge \ldots \wedge \frac{i}{2} \frac{dw_4}{\sqrt{\lambda_4}} \wedge \frac{d\bar{w}_4}{\sqrt{\lambda_4}} \\ &= \frac{1}{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \frac{i}{2} dw_1 \wedge d\bar{w}_1 \wedge \ldots \wedge \frac{i}{2} dw_4 \wedge d\bar{w}_4 \\ &= \frac{dw}{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \end{aligned}$$

so, since Lemma 1.46 and (1.9) imply

$$[\det \Lambda(t)] \lambda_1 \lambda_2 \lambda_3 \lambda_4 = t^{20} + \mathcal{O}\big(t^{23}\big)$$

$$\begin{aligned} J(t) &= \frac{t^{30} + \mathcal{O}\big(t^{33}\big)}{\pi^6 \det A(t) \det \Lambda(t) \prod \lambda_i} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j2} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| \left| \left| \sum_j u_{j3} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j4} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| e^{-\sum_{j=1}^4 |w_j|^2} dw \\ &= \frac{t^{30} + \mathcal{O}\big(t^{33}\big)}{\pi^6 \det A [t^{20} + \mathcal{O}\big(t^{23}\big)]} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j2} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| \cdot \left| \left| \sum_j u_{j3} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j4} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| e^{-\langle w, w\rangle}\, dw \\ &= \frac{t^{10} + \mathcal{O}\big(t^{13}\big)}{\pi^6 \det A (1 + \mathcal{O}\big(t^3\big))} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j2} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| \cdot \left| \left| \sum_j u_{j3} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j4} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| e^{-\langle w, w\rangle}\, dw \\ &= \frac{t^{10} + \mathcal{O}\big(t^{13}\big)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left| \left| \sum_j u_{j1} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j2} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| \cdot \left| \left| \sum_j u_{j3} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j4} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| e^{-\langle w, w\rangle}\, dw \\ &= \frac{1 + \mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} t^5 \left| \left| \sum_j u_{j1} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j2} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| \cdot t^5 \left| \left| \sum_j u_{j3} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 - \left| \sum_j u_{j4} \tfrac{w_j}{\sqrt{\lambda_j}} \right|^2 \right| e^{-\langle w, w\rangle}\, dw \\ &= \frac{1 + \mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left| \left| \sum_j \tfrac{t^{\frac{5}{2}} u_{j1}}{\sqrt{\lambda_j}} w_j \right|^2 - \left| \sum_j \tfrac{t^{\frac{5}{2}} u_{j2}}{\sqrt{\lambda_j}} w_j \right|^2 \right| \cdot \left| \left| \sum_j \tfrac{t^{\frac{5}{2}} u_{j3}}{\sqrt{\lambda_j}} w_j \right|^2 - \left| \sum_j \tfrac{t^{\frac{5}{2}} u_{j4}}{\sqrt{\lambda_j}} w_j \right|^2 \right| e^{-\langle w, w\rangle}\, dw \end{aligned}$$

Now

$$\begin{aligned}
\frac{t^{\frac{5}{2}}u_{11}}{\sqrt{\lambda_1}} &= -\frac{\sqrt{30}}{8640}t^{\frac{7}{2}} - \frac{7\sqrt{30}}{69120}t^{\frac{9}{2}} + \mathcal{O}\Big(t^{\frac{11}{2}}\Big) & \frac{t^{\frac{5}{2}}u_{21}}{\sqrt{\lambda_2}} &= -\frac{\sqrt{6}}{12}t - \frac{5\sqrt{6}}{96}t^2 + \mathcal{O}\big(t^3\big) \\
\frac{t^{\frac{5}{2}}u_{12}}{\sqrt{\lambda_1}} &= -\frac{\sqrt{30}}{720}t^{\frac{5}{2}} - \frac{\sqrt{30}}{1152}t^{\frac{7}{2}} + \mathcal{O}\Big(t^{\frac{9}{2}}\Big) & \frac{t^{\frac{5}{2}}u_{22}}{\sqrt{\lambda_2}} &= \frac{\sqrt{6}}{12}t - \frac{\sqrt{6}}{96}t^2 + \mathcal{O}\big(t^3\big) \\
\frac{t^{\frac{5}{2}}u_{13}}{\sqrt{\lambda_1}} &= \frac{\sqrt{30}}{8640}t^{\frac{7}{2}} + \frac{\sqrt{30}}{23040}t^{\frac{9}{2}} + \mathcal{O}\Big(t^{\frac{11}{2}}\Big) & \frac{t^{\frac{5}{2}}u_{23}}{\sqrt{\lambda_2}} &= -\frac{\sqrt{6}}{12}t - \frac{\sqrt{6}}{96}t^2 + \mathcal{O}\big(t^3\big) \\
\frac{t^{\frac{5}{2}}u_{14}}{\sqrt{\lambda_1}} &= \frac{\sqrt{30}}{720}t^{\frac{5}{2}} + \frac{\sqrt{30}}{5760}t^{\frac{7}{2}} + \mathcal{O}\Big(t^{\frac{9}{2}}\Big) & \frac{t^{\frac{5}{2}}u_{24}}{\sqrt{\lambda_2}} &= \frac{\sqrt{6}}{12}t + \frac{\sqrt{6}}{32}t^2 + \mathcal{O}\big(t^3\big)
\end{aligned}$$

and

$$\begin{aligned}
\frac{t^{\frac{5}{2}}u_{31}}{\sqrt{\lambda_3}} &= \frac{\sqrt{6}}{2}t^{\frac{1}{2}} + \frac{\sqrt{6}}{16}t^{\frac{3}{2}} + \mathcal{O}\Big(t^{\frac{5}{2}}\Big) \\
\frac{t^{\frac{5}{2}}u_{32}}{\sqrt{\lambda_3}} &= \frac{\sqrt{6}}{12}t^{\frac{3}{2}} + \frac{67\sqrt{6}}{576}t^{\frac{5}{2}} + \mathcal{O}\Big(t^{\frac{7}{2}}\Big) \\
\frac{t^{\frac{5}{2}}u_{33}}{\sqrt{\lambda_3}} &= -\frac{\sqrt{6}}{2}t^{\frac{1}{2}} - \frac{\sqrt{6}}{16}t^{\frac{3}{2}} + \mathcal{O}\Big(t^{\frac{5}{2}}\Big) \\
\frac{t^{\frac{5}{2}}u_{34}}{\sqrt{\lambda_3}} &= \frac{\sqrt{6}}{6}t^{\frac{3}{2}} + \frac{121\sqrt{6}}{576}t^{\frac{5}{2}} + \mathcal{O}\Big(t^{\frac{7}{2}}\Big)
\end{aligned}$$

$$\begin{aligned}
\frac{t^{\frac{5}{2}}u_{41}}{\sqrt{\lambda_4}} &= \frac{\sqrt{6}}{3} - \frac{7\sqrt{6}}{72}t - \frac{473\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big) \\
\frac{t^{\frac{5}{2}}u_{42}}{\sqrt{\lambda_4}} &= \frac{\sqrt{6}}{3} - \frac{\sqrt{6}}{72}t - \frac{29\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big) \\
\frac{t^{\frac{5}{2}}u_{43}}{\sqrt{\lambda_4}} &= \frac{\sqrt{6}}{3} + \frac{5\sqrt{6}}{72}t + \frac{799\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big) \\
\frac{t^{\frac{5}{2}}u_{44}}{\sqrt{\lambda_4}} &= \frac{\sqrt{6}}{3} + \frac{11\sqrt{6}}{72}t + \frac{91\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big)
\end{aligned}$$

so

$$J(t) = \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} |\alpha_{1t}(w)\alpha_{1t}(\bar{w}) - \beta_{1t}(w)\beta_{1t}(\bar{w})| \cdot |\gamma_{1t}(w)\gamma_{1t}(\bar{w}) - \delta_{1t}(w)\delta_{1t}(\bar{w})| \cdot e^{-\langle w,w\rangle}\, dw$$

where

$$\alpha_{1t}(w) = \mathcal{O}(t^{\frac{7}{2}})w_1 + \Big(-\tfrac{\sqrt{6}}{12}t - \tfrac{5\sqrt{6}}{96}t^2 + \mathcal{O}\big(t^3\big)\Big)w_2 + \bigg(\tfrac{\sqrt{6}}{2}t^{\frac{1}{2}} + \tfrac{\sqrt{6}}{16}t^{\frac{3}{2}} + \mathcal{O}(t^{\frac{5}{2}})\bigg)w_3 + \Big(\tfrac{\sqrt{6}}{3} - \tfrac{7\sqrt{6}}{72}t - \tfrac{473\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big)\Big)w_4$$

$$\beta_{1t}(w) = \mathcal{O}(t^{\frac{5}{2}})w_1 + \Big(\tfrac{\sqrt{6}}{12}t - \tfrac{\sqrt{6}}{96}t^2 + \mathcal{O}\big(t^3\big)\Big)\, w_2 + \Big(\tfrac{\sqrt{6}}{12}t^{\frac{3}{2}} + \mathcal{O}(t^{\frac{5}{2}})\Big)\, w_3 + \Big(\tfrac{\sqrt{6}}{3} - \tfrac{\sqrt{6}}{72}t - \tfrac{29\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big)\Big)\, w_4$$

$$\gamma_{1t}(w) = \mathcal{O}(t^{\frac{7}{2}})w_1 + \Big(-\tfrac{\sqrt{6}}{12}t - \tfrac{\sqrt{6}}{96}t^2 + \mathcal{O}\big(t^3\big)\Big)w_2 + \bigg(-\tfrac{\sqrt{6}}{2}t^{\frac{1}{2}} - \tfrac{\sqrt{6}}{16}t^{\frac{3}{2}} + \mathcal{O}(t^{\frac{5}{2}})\bigg)w_3 + \Big(\tfrac{\sqrt{6}}{3} + \tfrac{5\sqrt{6}}{72}t + \tfrac{799\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big)\Big)w_4$$

$$\delta_{1t}(w) = \mathcal{O}(t^{\frac{5}{2}})w_1 + \Big(\tfrac{\sqrt{6}}{12}t + \tfrac{\sqrt{6}}{32}t^2 + \mathcal{O}\big(t^3\big)\Big)\, w_2 + \Big(\tfrac{\sqrt{6}}{6}t^{\frac{3}{2}} + \mathcal{O}(t^{\frac{5}{2}})\Big)\, w_3 + \Big(\tfrac{\sqrt{6}}{3} + \tfrac{11\sqrt{6}}{72}t + \tfrac{91\sqrt{6}}{3456}t^2 + \mathcal{O}\big(t^3\big)\Big)\, w_4$$

So

$$J(t) = \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A}\int_{\mathbb{C}^4}\left|\alpha_2+\beta_2\sqrt{t}+\gamma_2 t+\delta_2 t^{\frac{3}{2}}+\sum\nolimits_{jk} {}^1\varepsilon_{jk}(t)w_j\bar{w}_k\right| \cdot \left|\alpha_3+\beta_3\sqrt{t}+\gamma_3 t+\delta_3 t^{\frac{3}{2}}+\sum\nolimits_{jk} {}^2\varepsilon_{jk}(t)w_j\bar{w}_k\right| e^{-\langle w,w\rangle}\, dw$$

where

$$
\begin{aligned}
\alpha_2(t) &= \frac{\sqrt{6}}{3}w_4\frac{\sqrt{6}}{3}\bar{w}_4 - \frac{\sqrt{6}}{3}w_4\frac{\sqrt{6}}{3}\bar{w}_4 = 0\\
&= \alpha_3(t)\\
\beta_2(t) &= \frac{\sqrt{6}}{2}w_3\frac{\sqrt{6}}{3}\bar{w}_4 + \frac{\sqrt{6}}{3}w_4\frac{\sqrt{6}}{2}\bar{w}_3 = (w_3\bar{w}_4 + \bar{w}_3 w_4) = 2\,\mathrm{Re}(w_3\bar{w}_4)\\
\beta_3(t) &= -\frac{\sqrt{6}}{2}w_3\frac{\sqrt{6}}{3}\bar{w}_4 + \left(-\frac{\sqrt{6}}{2}\right)\bar{w}_3\frac{\sqrt{6}}{3}w_4 = -\left(w_3\bar{w}_4 + \bar{w}_3 w_4\right) = -2\,\mathrm{Re}(w_3\bar{w}_4)\\
&= -\beta_2(t)\\
\gamma_2(t) &= -\frac{\sqrt{6}}{12}w_2\frac{\sqrt{6}}{3}\bar{w}_4 + \frac{\sqrt{6}}{2}w_3\frac{\sqrt{6}}{2}\bar{w}_3 + \frac{\sqrt{6}}{3}w_4\left(-\frac{\sqrt{6}}{12}\right)\bar{w}_2 + \frac{\sqrt{6}}{3}w_4(-\frac{7\sqrt{6}}{72})\bar{w}_4 - \frac{7\sqrt{6}}{72}w_4\frac{\sqrt{6}}{3}\bar{w}_4\\
&\quad - \frac{\sqrt{6}}{12}w_2\frac{\sqrt{6}}{3}\bar{w}_4 - \frac{\sqrt{6}}{3}w_4\frac{\sqrt{6}}{12}\bar{w}_2 - \frac{\sqrt{6}}{3}w_4\left(-\frac{\sqrt{6}}{72}\right)\bar{w}_4 + \frac{\sqrt{6}}{72}w_4\frac{\sqrt{6}}{3}\bar{w}_4\\
&= \frac{1}{6}\left(-2w_2\bar{w}_4 - 2\bar{w}_2 w_4 - 2w_4\bar{w}_4 + 9w_3\bar{w}_3\right) = \frac{1}{6}(-4\,\mathrm{Re}(w_2\bar{w}_4) + 9|w_3|^2 - 2|w_4|^2)\\
\gamma_3(t) &= -\frac{\sqrt{6}}{12}w_2\frac{\sqrt{6}}{3}\bar{w}_4 + (-\frac{\sqrt{6}}{2})w_3(-\frac{\sqrt{6}}{2})\bar{w}_3 + \frac{\sqrt{6}}{3}w_4(-\frac{\sqrt{6}}{12})\bar{w}_2 + \frac{\sqrt{6}}{3}w_4\frac{5\sqrt{6}}{72}\bar{w}_4 + \frac{5\sqrt{6}}{72}w_4\frac{\sqrt{6}}{3}\bar{w}_4\\
&\quad - \frac{\sqrt{6}}{12}w_2\frac{\sqrt{6}}{3}\bar{w}_4 - \frac{\sqrt{6}}{3}w_4\frac{\sqrt{6}}{12}\bar{w}_2 - \frac{\sqrt{6}}{3}w_4\frac{11\sqrt{6}}{72}\bar{w}_4 - \frac{11\sqrt{6}}{72}w_4\frac{\sqrt{6}}{3}\bar{w}_4\\
&= \frac{1}{6}\left(-2w_2\,\bar{w}_4 - 2\bar{w}_2 w_4 - 2w_4\,\bar{w}_4 + 9w_3\,\bar{w}_3\right) = \frac{1}{6}(-4\,\mathrm{Re}(w_2\bar{w}_4) + 9|w_3|^2 - 2|w_4|^2)\\
&= \gamma_2(t)\\
\delta_2(t) &= -\frac{1}{12}(3\bar{w}_2 w_3 + 3w_2\bar{w}_3 + 4\bar{w}_3 w_4 + 4w_3\bar{w}_4)t^{\frac{3}{2}} = -\frac{1}{12}(6\,\mathrm{Re}(w_2\bar{w}_3) + 8\,\mathrm{Re}(w_3\bar{w}_4))\\
\delta_3(t) &= \frac{1}{12}(3w_2\bar{w}_3 + 3\bar{w}_2 w_3 - 8w_3\bar{w}_4 - 8\bar{w}_3 w_4)t^{\frac{3}{2}} = \frac{1}{12}(6\,\mathrm{Re}(w_2\bar{w}_3) - 16\,\mathrm{Re}(w_3\bar{w}_4))\\
{}^{1}\varepsilon_{jk}(t) &= \mathcal{O}(t^2)\\
{}^{2}\varepsilon_{jk}(t) &= \mathcal{O}(t^2)
\end{aligned}
$$

notice $\alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_2, \gamma_3, \delta_2, \delta_3 \in \mathbb{R}$. This implies that since

$$\left(\alpha_2 + \beta_2\sqrt{t} + \gamma_2 t + \delta_2 t^{\frac{3}{2}} + \sum_{jk}{}^{1}\varepsilon_{jk}(t)w_j\bar{w}_k\right) \in \mathbb{R}$$

and

$$\left(\alpha_3 + \beta_3\sqrt{t} + \gamma_3 t + \delta_3 t^{\frac{3}{2}} + \sum_{jk}{}^{2}\varepsilon_{jk}(t)w_j\bar{w}_k\right) \in \mathbb{R}$$

by (1.11), $\left(\sum_{jk}{}^{1}\varepsilon_{jk}(t)w_j\bar{w}_k\right)$ and $\left(\sum_{jk}{}^{2}\varepsilon_{jk}(t)w_j\bar{w}_k\right)$ must be real, as well.

so

$$J(t) = \frac{1+\mathcal{O}(t^3)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left|\beta_2\sqrt{t} + \gamma_2 t + \delta_2 t^{\frac{3}{2}} + \sum_{jk} {}^1\varepsilon_{jk}(t) w_j \bar{w}_k\right| \left|-\beta_2\sqrt{t}+\gamma_2 t+\delta_3 t^{\frac{3}{2}}+\sum_{jk} {}^2\varepsilon_{jk}(t) w_j \bar{w}_k\right| e^{-\langle w,w\rangle}\, dw$$

$$= \frac{1+\mathcal{O}(t^3)}{\pi^6 \det A} \int_{\mathbb{C}^4} \Big| \underbrace{[-\beta_2{}^2]}_{\alpha_4(w)<0} t + [0] t^{\frac{3}{2}} + \underbrace{[\beta_2\delta_3+\gamma_2{}^2-\delta_2\beta_2]}_{\beta_4(w)} t^2 + \underbrace{\sum_{jk} {}^3\varepsilon_{jk}(t) w_j \bar{w}_k}_{\Sigma_3(w,t)} + \underbrace{\sum_{jk} {}^4\varepsilon_{jk\ell m}(t) w_j \bar{w}_k w_\ell \bar{w}_m}_{\Sigma_4(w,t)} \Big| e^{-\langle w,w\rangle} dw$$

where

$$\alpha_4 = -(\beta_2)^2 = -(2\,\mathrm{Re}(w_3\bar{w}_4))^2 = -4\,\mathrm{Re}(w_3\bar{w}_4)^2$$

$$\begin{aligned}
\beta_4 &= \beta_2\delta_3 + \gamma_2{}^2 - \delta_2\beta_2 \\
&= \frac{1}{6}\,\mathrm{Re}(w_3\bar{w}_4)\Big(6\,\mathrm{Re}(w_2\bar{w}_3) - 16\,\mathrm{Re}(w_3\bar{w}_4)\Big) \\
&\quad + \frac{1}{36}\Big(-4\,\mathrm{Re}(w_2\bar{w}_4) + 9|w_3|^2 - 2|w_4|^2\Big)^2 \\
&\quad + \frac{1}{6}\,\mathrm{Re}(w_3\bar{w}_4)\Big(6\,\mathrm{Re}(w_2\bar{w}_3) + 8\,\mathrm{Re}(w_3\bar{w}_4)\Big) \\
&= 2\,\mathrm{Re}(w_3\bar{w}_4)\,\mathrm{Re}(w_2\bar{w}_3) - \frac{4}{3}(\mathrm{Re}(w_3\bar{w}_4))^2 + \frac{4}{9}(\mathrm{Re}(w_2\bar{w}_4))^2 - 2\,\mathrm{Re}(w_2\bar{w}_4) w_3\bar{w}_3 \\
&\quad + \frac{4}{9}\,\mathrm{Re}(w_2\bar{w}_4) w_4\bar{w}_4 + \frac{9}{4} w_3{}^2\bar{w}_3^2 - w_3\bar{w}_3 w_4\bar{w}_4 + \frac{1}{9} w_4{}^2\bar{w}_4^2 \\
&= 2\,\mathrm{Re}(w_2\bar{w}_3)\,\mathrm{Re}(w_3\bar{w}_4) - \frac{4}{3}\,\mathrm{Re}(w_3\bar{w}_4)^2 + \frac{1}{9}\,\mathrm{Re}(w_2\bar{w}_4)\Big(4\,\mathrm{Re}(w_2\bar{w}_4) - 18|w_3|^2 + 4|w_4|^2\Big) \\
&\quad + \frac{9}{4}|w_3|^4 - |w_3|^2|w_4|^2 + \frac{1}{9}|w_4|^4 \\
&= 2\,\mathrm{Re}(w_2\bar{w}_3)\,\mathrm{Re}(w_3\bar{w}_4) - \frac{4}{3}\,\mathrm{Re}(w_3\bar{w}_4)^2 + \frac{1}{9}\,\mathrm{Re}(w_2\bar{w}_4)\Big(4\,\mathrm{Re}(w_2\bar{w}_4) - 18|w_3|^2 + 4|w_4|^2\Big) \\
&\quad + \frac{1}{36}\Big(9|w_3|^2 - 2|w_4|^2\Big)^2
\end{aligned}$$

$${}^3\varepsilon_{jk}(t) = \mathcal{O}\Big(t^{\frac{5}{2}}\Big)$$

$${}^4\varepsilon_{jk\ell m}(t) = {}^1\varepsilon_{jk}(t)\cdot {}^2\varepsilon_{\ell m}(t) = \mathcal{O}(t^4)$$

Again, notice that $\alpha_4(w)$, $\beta_4(w)$, $\Sigma_3(w,t)$, and $\Sigma_4(w,t)$ are all real.

So

$$J(t) = \frac{1+\mathcal{O}(t^3)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left|\alpha_4 t + \beta_4 t^2 + \Sigma_3(w,t) + \Sigma_4(w,t)\right| e^{-\langle w,w\rangle}\, dw$$

The following lemma will help take the error terms out of the absolute value.

Lemma 1.48. *For any $n > 1$, any continuous complex valued $f(w,t)$ and $g(w)$, and $\varepsilon(t) = \mathcal{O}(t^n)$*

$$|f(w,t)| = |f(w,t) - \varepsilon(t)g(w)| + \mathcal{O}(t^n)\,|g(w)|$$

Proof:

$$\Big||f(w,t)| - |f(w,t) - \varepsilon(t)g(w)|\Big| \le \Big|\varepsilon(t)g(w)\Big| = |\varepsilon(t)|\;|g(w)|$$

so

$$\underbrace{-|\varepsilon(t)|}_{\mathcal{O}(t^n)} \cdot |g(w)| \le |f(w,t)| - |f(w,t) - \varepsilon(t)g(w)| \le \underbrace{|\varepsilon(t)|}_{\mathcal{O}(t^n)} \cdot \ |g(w)|$$

so

$$|f(w,t)| - |f(w,t) - \varepsilon(t)g(w)| = \mathcal{O}(t^n)\,|g(w)|$$

so

$$|f(w,t)| = |f(w,t) - \varepsilon(t)g(w)| + \mathcal{O}(t^n)\,|g(w)|$$

Q.E.D.

Applying Lemma 1.48 repeatedly to $|\alpha_4 t + \beta_4 t^2 + \Sigma_3(w,t) + \Sigma_4(w,t)|$ and ${}^3\varepsilon_{jk}(t) w_j \bar{w}_k$ says that

$$J(t) = \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left(\big|\alpha_4 t + \beta_4 t^2 + \Sigma_4(w,t)\big| + \sum_{jk} \mathcal{O}\Big(t^{\frac{5}{2}}\Big)\,|w_j \bar{w}_k| \right) e^{-\langle w,w\rangle}\, dw$$

and applying Lemma 1.48 repeatedly to $|\alpha_4 t + \beta_4 t^2 + \Sigma_4(w,t)|$ and ${}^4\varepsilon_{jk\ell m}(t) w_j \bar{w}_k w_\ell \bar{w}_m$ says

$$J(t) = \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left(\big|\alpha_4 t + \beta_4 t^2\big| + \sum_{jk} \mathcal{O}\Big(t^{\frac{5}{2}}\Big)\,|w_j \bar{w}_k| + \sum_{jk\ell m} \mathcal{O}\big(t^4\big)\,|w_j \bar{w}_k w_\ell \bar{w}_m| \right) e^{-\langle w,w\rangle}\, dw$$

Finally, applying Lemma 1.48 to $|\alpha_4 t + \beta_4 t^2|$ and $\beta_4 t^2$ says

$$\begin{aligned}
J(t) =& \tfrac{1+\mathcal{O}(t^3)}{\pi^6 \det A(t)} \int_{\mathbb{C}^4} \left(|\alpha_4 t| + |\beta_4| \mathcal{O}(t^2) + \sum_{jk} \mathcal{O}\Big(t^{\frac{5}{2}}\Big)\,|w_j \bar{w}_k| + \sum_{jk\ell m} \mathcal{O}\big(t^4\big)\,|w_j \bar{w}_k w_\ell \bar{w}_m| \right) e^{-\langle w,w\rangle}\, dw \\
=& \quad \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \left(-t \int_{\mathbb{C}^4} \alpha_4\, e^{-\langle w,w\rangle}\, dw + \mathcal{O}\big(t^2\big) \int_{\mathbb{C}^4} |\beta_4|\, e^{-\langle w,w\rangle}\, dw \right) \\
& + \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \left(\sum_{jk} \mathcal{O}\Big(t^{\frac{5}{2}}\Big) \int_{\mathbb{C}^4} |w_j \bar{w}_k|\, e^{-\langle w,w\rangle}\, dw + \sum_{jk\ell m} \mathcal{O}\big(t^4\big) \int_{\mathbb{C}^4} |w_j \bar{w}_k w_\ell \bar{w}_m|\, e^{-\langle w,w\rangle}\, dw \right) \\
=& \quad \frac{1+\mathcal{O}\big(t^3\big)}{\pi^6 \det A(t)} \left(-t(-2\pi^4) + \mathcal{O}\big(t^2\big)\,\big[\text{finite}\big] + \sum_{jk} \mathcal{O}\Big(t^{\frac{5}{2}}\Big)\,\big[\text{finite}\big] + \sum_{jk\ell m} \mathcal{O}\big(t^4\big)\,\big[\text{finite}\big] \right) \\
=& \quad \frac{1+\mathcal{O}\big(t^3\big)}{\pi^2 \det A(t)} \Big(2t + \mathcal{O}\big(t^2\big) + \mathcal{O}\Big(t^{\frac{5}{2}}\Big) + \mathcal{O}\big(t^4\big) \Big) \\
=& \quad \frac{1+\mathcal{O}\big(t^3\big)}{\pi^2 \det A(t)} \Big(2t + \mathcal{O}\big(t^2\big) \Big) \\
=& \quad \frac{2t + \mathcal{O}\big(t^2\big) + \mathcal{O}\big(t^3\big) + \mathcal{O}\big(t^4\big) + \mathcal{O}\big(t^5\big)}{\pi^2 \det A(t)} \\
=& \quad \frac{2t + \mathcal{O}\big(t^2\big)}{\pi^2 \det A(t)} \\
=& \quad \frac{1}{\pi^2} \frac{2t + \mathcal{O}\big(t^2\big)}{3t + \mathcal{O}\big(t^2\big)} \\
=& \quad \frac{2}{3\pi^2} + \mathcal{O}(t)
\end{aligned}$$

so $J(r) = \dfrac{2}{3\pi^2} + \mathcal{O}\big(r^2\big)$. i.e. $J(r) \xrightarrow{r \searrow 0} \dfrac{2}{3\pi^2}$

Appendix or *Where did $U(t)$ and $D(t)$ come from?*

$D(t)$ and $U(t)$ naturally arise when calculating $J(t)$ by diagonalizing $Y(t)$.

$$\begin{aligned}
J(t) &= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||h_1|^2 - |x_1|^2\right| \cdot \left||h_2|^2 - |x_2|^2\right| e^{-\langle Y(t)(t^{-\frac{5}{2}} v),(t^{-\frac{5}{2}} v)\rangle}\, dv \\
&= \frac{1}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} t^5 \left||w_1|^2 - |w_2|^2\right| \cdot t^5 \left||w_3|^2 - |w_4|^2\right| e^{-\langle Y(t)w,w\rangle}\, t^{20} dw \\
&= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||w_1|^2 - |w_2|^2\right| \cdot \left||w_3|^2 - |w_4|^2\right| e^{-\langle Y(t)w,w\rangle}\, dw \\
&= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||w_1|^2 - |w_2|^2\right| \cdot \left||w_3|^2 - |w_4|^2\right| e^{-\langle U(t)^* D(t) U(t) w,w\rangle}\, dw \\
&= \frac{t^{30}}{\pi^6 \det A(t) \det \Lambda(t)} \int_{\mathbb{C}^4} \left||w_1|^2 - |w_2|^2\right| \cdot \left||w_3|^2 - |w_4|^2\right| e^{-\langle D(t) U(t) w,U(t)w\rangle}\, dw
\end{aligned}$$

where

$$D(t) = \begin{bmatrix} \lambda_1(t) & 0 & 0 & 0 \\ 0 & \lambda_2(t) & 0 & 0 \\ 0 & 0 & \lambda_3(t) & 0 \\ 0 & 0 & 0 & \lambda_4(t) \end{bmatrix}$$

with the $\lambda_i(t)$ being the eigenvalues of $Y(t)$ and

$$U(t) = \begin{bmatrix} \text{———} & v_1(t) & \text{———} \\ \text{———} & v_2(t) & \text{———} \\ \text{———} & v_3(t) & \text{———} \\ \text{———} & v_4(t) & \text{———} \end{bmatrix}$$

with the $v_i(t)$ being the associated normalized eigenvectors, making $U(t)$ real orthogonal.

So expanding the λ_i and v_i in t gives an expansion for $J(t)$. Maple outputs a 100 megabyte file for each eigenvalue when asked `eigenvalues(Y(t))` directly and crashes when asked to find an expansion for any individual eigenvalue. However, because the matrix is 4×4, the eigenvalues and eigenvectors can be calculated algebraically by applying the quartic formula [Car93] to $Y(t)$'s characteristic polynomial. Then these algebraic expressions can be expanded by maple.

$$Y(t) =: \begin{bmatrix} f_1(t) & f_2(t) & f_3(t) & f_4(t) \\ f_5(t) & f_6(t) & f_7(t) & f_8(t) \\ f_9(t) & f_{10}(t) & f_{11}(t) & f_{12}(t) \\ f_{13}(t) & f_{14}(t) & f_{15}(t) & f_{16}(t) \end{bmatrix}$$

$$\det(Y(t) - xI) =$$

$$x^4 + \underbrace{\begin{pmatrix} -f_6 \\ -f_1 \\ -f_{11} \\ -f_{16} \end{pmatrix}}_{F_3(t)} x^3 + \underbrace{\begin{pmatrix} -f_9 f_3 \\ +f_6 f_{16} \\ -f_{13} f_4 \\ +f_6 f_{11} \\ -f_5 f_2 \\ -f_{14} f_8 \\ +f_{11} f_{16} \\ -f_{12} f_{15} \\ -f_{10} f_7 \\ +f_1 f_6 \\ +f_1 f_{11} \\ +f_1 f_{16} \end{pmatrix}}_{F_2(t)} x^2 + \underbrace{\begin{pmatrix} -f_1 f_6 f_{16} \\ +f_{13} f_6 f_4 \\ -f_1 f_6 f_{11} \\ -f_5 f_{14} f_4 \\ +f_1 f_{10} f_7 \\ -f_{13} f_3 f_{12} \\ +f_{13} f_4 f_{11} \\ +f_5 f_2 f_{11} \\ -f_9 f_2 f_7 \\ +f_9 f_3 f_{16} \\ -f_9 f_4 f_{15} \\ -f_5 f_{10} f_3 \\ +f_5 f_2 f_{16} \\ -f_1 f_{11} f_{16} \\ +f_1 f_{12} f_{15} \\ -f_{13} f_2 f_8 \\ +f_9 f_6 f_3 \\ +f_1 f_{14} f_8 \\ -f_6 f_{11} f_{16} \\ +f_6 f_{12} f_{15} \\ +f_{10} f_7 f_{16} \\ -f_{10} f_8 f_{15} \\ -f_{14} f_7 f_{12} \\ +f_{14} f_8 f_{11} \end{pmatrix}}_{F_1(t)} x + \underbrace{\begin{pmatrix} -f_5 f_2 f_{11} f_{16} \\ -f_5 f_2 f_{12} f_{15} \\ -f_5 f_{10} f_3 f_{16} \\ +f_5 f_{10} f_4 f_{15} \\ +f_5 f_{14} f_3 f_{12} \\ -f_5 f_{14} f_4 f_{11} \\ -f_9 f_2 f_7 f_{16} \\ +f_9 f_2 f_8 f_{15} \\ +f_9 f_6 f_3 f_{16} \\ -f_9 f_6 f_4 f_{15} \\ -f_9 f_{14} f_3 f_8 \\ +f_9 f_{14} f_4 f_7 \\ +f_{13} f_2 f_7 f_{12} \\ -f_{13} f_2 f_8 f_{11} \\ -f_{13} f_6 f_3 f_{12} \\ +f_{13} f_6 f_4 f_{11} \\ +f_{13} f_{10} f_3 f_8 \\ -f_{13} f_{10} f_4 f_7 \\ -f_1 f_6 f_{11} f_{16} \\ +f_1 f_6 f_{12} f_{15} \\ +f_1 f_{10} f_7 f_{16} \\ -f_1 f_{10} f_8 f_{15} \\ -f_1 f_{14} f_7 f_{12} \\ +f_1 f_{14} f_8 f_{11} \end{pmatrix}}_{F_0(t)}$$

Because Y is hermitian, it's eigenvalues will all be real. In fact, because Y is positive definite, they will all be positive. In the calculation below, things can be complex (e.g. R) but all of the imaginary parts will go away by the end.

$$F_3(t) = -8640 + \mathcal{O}(t)$$

$$F_2(t) = 51840\, t^3 + \mathcal{O}\big(t^4\big)$$

$$F_1(t) = -17280\, t^7 + \mathcal{O}\big(t^8\big)$$

$$F_0(t) = 6480\, t^{12} + \mathcal{O}\big(t^{13}\big)$$

Solving the general quartic $x^4 + Bx^3 + Cx^2 + Dx + E = 0$ requires some simplifying definitions and a few choices.

$$\alpha := -\frac{3B^2}{8} + C = -27993600 + \mathcal{O}(t) \qquad (< 0 \text{ for small } t \text{ as } t \searrow 0)$$

$$\beta := \frac{B^3}{8} - \frac{BC}{2} + D = -80621568000 + \mathcal{O}(t) \qquad (< 0 \text{ for small } t \text{ as } t \searrow 0)$$

$$\gamma := -\frac{3B^4}{256} + \frac{CB^2}{16} - \frac{BD}{4} + E = -65303470080000 + \mathcal{O}(t)$$

$$P := -\frac{\alpha^2}{12} - \gamma = -223948800\, t^6 + \mathcal{O}\big(t^7\big)$$

$$Q := -\frac{\alpha^3}{108} + \frac{\alpha\gamma}{3} - \frac{\beta^2}{8} = -1289945088000\, t^9 + \mathcal{O}\big(t^{10}\big)$$

$$R = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}$$

(Choose either root.)

$$= 644972544000\, t^9 + \mathcal{O}\big(t^{10}\big) \pm \underbrace{\sqrt{-866644963570483200000\, t^{20} + \mathcal{O}(t^{21})}}_{i\mathcal{O}(t^{10})}$$

$$U = \sqrt[3]{R} \qquad \text{(Choose any of the three roots.)}$$

$$y := -\frac{5}{6}\alpha + U - \frac{P}{3U}$$

$$W := \sqrt{\alpha + 2y} \qquad \text{(Both roots come up in } \pm_s\text{)}$$

With those definitions, the roots should be

$$-\frac{B}{4} + \frac{\pm_s W \pm_t \sqrt{-\left(3\alpha + 2y \pm_s \frac{2\beta}{W}\right)}}{2}$$

where the $\pm_s$'s are dependent and the $\pm_t$ is independent.

Maple can expand α, β, and B easily. We need to carefully intervene to get it to expand W, y, and $\frac{2\beta}{W}$ which require R and U.

$$-\frac{Q^2}{4}-\frac{P^3}{27}=866644963570483200000\,t^{20}+\cdots+\frac{563370492281772061551028318944792605312337}{7662929083743683750}\,t^{40}+\mathcal{O}\big(t^{41}\big)$$

$$\begin{aligned}\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}} =&\sqrt{866644963570483200000t^{20}+\ldots+\mathcal{O}(t^{41})}\\ =&\sqrt{866644963570483200000}\,t^{10}\cdot\sqrt{1+\underbrace{\ldots+\mathcal{O}(t^{21})}_{x}}\\ =&\sqrt{866644963570483200000}\,t^{10}\cdot\left(\overbrace{1+\frac{1}{2}x+\ldots+\mathcal{O}\big(x^{21}\big)}^{\sqrt{1+x}}\right)\\ =&\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)\end{aligned}$$

$$-\frac{Q}{2}=644972544000\,t^{9}+\ldots+\mathcal{O}\big(t^{30}\big)$$

$$\begin{aligned}\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}} =&\frac{\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)}{644972544000\,t^{9}+\ldots+\mathcal{O}(t^{30})}\\ =&\frac{1}{644972544000\,t^{9}}\,\frac{1}{1+\underbrace{\ldots+\mathcal{O}(t^{21})}_{x}}\cdot\Big(\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)\Big)\\ =&\frac{1}{644972544000\,t^{9}}\Big(\overbrace{1-x+\ldots+\mathcal{O}\big(x^{21}\big)}^{\frac{1}{1+x}}\Big)\cdot\Big(\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)\Big)\\ =&\frac{1}{644972544000\,t^{9}}\Big(1+\ldots+\mathcal{O}\big(t^{21}\big)\Big)\Big(\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)\Big)\\ =&\frac{1}{644972544000\,t^{9}}\Big(\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)\Big)\\ =&\frac{\sqrt{3}}{12}\,t+\ldots+\mathcal{O}\big(t^{22}\big)\end{aligned}$$

Choose R in the I quadrant, namely

$$\begin{aligned}R:=&-\frac{Q}{2}+i\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}\\ =&644972544000\,t^{9}+\ldots+\mathcal{O}\big(t^{30}\big)+i\Big(\sqrt{866644963570483200000}\,t^{10}+\ldots+\mathcal{O}\big(t^{31}\big)\Big)\\ =&r_Re^{i\theta_R}\end{aligned}$$

where

$$r_R=\sqrt{\frac{-P^3}{27}}\quad\text{and}\quad\theta_R=\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right)$$

Choose U in the I quadrant, namely

$$U:=\sqrt[3]{R}=\sqrt[3]{r_R}e^{i\frac{1}{3}\theta_R}=r_Ue^{i\theta_U}$$

where

$$\theta_U = \frac{1}{3}\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right) \quad \text{and} \quad r_U = \left(\frac{(-P)^3}{27}\right)^{\frac{1}{6}} = \frac{\sqrt{-P}}{\sqrt[6]{27}}$$

$$y = -\frac{5}{6}\alpha + U - \frac{P}{3U} = -\frac{5}{6}\alpha + r_U e^{i\theta_U} - \frac{P}{3r_U e^{i\theta_U}} = -\frac{5}{6}\alpha + r_U e^{i\theta_U} - \frac{Pe^{-i\theta_U}}{3r_U} = -\frac{5}{6}\alpha + \frac{r_U^2 e^{i\theta_U} - \frac{P}{3}e^{-i\theta_U}}{r_U}$$

$$= -\frac{5}{6}\alpha + \frac{r_U^2(\cos\theta_U + i\sin\theta_U) - \frac{P}{3}(\cos\theta_U - i\sin\theta_U)}{r_U} = -\frac{5}{6}\alpha + \frac{\left(r_U^2 - \frac{P}{3}\right)}{r_U}\cos\theta_U + i\left[\frac{\left(r_U^2 + \frac{P}{3}\right)}{r_U}\right]\sin\theta_U$$

$$= -\frac{5}{6}\alpha + \left[\frac{\left(\frac{\sqrt{-P}}{\sqrt[6]{27}}\right)^2 - \frac{P}{3}}{r_U}\right]\cos\theta_U + i\left[\frac{\left(\frac{\sqrt{-P}}{\sqrt[6]{27}}\right)^2 + \frac{P}{3}}{r_U}\right]\sin\theta_U = -\frac{5}{6}\alpha + \left(\frac{\frac{-P}{3} - \frac{P}{3}}{r_U}\right)\cos\theta_U + i\left(\frac{\frac{-P}{3} + \frac{P}{3}}{r_U}\right)\sin\theta_U$$

$$= -\frac{5}{6} - \frac{2P}{3r_U}\cos\theta_U + i\left(\quad 0 \quad\right)\sin\theta_U = -\frac{5}{6}\alpha - \frac{2P}{3\left(\frac{\sqrt{-P}}{\sqrt[6]{27}}\right)}\cos\theta_U = -\frac{5}{6}\alpha + \frac{2}{\sqrt{3}}\sqrt{-P}\cos\theta_U$$

$$= -\frac{5}{6}\alpha + \frac{2}{\sqrt{3}}\sqrt{-P}\cos\left(\frac{1}{3}\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right)\right)$$

$$\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right) = \arctan\Bigg(\underbrace{\frac{\sqrt{3}}{12}t + \ldots + \mathcal{O}(t^{22})}_{x}\Bigg) = \Bigg(\underbrace{x - \frac{1}{3}x^2 + \ldots + \mathcal{O}(x^{22})}_{\arctan x}\Bigg)$$

$$= \frac{\sqrt{3}}{12}t + \ldots + \mathcal{O}(t^{22})$$

so

$$\frac{1}{3}\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right) = \frac{\sqrt{3}}{36}t + \ldots + \mathcal{O}(t^{22})$$

$$\cos\frac{1}{3}\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right) = \cos\Bigg(\underbrace{\frac{\sqrt{3}}{36}t + \ldots + \mathcal{O}(t^{22})}_{x}\Bigg) = \underbrace{1 + \ldots + \mathcal{O}(x^{22})}_{\cos x} = 1 + \ldots + \mathcal{O}(t^{22})$$

$$y = -\frac{5}{6}\alpha + \frac{2}{\sqrt{3}}\sqrt{-P}\cos\frac{1}{3}\arctan\left(\frac{\sqrt{-\frac{Q^2}{4}-\frac{P^3}{27}}}{-\frac{Q}{2}}\right)$$

$$= -\frac{5}{6}\left(-27993600 + \ldots + \mathcal{O}(t^{25})\right) + \frac{2}{\sqrt{3}}\left(8640\sqrt{3}t^3 + \ldots\mathcal{O}(t^{25})\right)\left(1 + \ldots + \mathcal{O}(t^{22})\right)$$

$$= \frac{5\cdot 27993600}{6} + \ldots + \mathcal{O}(t^{25}) + \frac{2\cdot 8640\sqrt{3}}{\sqrt{3}}t^3 + \ldots + \mathcal{O}(t^{25}) = \frac{5\cdot 27993600}{6} + \ldots + \mathcal{O}(t^{25})$$

$$W=\sqrt{\alpha+2y}=\sqrt{(-27993600+\ldots+\mathcal{O}(t^{25}))+2\left(\frac{5\cdot 27993600}{6}+\ldots+\mathcal{O}(t^{25})\right)}$$
$$=\sqrt{\frac{2}{3}\cdot 27993600+\ldots+\mathcal{O}(t^{25})}=\sqrt{\frac{2}{3}\cdot 27993600}\sqrt{1+\underbrace{\ldots+\mathcal{O}(t^{25})}_{x}}$$
$$=\sqrt{\frac{2}{3}\cdot 27993600}\underbrace{\left(1+\ldots+\mathcal{O}\left(x^{25}\right)\right)}_{\sqrt{1+x}}=\sqrt{\frac{2}{3}\cdot 27993600}\left(1+\ldots+\mathcal{O}\left(t^{25}\right)\right)$$
$$=\sqrt{\frac{2}{3}\cdot 27993600}+\ldots+\mathcal{O}\left(t^{25}\right)=4320+\ldots+\mathcal{O}\left(t^{25}\right)$$

$$\frac{2\beta}{W}=\frac{-161243136000+\ldots+\mathcal{O}\left(t^{25}\right)}{4320+\ldots+\mathcal{O}(t^{25})}=\left(-161243136000+\ldots+\mathcal{O}\left(t^{25}\right)\right)\cdot\frac{1}{4320}\cdot\frac{1}{1+\underbrace{\ldots+\mathcal{O}(t^{25})}_{x}}$$
$$=\frac{1}{4320}\left(-161243136000+\ldots+\mathcal{O}\left(t^{25}\right)\right)\overbrace{\left(1+\ldots+\mathcal{O}\left(x^{25}\right)\right)}^{\frac{1}{1+x}}=-\frac{161243136000}{4320}+\ldots+\mathcal{O}\left(t^{25}\right)$$
$$=-37324800+\ldots+\mathcal{O}\left(t^{25}\right)$$

So

$$\lambda_1:=\lambda_{+,+}=-\frac{B}{4}+\frac{W+\sqrt{-(3\alpha+2y+\frac{2\beta}{W})}}{2}=\frac{-B+2\sqrt{-3\alpha-2y-\frac{2\beta}{W}}}{4}$$
$$=\frac{-(-8640+\ldots+\mathcal{O}\left(t^{25}\right))+2W+2\sqrt{-3(-27993600+\ldots+\mathcal{O}\left(t^{25}\right))-2(\frac{5\cdot 27993600}{6}+\ldots+\mathcal{O}\left(t^{25}\right))-(\frac{-161243136000}{4320}+\ldots+\mathcal{O}\left(t^{25}\right))}}{4}$$
$$=\frac{(8640+\ldots+\mathcal{O}\left(t^{25}\right))+2(4320+\ldots+\mathcal{O}\left(t^{25}\right))+2\sqrt{74649600+\ldots+\mathcal{O}(t^{25})}}{4}$$
$$=\frac{17280+\ldots+\mathcal{O}\left(t^{25}\right)+2\cdot 8640\sqrt{1+\underbrace{\ldots+\mathcal{O}(t^{25})}_{x}}}{4}$$
$$=\frac{17280+\ldots+\mathcal{O}\left(t^{25}\right)+17280\underbrace{(1+\ldots+\mathcal{O}\left(x^{25}\right))}_{\sqrt{1+x}}}{4}=8640+\ldots+\mathcal{O}\left(t^{25}\right)$$

$$\lambda_2:=\lambda_{+,-}=\frac{-B+2W-2\sqrt{-3\alpha-2y-\frac{2\beta}{W}}}{4}=\frac{17280+\ldots+\mathcal{O}\left(t^{25}\right)-17280(1+\ldots+\mathcal{O}\left(t^{25}\right))}{4}$$
$$=6t^3+\ldots+\mathcal{O}\left(t^{25}\right)$$

$$\lambda_3:=\lambda_{-,+}=\frac{-B-2W+2\sqrt{-3\alpha-2y+\frac{2\beta}{W}}}{4}=\frac{-B-2W+2\sqrt{\frac{1}{9}t^8+\ldots+\mathcal{O}(t^{25})}}{4}$$
$$=\frac{-B-2W+\frac{2}{3}t^4\sqrt{1+\underbrace{\ldots+\mathcal{O}(t^{17})}_{x}}}{4}=\frac{1}{4}\left[-B-2W+\frac{2}{3}t^4\underbrace{\left(1+\ldots+\mathcal{O}\left(x^{17}\right)\right)}_{\sqrt{1+x}}\right]$$
$$=\frac{-B-2W+\frac{2}{3}t^4+\ldots+\mathcal{O}\left(t^{21}\right)}{4}=\frac{1}{3}t^4+\ldots+\mathcal{O}\left(t^{21}\right)$$

$$\lambda_4 := \lambda_{-,-} = \frac{-B - 2W - \sqrt{-3\alpha - 2y + \frac{2\beta}{W}}}{4} = \frac{-B-2W-\frac{2}{3}t^4(1+\ldots+\mathcal{O}(t^{17}))}{4} = \frac{3}{8}t^5 + \ldots + \mathcal{O}(t^{21})$$

We can check that these expansions for the eigenvalues are correct by checking that evaluating the elementary symmetric polynomials in four variables on them gives the coefficients of the characteristic polynomial. i.e.

characteristic polynomial =

$$x^4 - \underbrace{e_1(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}_{\text{tr}}x^3 + e_2(\lambda_1,\lambda_2,\lambda_3,\lambda_4)x^2 - e_3(\lambda_1,\lambda_2,\lambda_3,\lambda_4)x + \underbrace{e_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}_{\text{det}}$$

Maple confirms that

$$e_1(\lambda_1,\lambda_2,\lambda_3,\lambda_4) + B = \mathcal{O}(t^{21})$$

$$e_2(\lambda_1,\lambda_2,\lambda_3,\lambda_4) - C = \mathcal{O}(t^{21})$$

$$e_3(\lambda_1,\lambda_2,\lambda_3,\lambda_4) + D = \mathcal{O}(t^{24})$$

$$e_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4) - E = \mathcal{O}(t^{28})$$

To find the associated eigenvectors, we need to find the kernel of $A := Y - \lambda_j I$. i.e. we want to find v_j such that $Av_j = 0$. If the last row of A isn't a row of zeroes, which is the case for the four A's that we examine here, then a sequence of elementary row operations represented by multiplication by an invertible J can take A to a matrix with a row of 0's on the bottom.

$$JA = \begin{bmatrix} & & & a \\ & B & & b \\ & & & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then

$$JA \begin{bmatrix} -B^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} & & & a \\ & B & & b \\ & & & c \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -B^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} -BB^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A \begin{bmatrix} -B^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ 1 \end{bmatrix} = J^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} -B^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ 1 \end{bmatrix}$ is an unnormalized eigenvector. We need $-B^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for each A. To expand B^{-1} we just expand $\operatorname{adj} B$ and $\det B$ and divide to get $B^{-1} = \frac{1}{\det B}\operatorname{adj} B$.

v_1 :

$$A = \begin{bmatrix} -8640 + \ldots + O\left(t^{25}\right) & 360\,t + \ldots + O\left(t^{25}\right) & -30\,t^2 + \ldots + O\left(t^{25}\right) & -360\,t + \ldots + O\left(t^{25}\right) \\ 360\,t + \ldots + O\left(t^{25}\right) & -4320 + \ldots + O\left(t^{25}\right) & -360\,t + \ldots + O\left(t^{25}\right) & -4320 + \ldots + O\left(t^{25}\right) \\ -30\,t^2 + \ldots + O\left(t^{25}\right) & -360\,t + \ldots + O\left(t^{25}\right) & -8640 + \ldots + O\left(t^{25}\right) & 360\,t + \ldots + O\left(t^{25}\right) \\ -360\,t + \ldots + O\left(t^{25}\right) & -4320 + \ldots + O\left(t^{25}\right) & 360\,t + \ldots + O\left(t^{25}\right) & -4320 + \ldots + O\left(t^{25}\right) \end{bmatrix}$$

We get $v_1 =$

$$\begin{bmatrix} -\frac{\sqrt{2}}{24}t - \frac{\sqrt{2}}{48}t^2 + \cdots + \mathcal{O}\left(t^{25}\right) \\ -\frac{\sqrt{2}}{38} - \frac{\sqrt{2}}{8}t + \cdots + \mathcal{O}\left(t^{25}\right) \\ \frac{\sqrt{2}}{24}t + 0\,t^2 + \cdots + \mathcal{O}\left(t^{25}\right) \\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}t + \cdots + \mathcal{O}\left(t^{25}\right) \end{bmatrix}$$

Maple confirms that

$$Yv_1 - \lambda_1 v_1 = \begin{bmatrix} \mathcal{O}\left(t^{25}\right) \\ \mathcal{O}\left(t^{25}\right) \\ \mathcal{O}\left(t^{25}\right) \\ \mathcal{O}\left(t^{25}\right) \end{bmatrix}$$

v_2 :

$$A = \begin{bmatrix} 30\,t^2 + \ldots + O\left(t^{25}\right) & 360\,t + \ldots + O\left(t^{25}\right) & -30\,t^2 + \ldots + O\left(t^{25}\right) & -360\,t + \ldots + O\left(t^{25}\right) \\ 360\,t + \ldots + O\left(t^{25}\right) & 4320 + \ldots + O\left(t^{25}\right) & -360\,t + \ldots + O\left(t^{25}\right) & -4320 + \ldots + O\left(t^{25}\right) \\ -30\,t^2 + \ldots + O\left(t^{25}\right) & -360\,t + \ldots + O\left(t^{25}\right) & 30\,t^2 + \ldots + O\left(t^{25}\right) & 360\,t + \ldots + O\left(t^{25}\right) \\ -360\,t + \ldots + O\left(t^{25}\right) & -4320 + \ldots + O\left(t^{25}\right) & 360\,t + \ldots + O\left(t^{25}\right) & 4320 + \ldots + O\left(t^{25}\right) \end{bmatrix}$$

We get $v_2 =$

$$\begin{bmatrix} -\frac{1}{2} - \frac{3}{16}\,t + \cdots + \mathcal{O}\left(t^{18}\right) \\ \frac{1}{2} - \frac{3}{16}t + \cdots + \mathcal{O}\left(t^{18}\right) \\ -\frac{1}{2} + \frac{1}{16}t + \cdots + \mathcal{O}\left(t^{18}\right) \\ \frac{1}{2} + \frac{1}{16}t + \cdots + \mathcal{O}\left(t^{18}\right) \end{bmatrix}$$

again, maple confirms that

$$Yv_2 - \lambda_2 v_2 = \begin{bmatrix} \mathcal{O}\left(t^{18}\right) \\ \mathcal{O}\left(t^{18}\right) \\ \mathcal{O}\left(t^{18}\right) \\ \mathcal{O}\left(t^{18}\right) \end{bmatrix}$$

v_3:

$$A = \begin{bmatrix} 30\,t^2 + \ldots + O\left(t^{21}\right) & 360\,t + \ldots + O\left(t^{21}\right) & -30\,t^2 + \ldots + O\left(t^{21}\right) & -360\,t + \ldots + O\left(t^{21}\right) \\ 360\,t + \ldots + O\left(t^{21}\right) & 4320 + \ldots + O\left(t^{21}\right) & -360\,t + \ldots + O\left(t^{21}\right) & -4320 + \ldots + O\left(t^{21}\right) \\ -30\,t^2 + \ldots + O\left(t^{21}\right) & -360\,t + \ldots + O\left(t^{21}\right) & 30\,t^2 + \ldots + O\left(t^{21}\right) & 360\,t + \ldots + O\left(t^{21}\right) \\ -360\,t + \ldots + O\left(t^{21}\right) & -4320 + \ldots + O\left(t^{21}\right) & 360\,t + \ldots + O\left(t^{21}\right) & 4320 + \ldots + O\left(t^{21}\right) \end{bmatrix}$$

In this case, the unnormalized eigenvector would have negative powers of t, so in the maple calculations, we find the normalized version of tv which is the same. i.e. we use the fact that

$$\frac{t\tilde{v}}{\|t\tilde{v}\|} = \frac{\tilde{v}}{\|\tilde{v}\|}$$

We get $v_3 =$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} + 0\,t + \cdots + \mathcal{O}(t^9) \\ \frac{\sqrt{2}}{12}t + \frac{61\sqrt{2}}{576}t^2 + \cdots + \mathcal{O}(t^9) \\ -\frac{\sqrt{2}}{2} + 0\,t + \cdots + \mathcal{O}(t^9) \\ \frac{\sqrt{2}}{6}t + \frac{109\sqrt{2}}{576}t^2 + \cdots + \mathcal{O}(t^9) \end{bmatrix}$$

again maple confirms that

$$Yv_3 - \lambda_3 v_3 = \begin{bmatrix} \mathcal{O}(t^9) \\ \mathcal{O}(t^9) \\ \mathcal{O}(t^9) \\ \mathcal{O}(t^9) \end{bmatrix}$$

---

v_4:

$$A = \begin{bmatrix} 30\,t^2 + \ldots + O\left(t^{21}\right) & 360\,t + \ldots + O\left(t^{21}\right) & -30\,t^2 + \ldots + O\left(t^{21}\right) & -360\,t + \ldots + O\left(t^{21}\right) \\ 360\,t + \ldots + O\left(t^{21}\right) & 4320 + \ldots + O\left(t^{21}\right) & -360\,t + \ldots + O\left(t^{21}\right) & -4320 + \ldots + O\left(t^{21}\right) \\ -30\,t^2 + \ldots + O\left(t^{21}\right) & -360\,t + \ldots + O\left(t^{21}\right) & 30\,t^2 + \ldots + O\left(t^{21}\right) & 360\,t + \ldots + O\left(t^{21}\right) \\ -360\,t + \ldots + O\left(t^{21}\right) & -4320 + \ldots + O\left(t^{21}\right) & 360\,t + \ldots + O\left(t^{21}\right) & 4320 + \ldots + O\left(t^{21}\right) \end{bmatrix}$$

We get $v_4 =$

$$\begin{bmatrix} \frac{1}{2} - \frac{3}{16}t + \cdots + \mathcal{O}(t^{12}) \\ \frac{1}{2} - \frac{1}{16}t + \cdots + \mathcal{O}(t^{12}) \\ \frac{1}{2} + \frac{1}{16}t + \cdots + \mathcal{O}(t^{12}) \\ \frac{1}{2} + \frac{3}{16}t + \cdots + \mathcal{O}(t^{12}) \end{bmatrix}$$

maple confirms that

$$Yv_4 - \lambda_4 v_4 = \begin{bmatrix} \mathcal{O}(t^{12}) \\ \mathcal{O}(t^{12}) \\ \mathcal{O}(t^{12}) \\ \mathcal{O}(t^{12}) \end{bmatrix}$$

All collected together, we have

$$\lambda_1 = 8640 + \ldots + \mathcal{O}(t^{25})$$

$$\lambda_2 = 6t^3 + \ldots + \mathcal{O}(t^{25})$$

$$\lambda_3 = \frac{1}{3}t^4 + \ldots + \mathcal{O}(t^{21})$$

$$\lambda_4 = \frac{3}{8}t^5 + \ldots + \mathcal{O}(t^{21})$$

and $U =$

$$\begin{bmatrix} -\frac{\sqrt{2}}{24}t - \frac{\sqrt{2}}{48}t^2 + \cdots + \mathcal{O}(t^{25}) & -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}t + \cdots + \mathcal{O}(t^{25}) & \frac{\sqrt{2}}{24}t + 0\,t^2 + \cdots + \mathcal{O}(t^{25}) & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}t + \cdots + \mathcal{O}(t^{25}) \\ -\frac{1}{2} - \frac{3}{16}t + \cdots + \mathcal{O}(t^{18}) & \frac{1}{2} - \frac{3}{16}t + \cdots + \mathcal{O}(t^{18}) & -\frac{1}{2} + \frac{1}{16}t + \cdots + \mathcal{O}(t^{18}) & \frac{1}{2} + \frac{1}{16}t + \cdots + \mathcal{O}(t^{18}) \\ \frac{\sqrt{2}}{2} + 0\,t + \cdots + \mathcal{O}(t^9) & \frac{\sqrt{2}}{12}t + \frac{61\sqrt{2}}{576}t^2 + \cdots + \mathcal{O}(t^9) & -\frac{\sqrt{2}}{2} + 0\,t + \cdots + \mathcal{O}(t^9) & \frac{\sqrt{2}}{6}t + \frac{109\sqrt{2}}{576}t^2 + \cdots + \mathcal{O}(t^9) \\ \frac{1}{2} - \frac{3}{16}t + \cdots + \mathcal{O}(t^{12}) & \frac{1}{2} - \frac{1}{16}t + \cdots + \mathcal{O}(t^{12}) & \frac{1}{2} + \frac{1}{16}t + \cdots + \mathcal{O}(t^{12}) & \frac{1}{2} + \frac{3}{16}t + \cdots + \mathcal{O}(t^{12}) \end{bmatrix}$$

The calculation only requires the λ_i be expanded to t^{12} and U be expanded to t^3.

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Curriculum Vitæ

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