ON A COHOMOLOGICAL STUDY OF HEISENBERG GROUPS OVER THE RING OF ALGEBRAIC INTEGERS

by

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Abstract

Henri Poincaré proved an important result for modular forms, namely that the space of cusp forms is generated by Poincaré sums. Taking inspiration from his work, we generalize to create a triple system $(\mathfrak{g}, (G, M))$, and we concern ourselves with determining whether analogous results to that of Poincaré's occur in other situations.

We will consider two cases in particular. The first is the case where $\mathfrak{g}$ is the Galois group of some finite Galois extension K over $\mathbb{Q}$, G is the ring of algebraic integers of this extension $\mathcal{O}_K$ viewed additively, and $M = \mathcal{O}_K^2$. Using a result of Takashi Ono, we are able to characterize the index i_γ for all quadratic extensions.

In the second case, we take $\mathfrak{g}$ and K as before, but now we consider $G = H_3(\mathcal{O}_K)$, the Heisenberg group of algebraic integers, and $M = \mathcal{O}_K^3$. We are able to prove a generalization of Ono's Theorem in this case using two $\mathbb{Z}$-modules, Ξ_K and Θ_η, and we can use our theorem to determine i_γ for all quadratic extensions in this case as well.

Advisor: Dr. Takashi Ono.

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Contents

Chapter 1

Introduction

1.1 Introduction

We gain the inspiration for our work from that of Henri Poincaré. In his work, he studied modular forms and the relation between the space of cusp forms and the space of Poincaré series. By looking at his work in a cohomological setting, it is both easy and natural to attempt to generalize his work.

To do this in a general setting, we take a system $(\mathfrak{g}, (G, M))$, letting G be a group, M be a left G-module, and $\mathfrak{g}$ a finite group which acts naturally on both G and M. Then, let $C \in Z^1(\mathfrak{g}, G)$ be a cocycle. We can define:

$$M_C = \{x \in M : C_s{}^s x = x, \forall s \in \mathfrak{g}\},$$

and

$$P_C = \{p_C(x) : p_C(x) = \sum_{t \in \mathfrak{g}} C_t{}^t x, x \in M\}.$$

M_C and P_C are both additive groups, and we will show in Chapter 2 that we have $|\mathfrak{g}| M_C \subseteq P_C \subseteq M_C$, meaning that we can speak about the quotient group M_C/P_C, and the index $|M_C/P_C|$ is finite. Further, we will show that this index is dependent only on the cohomology class $\gamma \in H^1(\mathfrak{g}, G)$. That is, if C and C' are cohomologous, then $M_C/P_C = M_{C'}/P_{C'}$. For any cohomology class γ, we can think of M_C/P_C as the twisted cohomology $\widehat{H}^0(\mathfrak{g}, M)_\gamma$.

We will study two cases. The first is the case where $\mathfrak{g}$ is the Galois group of a finite Galois extension K over $\mathbb{Q}$, G is the ring of integers of that extension $\mathcal{O}_K$ viewed additively, and $M = \mathcal{O}_K^2$. The second case is similar, with our $\mathfrak{g}$ still the same Galois group, but G is now considered to be the Heisenberg group $H_3(\mathcal{O}_K)$, and $M = \mathcal{O}_K^3$.

When considering Poincaré's work, he showed that when dealing with cusp forms and Poincaré series $M_C = P_C$, i.e. $M_C/P_C = 0$. It is natural to wonder if this would be the case in the generalized setting. However, it is not in general true that $M_C = P_C$. Therefore, we wish to determine the index $i_\gamma = |M_C/P_C|$ for each cohomology class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, and to study under what conditions does one actually attain $M_C/P_C = 0$.

In Chapter 3, we look at the index of M_C/P_C in the case where $M = \mathcal{O}_K^2$. We introduce Ξ_K as follows:

$$\Xi_K := \{\xi \in \mathcal{O}_K : \xi \equiv {}^\sigma\xi \bmod n, \forall \sigma \in \mathfrak{g}\}.$$

This Ξ_K is important as it allows us to explicitly determine the form of all cocycles $C \in Z^1(\mathfrak{g}, G)$. With the help of Ξ_K, we have the following theorem due to T. Ono that relates M_C/P_C to $\widehat{H}^0(\mathfrak{g}, A^{-1}M)$, where A is a matrix determined by ξ:

Theorem 3.2.2 (T. Ono) Let $K/\mathbb{Q}$ be a finite Galois extension of degree n, and let $\mathfrak{g}$ be its Galois group. Let $G = \mathcal{O}_K$, and $M = \mathcal{O}_K^2$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we have $i_\gamma(\mathfrak{g}, M) = |\widehat{H}^0(\mathfrak{g}, A(\xi)^{-1}M)|$.

In Chapter 4, I prove an explicit determination of $(A^{-1}M)^{\mathfrak{g}}$, the group of elements in $A^{-1}M$ that are fixed by all elements of $\mathfrak{g}$:

Theorem 4.1.1 Let K be a finite Galois extension over $\mathbb{Q}$ of degree n. Consider the system $(\mathfrak{g}, (G, M))$, where $\mathfrak{g} = \mathrm{Gal}(K, \mathbb{Q})$, $G = \mathcal{O}_K$, and $M = \mathcal{O}_K^2$. Let $\xi \in \Xi_K/\mathbb{Z} \subseteq \mathcal{O}_K/\mathbb{Z}$, with $\xi = \alpha_2\omega_2 + \cdots + \alpha_n\omega_n$. Let $d = gcd(\alpha_2, \ldots, \alpha_n, n)$, and let d' be the number such that $n = d \cdot d'$. Then,

$$(A^{-1}M)^{\mathfrak{g}} = \begin{pmatrix} \mathbb{Z} \\ d'\mathbb{Z} \end{pmatrix}.$$

We use both of these theorems to determine our index i_γ in the quadratic extension case:

Theorem 4.2.1 Let $\mathfrak{g} = \mathrm{Gal}(\mathbb{Q}(\sqrt{m})/\mathbb{Q}), G = \mathcal{O}_K, M = \mathcal{O}_K^2$ and $m \equiv 1 \bmod 4$. Let $\gamma = [\alpha\omega]$ be a cohomology class in $H^1(\mathfrak{g}, G)$. Then:

$$i_\gamma = 1, \text{ for any choice of } \gamma.$$

Theorem 4.2.2 Let $\mathfrak{g} = \mathrm{Gal}(\mathbb{Q}(\sqrt{m})/\mathbb{Q}), G = \mathcal{O}_K, M = \mathcal{O}_K^2$ and $m \equiv 2, 3 \bmod 4$. Let $\gamma = [\alpha\omega]$ be a cohomology class in $H^1(\mathfrak{g}, G)$. Then:

$$i_\gamma = \begin{cases} 4 & \text{if } \alpha \equiv 0 \bmod 2, \\ 2 & \text{if } \alpha \equiv 1 \bmod 2, m \equiv 2 \bmod 4, \\ 1 & \text{if } \alpha \equiv 1 \bmod 2, m \equiv 3 \bmod 4. \end{cases}$$

In Chapter 5, we concentrate on the second case, namely where $\mathfrak{g}$ is the Galois group of a finite Galois extension K over $\mathbb{Q}$, G is the Heisenberg group $H_3(\mathcal{O}_K)$, and $M = \mathcal{O}_K^3$. We wish to have a theorem analogous to that of Theorem 3.2.2 in this case. However the module Ξ_K is not enough for us to determine the form of all cocycles, so we introduce Θ_η:

$$\Theta_\eta := \{(\xi, \theta) \in \Xi_K \times \mathcal{O}_K : \theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)^\sigma\eta \bmod n^2, \forall\sigma \in \mathfrak{g}\}.$$

We are able to show that every cocycle $C \in Z^1(\mathfrak{g}, H_3(\mathcal{O}_K))$ can be determined by elements of Ξ_K and Θ_η, and this enables us to prove an analog to Theorem 3.2.2:

Theorem 5.6.1 Let $K/\mathbb{Q}$ be a finite Galois extension of degree n, and let $\mathfrak{g}$ be its Galois group. Let $G = H_3(\mathcal{O}_K)$, and $M = \mathcal{O}_K^3$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we have $i_\gamma(\mathfrak{g}, M) = |\widehat{H}^0(\mathfrak{g}, A(\xi, \eta, \theta)^{-1}M)|$.

In Chapter 6, we study this case in more detail, determining our index i_γ in the case of quadratic extensions, and we conclude with a nice theorem that determines i_γ any time we have $\Xi_K = \mathbb{Z} + n\mathcal{O}_K$.

Theorem 6.1.1 Let $K = \mathbb{Q}(\sqrt{m}), m \equiv 3 \bmod 4$. Let $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q}), G = H_3(\mathcal{O}_K), M = \mathcal{O}_K^3$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we have the following explicit determination of i_γ :

$$i_\gamma = \begin{cases} 1 & \text{if } \eta \equiv 1 + \omega \bmod 2\mathcal{O}_K, \theta \equiv \omega \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 0 \bmod 2\mathcal{O}_K, \theta \equiv \xi \equiv 0 \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 1 \bmod 2\mathcal{O}_K, \theta \equiv \xi \bmod 2\mathbb{Z}\omega, \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 6.2.1 Let $K = \mathbb{Q}(\sqrt{m}), m \equiv 2 \bmod 4$. Let $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q}), G = H_3(\mathcal{O}_K), M = \mathcal{O}_K^3$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we have the following explicit determination of i_γ :

$$i_\gamma = \begin{cases} 2 & \text{if } \eta \equiv \omega \bmod 2\mathcal{O}_K, \xi \equiv \omega \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 0 \bmod 2\mathcal{O}_K, \theta \equiv \xi \equiv 0 \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 1 \bmod 2\mathcal{O}_K, \theta \equiv \xi \bmod 2\mathbb{Z}\omega, \\ 4 & \text{otherwise.} \end{cases}$$

Theorem 6.3.1 Let $(\mathfrak{g}, (G, M))$ be a system as above. Then, if $\Xi_K = \mathbb{Z} + n\mathcal{O}_K$, we have:

$$|M_C/P_C| = |\mathbb{Z}/T(\mathcal{O}_K)|^3.$$

Corollary 6.3.2 Let $K = \mathbb{Q}(\sqrt{m}), m \equiv 1 \text{ mod } 4$. Let $\mathfrak{g} = \text{Gal}(K/\mathbb{Q}), G = H_3(\mathcal{O}_K), M = \mathcal{O}_K^3$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we have the following determination of i_γ :

$$i_\gamma = 1 \text{ for all choices of } (\eta, \xi, \theta).$$

Chapter 2

Background

2.1 Cohomology

The concepts of the cohomology of finite groups are covered extensively in [1], [10], [16], and [17]. We refer to these sources for our definitions and properties throughout this section.

2.1.1 Definition of a Group Cohomology

Let G be a group, written multiplicatively, and let A be a G-module. We let G act on A from the left, namely $(g, a) \to {}^g a$ for $g \in G, a \in A$. By the definition of module, we know that, for $g, h \in G$ and $a, b \in A$:

$$
\begin{gathered}
{}^{1_G} a = a, \\
{}^g(a+b) = {}^g a + {}^g b,
\end{gathered}
$$

and

$$
{}^{gh} a = {}^g({}^h a).
$$

Finally, let A^G be the submodule of A that contains all elements fixed by G, i.e.

$$A^G = \{a \in A : {}^g a = a, \forall g \in G\}. \tag{2.1}$$

Definition 2.1.1. Let $n \in \mathbb{Z}, n \geqslant 0$. By $C^n(G, A)$, we mean the set of all maps of $G^n \to A$, which we shall call *the set of n-cochains* of G with coefficients in A. In the case where $n = 0$, we let $G^0 = 1_G$, and we use the convention of defining the set of 0-cochains to be A; that is, $C^0(G, A) = A$.

We now define the coboundary map d_{n+1}:

$$d_{n+1} : C^n(G, A) \to C^{n+1}(G, A)$$

where

$$\begin{aligned}(d_{n+1}f)(g_1, g_2, \ldots, g_{n+1}) = & {}^{g_1} f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n).\end{aligned}$$

It is easily verified that $d_{n+1} \circ d_n = 0$, and im $d_n \subset \ker d_{n+1}$, and this gives us the long exact sequence:

$$0 \to C^0(G, A) \to C^1(G, A) \to C^2(G, A) \to \cdots .$$

Definition 2.1.2. The *n-th cohomology group* is defined to be

$$H^n(G, A) = Z^n(G, A)/B^n(G, A), \tag{2.2}$$

where $Z^n(G, A) = \ker d_{n+1}$ is known as the *group of cocycles* and $B^n(G, A) = \text{im } d_n$ is called the *group of coboundaries.*

Remark 2.1.3. (a) $H^0(G, A) = A^G$.

(b) $H^1(G, A)$ is the group of equivalent classes of crossed-homomorphisms of G into A:

$$Z^1(G, A) = \{f : G \to A : f(st) = f(s) + {}^s f(t)\}. \tag{2.3}$$

$$B^1(G, A) = \{g : G \to A : g(s) = {}^s a - a, a \in A\}. \tag{2.4}$$

(c) $H^2(G, A)$ is obtained by:

$$Z^2(G, A) = \{f : G \times G \to A : f(st, u) + f(s, t) = f(s, tu) + {}^s f(t, u)\} \tag{2.5}$$

and

$$B^2(G, A) = \{g : G \times G \to A : g(s, t) = h(s) + {}^s h(t) - h(st), \text{ where } h : G \to A\}. \tag{2.6}$$

2.1.2 Nonabelian Cohomology

We again let G be a group, but we now allow for the case where A is a nonabelian group, on which G acts from the left. In the nonabelian case, we can only talk about $H^0(G, A)$ and $H^1(G, A)$. We again set $H^0(G, A) = A^G$, the G-fixed elements of A. We express a cocycle as a map from G to A defined as $s \mapsto C_s$, with the condition that $C_{st} = C_s {}^s C_t$. As before, we let $Z^1(G, A)$ be the set of all cocycles.

Definition 2.1.4. Let $C, C' \in Z^1(G, A)$. We say that C and C' are *cohomologous*, written $C \sim C'$, if there exists a $U \in A$ such that, for all $s \in G$, we have the relation:

$$C'_s = U^{-1} C_s {}^s U. \tag{2.7}$$

It is clear that this is an equivalence relation on the set of cocycles. Therefore, we can define the *cohomology set of G with values in A* as the quotient set:

$$H^1(G, A) = Z^1(G, A)/\sim . \tag{2.8}$$

Remark 2.1.5. (a) We can easily see that if A is abelian, this definition is analogous with the first cohomology group defined in Remark 2.1.3.
(b) Note that when A is not abelian, $Z^1(G, A)$ and $H^1(G, A)$ do not have a natural group structure. However, there is a distinguished element, which is the class of the unit cocycle $C_s = 1$, and we consider $H^1(G, A)$ as a pointed set.

If $f : A \to B$ is a G-group homomorphism, we then define

$$f_0 : H^0(G, A) \to H^0(G, B)$$

and

$$f_1 : H^1(G, A) \to H^1(G, B)$$

in the following way: f_0 is the restriction of f to A^G, therefore the image of f_0 is the set of fixed elements. We define f_1 by $f_1(C)_s = f(C_s)$. This is compatible with the equivalence relation. We see that f_0 is a group homomorphism, and f_1 is a morphism of pointed sets, which means f sends the unit cocycle of A to the unit cocycle of B.

We define the *kernel* of a morphism of pointed sets as the pre-image of the distinguished element. Now, we are able to consider exact sequences of pointed sets. We can also have a long exact sequence for the cohomology sets, but in general we cannot extend the sequence to $H^2(G, A)$.

Proposition 2.1.6. *Let* $1 \to L \to M \to N \to 1$ *be an exact sequence of nonabelian* G*-groups. Then, the following sequence of pointed sets is also exact:*

$$\begin{aligned} 1 \to H^0(G, L) \to H^0(G, M) \to H^0(G, N) \\ \to H^1(G, L) \to H^1(G, M) \to H^1(G, N). \end{aligned} \tag{2.9}$$

The proof can be found in [17].

2.1.3 Twisting

Definition 2.1.7. Let A be a G-group. We define a new action of G on A twisted by C, a 1-cocycle, in the following way. Let $C \in Z^1(G, A)$. By ${}_CA$, we mean the set A on which G acts by the formula:

$$ {}^{s'}a := C_s {}^s a C_s^{-1}. \tag{2.10}$$

We say that ${}_CA$ is obtained by *twisting* A using the cocycle C.

Proposition 2.1.8. *Let* $C \in Z^1(G, A)$ *and* $A' = {}_CA$*. To every cocycle* D_s *in* A' *associate a cocycle of* G *in* A*,* D_sC_s*. Then, we have the following bijections:*

$$t_C : Z^1(G, A') \to Z^1(G, A)$$

and

$$\tau_C : H^1(G, A') \to H^1(G, A)$$

induced by t_C*, mapping the neutral element of* $H^1(G, A')$ *into the class of* C*.*

For a proof, see [16].

2.2 Poincaré Series

The definitions and theorems in this section are from [4].

2.2.1 Modular Forms

For a thorough study of modular forms, or for a introduction to linear fractional transformations, see [5] or [3], respectively.

Let $\mathfrak{H}$ denote the upper half plane, i.e. $\mathfrak{H} = \{z \in \mathbb{C} : Imz > 0\}$. We know from complex analysis that the only conformal automorphisms of $\mathfrak{H}$ are the linear fractional transformations:

$$T : z \mapsto \frac{az+b}{cz+d}, \tag{2.11}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, i.e., a matrix of real coefficients with determinant equal to one. We now define the *inhomogeneous modular group* Γ to be the group consisting of linear fractional transformations associated to integral matrices. We then get that Γ is isomorphic to $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I_2$.

Definition 2.2.1. Let G be a subgroup of Γ with finite index. We say that a transformation T is *parabolic* if it has only one fixed point on the real line or at infinity. By fixed point, we mean a point p that satisfies $T(p) = p$. A fixed point of a parabolic transformation in G is called a *cusp* of G.

Definition 2.2.2. Let k be an integer. An *unrestricted modular form of weight 2k for G* is a meromorphic function $f(z)$ on $\mathfrak{H}$ such that, for all transformations T in the form of equation (2.11) belonging to G, we have:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z). \tag{2.12}$$

Let $J_T(a) := \frac{dT}{dz} = (cz+d)^{-2}$. Then, equation (2.12) can be expressed as $f(T(z)) = J_T(z)^{-k} f(z)$. The local coordinate at $i\infty$ is $\zeta = e^{2\pi iz/q}$, where q is the smallest positive integer such that the translation $z \mapsto z+q$ is in G.

Definition 2.2.3. Let $\widehat{f}(\zeta) = f(z)$. An unrestricted modular form $f(z)$ is called *holomorphic at* ∞ if $\widehat{f}(\zeta)$ is holomorphic for $|\zeta| < 1$.

This condition implies that we have a Taylor expansion in ζ for $\widehat{f}(\zeta)$:

$$\widehat{f}(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n,$$

which, in turn, induces a Fourier expansion for $f(z)$:

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi izn/q}.$$

Definition 2.2.4. Let p be a parabolic fixed point of G, $p \neq \infty$. Also, let $S \in \Gamma$ be a transformation that maps p to ∞, and $g(z) = J_{S^{-1}}(z)^k f(S^{-1}z)$. We say that $f(z)$ is *holomorphic at* p if $g(z)$ is holomorphic at ∞.

Definition 2.2.5. A *modular form* is an unrestricted modular form that is holomorphic at all point in $\mathfrak{H}$ and at all cusps of the group.

Definition 2.2.6. A *cusp form of weight 2k for G* is an modular form of weight $2k$ that vanishes all cusps of the group.

Definition 2.2.7. The *Poincaré series of weight 2k and of character* ν *for* G is series:

$$\phi_\nu(z) = \sum_{T\in\mathcal{R}} J_T(z)^k e^{2\pi i\nu T(z)/q}$$

where ν is a nonnegative integer, G_0 is the infinite cyclic subgroup of translation in G, generated by the least translation $T : z \mapsto z + q$ in G, and $\mathcal{R}$ is the set of coset representatives of G mod G_0.

We now have the following two theorems. We will use these results in the next section to develop the cohomology theory which will inspire our work for the remainder of the paper.

Theorem 2.2.8. *The Poincaré series*

$$\phi_\nu(z) = \sum_{T\in\mathcal{R}} e^{2\pi i\nu T(z)/q}(cz + d)^{-2k}$$

converges absolutely uniformly on compact subsets $\mathfrak{H}$ *for* $\nu > 0$ *and* $k \geqslant 1$, *and for* $\nu = 0$, *and* $k > 1$. *Also,* $\phi_\nu(z)$ *converges absolutely uniformly on every fundamental domain* D *in* G *and represents a modular form of weight* $2k$ *for* G.
In addition:
(a) $\phi_0(z)$ *is zero at all finite, parabolic vertices, and nonzero at* $i\infty$.
(b) $\phi_\nu(z)$ *is a cusp form for* $\nu \geqslant 1$.

Theorem 2.2.9. *Every cusp form is a linear combination of the Poincaré series* $\phi_\nu(z), \nu \geqslant 1$.

2.3 The System $(\mathfrak{g}, (G, M))$

2.3.1 The System $(\mathfrak{g}, (G, M))$

The definitions and theorems in this section are from [15].

Let G be a group, and let M be a left G-module. We now introduce a finite group $\mathfrak{g}$ that acts naturally on the pair (G, M). That is to say, G is a $\mathfrak{g}$-group and M is a $\mathfrak{g}$-module such that, for $\sigma \in \mathfrak{g}, s \in G, x \in M$, we have the relation ${}^{\sigma}(sx) = {}^{\sigma}s{}^{\sigma}x$.

We now consider a cocycle C of $\mathfrak{g}$ in G, i.e. $C \in Z^1(\mathfrak{g}, G)$. By the definition of $Z^1(\mathfrak{g}, G)$, C is a map from $\mathfrak{g}$ to G that satisfies the relation $C(\sigma\tau) = C(\sigma){}^{\sigma}C(\tau)$ for all $\sigma, \tau \in \mathfrak{g}$. For a cocycle C, we associate M_C, a subgroup of M, by:

$$M_C := \{x \in M : C(\sigma){}^{\sigma}x = x, \forall \sigma \in \mathfrak{g}\}. \tag{2.13}$$

Let $p_C(x) = \sum_{\tau \in \mathfrak{g}} C(\tau){}^{\tau}x, x \in M$. Notice that this sum is convergent since $\mathfrak{g}$ is finite. We then define:

$$P_C = \{y : y = p_C(x), x \in M\}. \tag{2.14}$$

Remark 2.3.1. (a) $P_C \subseteq M_C$.

We can see this fact clearly, for if $y \in P_C$, $\sigma \in \mathfrak{g}$:

$$\begin{aligned} C(\sigma){}^{\sigma}y &= C(\sigma){}^{\sigma}p_C(x) \\ &= C(\sigma)^{\sigma}\left(\sum_{\tau \in \mathfrak{g}} C(\tau){}^{\tau}x\right) \\ &= C(\sigma)\sum_{\tau \in \mathfrak{g}} {}^{\sigma}C(\tau){}^{\sigma\tau}x \\ &= \sum_{\tau \in \mathfrak{g}} C(\sigma){}^{\sigma}C(\tau){}^{\sigma\tau}x. \end{aligned}$$

Here, we use the identity $C(\sigma)^{\sigma}C(\tau) = C(\sigma\tau)$ and the fact that the sum over τ is equivalent to the sum over $u = \sigma\tau$ to get:

$$C(\sigma)^{\sigma}y = \sum_{u \in \mathfrak{g}} C(u)^{u}x = y.$$

(b) $|\mathfrak{g}|M_C \subseteq P_C \subseteq M_C$. The first inclusion comes from the fact that, for any $x \in M$ and for any integer j, ${}^{\sigma}(jx) = j^{\sigma}x$. Then, if we let $|\mathfrak{g}| = n$, and $x \in M_C$, we see:

$$\sum_{\tau \in \mathfrak{g}} C(\tau)^{\tau}x = \sum_{\tau \in \mathfrak{g}} x = nx.$$

(c) Recall that two cocycles C, C' are cohomologous, i.e. $C \sim C'$, if there exists a $U \in G$ such that $C'(\sigma) = U^{-1}C(\sigma)^{\sigma}U$ for all $\sigma \in \mathfrak{g}$. A simple calculation shows that:

$$M_{C'} = U^{-1}M_C, \qquad P_{C'} = U^{-1}P_C,$$

From this, we get that $M_C/P_C \cong M_{C'}/P_{C'}$. This implies that the structure of M_C/P_C depends solely on the cohomology class $\gamma = [C]$ in the cohomology set $H^1(\mathfrak{g}, G) = Z^1(\mathfrak{g}, G)/\sim$.

Definition 2.3.2. Let $Tx := \sum_{\tau \in \mathfrak{g}} {}^{\tau}x$ denote the *trace* of an element x. Also, we let $T(M) = \{Tx : x \in M\}$.

Now, setting $C = 1$ in our definitions of M_C and P_C, we obtain:

$$M_1 = M^{\mathfrak{g}}, \qquad P_1 = T(M),$$

which gives us:

$$M_1/P_1 = \widehat{H}^0(\mathfrak{g}, M). \tag{2.15}$$

For an arbitrary $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we identify $M_C/P_C = \widehat{H}^0(\mathfrak{g}, M)_\gamma$, known as *the Tate group twisted by* γ.

Finally, we define:

$$i_\gamma(\mathfrak{g}, M) = [M_C : P_C], \text{ where } \gamma = [C] \in H^1(\mathfrak{g}, G). \tag{2.16}$$

It is this determination of $i_\gamma(\mathfrak{g}, M)$, inspired by Poincaré, that will be our primary focus in the continuation.

2.3.2 Cohomological View

We are able to look at Theorem 2.2.9 in terms of our system $(\mathfrak{g}, (G, M))$ using group cohomology. Let R be the ring of holomorphic functions on $\mathfrak{H}$. Let $\mathfrak{g}$ be a subgroup of finite index in $PSL_2(\mathbb{Z})$, let G be $R^\times$, the group of invertible elements in R, and let M be the subring of all the elements in R that vanish at all cusps. For the sake of clarity, we let $C_s := C(s)$.

Let $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \mathfrak{g}$. Then, if we take f as an element in either G or M, $\mathfrak{g}$ has a natural action on both G and M:

$$(s, f(z)) \mapsto {}^s f(z) := f\left(\frac{az+b}{cz+d}\right). \tag{2.17}$$

We define $C : \mathfrak{g} \to G$ by

$$C_s(z) = (cz+d)^{-2k} \text{ for } z \in \mathfrak{H}. \tag{2.18}$$

We see that C is a 1-cocycle in $Z^1(\mathfrak{g}, G)$:

Let $t = \begin{pmatrix} e & f \\ g & h \end{pmatrix}^{-1} \in \mathfrak{g}$.

Then: ${C_s}^s C_t = [(c + dz)^{-2k}]^s[(gz + h)^{-2k}]$. We now apply the action of s to get:

$$\begin{aligned} C_s(z)^s C_t(z) &= (c + dz)^{-2k}[(g\left(\frac{az + b}{cz + d}\right) + h)]^{-2k} \\ &= [g(az + b) + h(cz + d)]^{-2k} \\ &= [(ga + ch)z + (bg + dh)]^{-2k} \\ &= C_{st}(z), \end{aligned}$$

where the last equality is due to the fact that

$$\begin{aligned} (st)^{-1} = t^{-1}s^{-1} &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix}. \end{aligned}$$

Next, define M_C as the space of cusp forms of weight $2k$:

$$M_C = \left\{ f \in R : (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right) = f(z), f \text{ vanishes at all cusps} \right\}.$$

We see that the condition on elements of M_C, namely $(cz + d)^{-2k} f(\frac{az+b}{cz+d}) = f(z)$, is equivalent to ${C_s}^s f(z) = f(z)$. We also define P_C as the set of all Poincaré series:

$$P_C = \left\{ \phi_\nu : \phi_\nu = \sum_{T \in \mathcal{R}} (cz + d)^{-2k} e^{2\pi i \nu T(z)/q} \right\}.$$

Again, expressing the condition in terms of C_s, we see:

$$\phi_\nu = \sum_{s \in \mathcal{R}} C_s(z)^s g(z),$$

where $g(z) = e^{2\pi i \nu z/q}, \nu \geqslant 1$. By Theorem 2.2.9, we see that $M_C = P_C$, which makes the quotient group trivial, i.e. $M_C/P_C = 0$.

It is the index of this M_C/P_C that we wish to study, as in general, we do not have $M_C/P_C = 0$.

Remark 2.3.3. The determination of the order of M_C/P_C in the case where G is the Galois group of a Galois field extension K/k and where R is the ring of integers $O_K^\times$ viewed as a group with multiplicative structure was studied by Souk-Min Lee in [7] and by E.K. Lee in [6]. In the next chapter, we will concentrate instead on the ring of integers viewed additively.

Chapter 3

$\mathfrak{g} = \textbf{Gal}(K/\mathbb{Q})$, $G = \mathcal{O}_K$, $M = (\mathcal{O}_K)^2$

We now narrow our focus onto a system $(\mathfrak{g}, (G, M))$ with particular interest to us, namely where $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$ for some Galois extension K, $G = \mathcal{O}_K$, viewed as an additive group, and $M = (\mathcal{O}_K)^2$. In order to study this group, we will have to introduce an important module.

3.1 The Module Ξ_K

The results and theorems from this section can be found in [15].

Let $\mathbb{Q}$ denote the rational numbers. We let K be a finite, Galois extension over $\mathbb{Q}$, $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$, and $|\mathfrak{g}| = n$. We take $\mathcal{O}_K$ to be the ring of algebraic integers, i.e. the ring of all elements $x \in K$ such that x is the root of a monic polynomial with coefficients in $\mathbb{Z}$. We now introduce the following module.

$$\Xi_K := \{\xi \in \mathcal{O}_K : \xi \equiv {}^{\sigma}\xi \bmod n, \forall \sigma \in \mathfrak{g}\}. \tag{3.1}$$

We see that Ξ_K is a $\mathbb{Z}-$module in $\mathcal{O}_K$ and is $\mathfrak{g}$-stable. It is trivial to show that Ξ_K

contains both $\mathbb{Z}$ and $n\mathcal{O}_K$. We have the relation:

$$\Xi_k/n\mathcal{O}_K = (\mathcal{O}_K/n\mathcal{O}_K)^{\mathfrak{g}}, \tag{3.2}$$

where, as before, $M^{\mathfrak{g}}$ denotes the elements of M fixed by all $\sigma \in \mathfrak{g}$.

Let $\xi \in \Xi_K$ and $\sigma \in \mathfrak{g}$. We consider an element $t(\xi)_\sigma$ defined as follows:

$$t(\xi)_\sigma = \frac{1}{n}(\xi - {}^\sigma\xi). \tag{3.3}$$

We will write $t_\sigma = t(\xi)_\sigma$, when the context is clear. Since $\xi \in \Xi_K$, we know that $\xi - {}^\sigma\xi \equiv 0 \mod n$ for all $\sigma \in \mathfrak{g}$, which means $\xi - {}^\sigma\xi \in n\mathcal{O}_K$, i.e. $\frac{1}{n}(\xi - {}^\sigma\xi) \in \mathcal{O}_K$.

Since $\mathfrak{g}$ acts on the additive group of the ring of integers $\mathcal{O}_K^+$, we can study the first cohomology group $H^1(\mathfrak{g}, \mathcal{O}_K)$. We see that under our current setup, t_σ is an additive cocycle in $Z^1(\mathfrak{g}, \mathcal{O}_K)$:

$$t_\sigma + {}^\sigma t_\tau = \frac{1}{n}(\xi - {}^\sigma\xi) + {}^\sigma\Big(\frac{1}{n}(\xi - {}^\tau\xi)\Big) = \frac{1}{n}(\xi - {}^\sigma\xi + {}^\sigma\xi - {}^{\sigma\tau}\xi) = t_{\sigma\tau}. \tag{3.4}$$

We now state the following propositions and theorem. Proofs can be found in [15].

Proposition 3.1.1. *Let $t : \Xi_K \to Z^1(\mathfrak{g}, \mathcal{O}_K)$, defined by (3.3). Through the map t we have the isomorphism:*

$$\Xi_K/\mathbb{Z} \cong Z^1(\mathfrak{g}, \mathcal{O}_K). \tag{3.5}$$

Proposition 3.1.2. *Let $K/\mathbb{Q}$ be a finite Galois extension of degree n, let $\mathfrak{g} =$Gal$(K/\mathbb{Q})$, and let Ξ_K be the module defined in (3.1). We then have:*

$$H^1(\mathfrak{g}, \mathcal{O}_K) \cong \Xi_K/(\mathbb{Z} + n\mathcal{O}_K). \tag{3.6}$$

Theorem 3.1.3. *Let $K/\mathbb{Q}$ be a fine Galois extension of degree n, and let $\mathfrak{g} =$ Gal $(K/\mathbb{Q})$. Then:*

$$|H^1(\mathfrak{g}, \mathcal{O}_K)| = |(\mathcal{O}_K/n\mathcal{O}_K)^{\mathfrak{g}}|/n. \tag{3.7}$$

Remark 3.1.4. Note that Proposition 3.1.1 tells us that we need only consider $\xi \in \Xi_K/\mathbb{Z}$, i.e. we can choose ξ with no integral part. This will be helpful in our calculations later.

3.2 A Result Due to Ono

We now turn our attention to the case that we will concern ourselves with for the remainder of this chapter.

Consider a finite, Galois field extension K over $\mathbb{Q}$, where $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$ and $|\mathfrak{g}| = n$. Our goal is to deal with the case where $G = \mathcal{O}_K$, viewed additively. To do this we begin by letting:

$$M = \mathcal{O}_K^2 = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_i \in \mathcal{O}_K \right\}.$$

We know that $\mathfrak{g}$ acts on $\mathcal{O}_K$ and $\mathcal{O}_K^2$ in a natural way. We will now define an appropriate action for $\mathcal{O}_K$ on $\mathcal{O}_K^2$.

Let $t \in \mathcal{O}_K$. We can express this t as:

$$T_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

We then define the action of $\mathcal{O}_K$ on $\mathcal{O}_K^2$ as follows. Let $t \in \mathcal{O}_K$ and $x \in \mathcal{O}_K^2$. Then,

$$t \circ \mathbf{x} := T_t\mathbf{x} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + tx_2 \\ x_2 \end{pmatrix}. \tag{3.8}$$

It is simple to verify that ${}^{\sigma}(t \circ \mathbf{x}) = {}^{\sigma}t \circ {}^{\sigma}\mathbf{x}$. Therefore, we have an admissible system $(\mathfrak{g}, (\mathcal{O}_K, \mathcal{O}_K^2))$.

The benefit of the action from (3.8) is we have given $\mathcal{O}_K^+$, which is an additive group, a multiplicative structure. We now see that:

$$G = \mathcal{O}_K \cong \left\{ T_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in GL_2(\mathcal{O}_K) \right\}. \tag{3.9}$$

Call the isomorphism ϕ, so that $\phi(t) = T_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then clearly we see that under this isomorphism addition becomes multiplication:

$$\phi(t+s) = T_{t+s} = \begin{pmatrix} 1 & t+s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = T_t T_s.$$

This also means that the cocycle relation goes from additive to multiplicative:

$$t_{\sigma\tau} = t_\sigma + {}^{\sigma}t_\tau \longleftrightarrow C_{\sigma\tau} = C_\sigma {}^{\sigma}C_\tau,$$

where

$$C_\sigma = \begin{pmatrix} 1 & t_\sigma \\ 0 & 1 \end{pmatrix}.$$

Recall from (3.3) that $t(\xi)_\sigma = \frac{1}{n}(\xi - {}^{\sigma}\xi)$. In our C_σ notation, this becomes:

$$C_\sigma = \begin{pmatrix} 1 & t_\sigma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{n}(\xi - {}^\sigma\xi) \\ 0 & 1 \end{pmatrix} = A(\xi)^\sigma A(\xi)^{-1},$$

where

$$A(\xi) = \begin{pmatrix} 1 & \frac{\xi}{n} \\ 0 & 1 \end{pmatrix} \in GL_2(K).$$

Remark 3.2.1. Note that $n\mathcal{O}_K$ is contained in Ξ_K, but not equal. That is to say, in the case where $\mathbb{Z} + n\mathcal{O}_K \subsetneq \Xi_K$, $\xi \in \Xi_K/\mathbb{Z}$ is not necessarily an element of $n\mathcal{O}_K$. This means that $\frac{\xi}{n} \notin \mathcal{O}_K$, which in turn implies that the matrix $A(\xi)$ from above is in $GL_2(K)$, and not, de facto, in $GL_2(\mathcal{O}_K)$.

When we have the latter condition will depend on our choice of ξ. In the cases where $\mathbb{Z} + n\mathcal{O}_K = \Xi_K$, i.e., where for any choice of $\xi \in \Xi_K/\mathbb{Z}$ we can write $\xi = nx, x \in \mathcal{O}_K$, we obtain nice results, as we shall see.

Recall that $M = \mathcal{O}_K^2$. We express M_C and P_C in these terms:

(3.10) $$M_C = \left\{\mathbf{x} \in M : C_\sigma{}^\sigma\mathbf{x} = \mathbf{x}, \forall\sigma \in \mathfrak{g}\right\}.$$

(3.11) $$P_C = \left\{\mathbf{y} = p_C(\mathbf{x}) = \sum_{\tau\in\mathfrak{g}} C_\tau{}^\tau\mathbf{x}, \mathbf{x} \in M\right\}.$$

Now, we state the theorem.

Theorem 3.2.2. ***(T. Ono)*** *Let $K/\mathbb{Q}$ be a finite Galois extension of degree n, and let $\mathfrak{g}$ be its Galois group. Let $G = \mathcal{O}_K$, and $M = \mathcal{O}_K^2$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, corresponding to element $\xi \in \Xi_K$ by equation (3.5) from Proposition 3.1.1, we have $i_\gamma(\mathfrak{g}, M) = |\widehat{H}^0(\mathfrak{g}, A(\xi)^{-1}M)|$.*

For the proof, see [15].

Remark 3.2.3. Here, we note that $\widehat{H}^0(G, A) = M^G/T(M)$, where M^G is the group of all elements in M that are fixed by all elements of G, and $T(M)$ is the group formed by all sums $\sum_{t\in G} {}^t x$, for any $x \in M$. Therefore, Theorem 3.2.2 states that:

$$i_\gamma(\mathfrak{g}, M) = |\widehat{H}^0(\mathfrak{g}, A(\xi)^{-1}M)| = |(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)|.$$

Chapter 4

The Determination of $(A^{-1}M)^{\mathfrak{g}}$ and the Quadratic Case

In this chapter, we tackle the problem of determining i_γ when $K = \mathbb{Q}(\sqrt{m})$, for m a square-free integer. In order to facilitate this process, we will study $(A^{-1}M)^{\mathfrak{g}}$, finding a nice determination, since, as Theorem 3.2.2 and Remark 3.2.3 state, this is equivalent to M_C and critical to determining our i_γ.

4.1 An Explicit Determination of $(A^{-1}M)^{\mathfrak{g}}$

We start by computing $A^{-1}M$, and then we shall look at its fixed group.

Recall:

$$A^{-1}M = \left\{ \begin{pmatrix} 1 & -\frac{\xi}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 - \frac{\xi x_2}{n} \\ x_2 \end{pmatrix} \right\}.$$

Our goal, therefore, is to determine the conditions under which the following system of equations is satisfied:

$$x_1 - \frac{\xi x_2}{n} \in \mathbb{Q}. \tag{4.1}$$

$$x_2 \in \mathbb{Q}. \tag{4.2}$$

We notice first that for equation (4.2), we know that $x_2 \in \mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}$. Now, we turn our attention to our first equation. We know that we can express $\mathcal{O}_K$ as an n-dimensional vector field over $\mathbb{Z}$, with one of our basis elements 1, i.e. $\mathcal{O}_K = \langle 1, \omega_2, \ldots, \omega_n \rangle_{\mathbb{Z}}$. Now, since $x_1, \xi \in \mathcal{O}_K$, and since we know the integer part of $\xi = 0$, we can express them as follows:

$$x_1 = a_1 + a_2\omega_2 + \cdots + a_n\omega_n, a_i \in \mathbb{Z} \tag{4.3}$$

$$\xi = 0 + \alpha_2\omega_2 + \cdots + \alpha_n\omega_n, \alpha_i \in \mathbb{Z}. \tag{4.4}$$

Now, we look at (4.1):

$$\begin{aligned} x_1 - \frac{\xi x_2}{n} =& a_1 + a_2\omega_2 + \cdots + a_n\omega_n \\ & - \frac{x_2\alpha_2\omega_2}{n} - \cdots - \frac{x_2\alpha_n\omega_n}{n} \\ =& a_1 + (a_2 - \frac{x_2\alpha_2}{n})\omega_2 + \cdots + (a_n - \frac{x_2\alpha_n}{n})\omega_n. \end{aligned} \tag{4.5}$$

Since we know this term must be a rational number and $a_i \in \mathbb{Z}$, this implies that (4.1) must, in fact, be an integer. This means that the ω_i terms all must cancel out.

That is:

$$a_i = \frac{x_2\alpha_i}{n}, 2 \leqslant i \leqslant n. \tag{4.6}$$

Since $a_i \in \mathbb{Z}$ and we need to satisfy (4.6), this implies that $\frac{x_2\alpha_i}{n} \in \mathbb{Z}$ for every $2 \leqslant i \leqslant n$, i.e. $n \Big| x_2\alpha_i$. Let $d = \text{g.c.d.}(\alpha_2, \alpha_3, \ldots, \alpha_n, n)$, i.e. the largest integer that divides all the coefficients of ξ and n.

Now, let $n = p_1^{r_1} \cdots p_s^{r_s}$ be the prime factorization of n, where all p_i are prime and all $r_i \geqslant 1$. Since $d|n$, we can write $d = p_1^{k_1} \cdots p_s^{k_s}$, where $0 \leqslant k_i \leqslant r_i$. Consider $p_1^{k_1}$. This is the largest power of p_1 that divides every α_i. This means, for at least one α_j, $p_1^{k_1} \Big| \alpha_j$ but $p_1^{k_1+1} \not\Big| \alpha_j$. Therefore, in order for $x_2\alpha_j$ to be divisible by n, $p_1^{r_1-k_1}$ must divide x_2. Similar logic applies to all the p_i's. Let $d' = p_1^{r_1-k_1} \cdots p_s^{r_s-k_s}$. Then, this implies that d' must divide x_2. Therefore, $x_2 \in d'\mathbb{Z}$. Finally, we see that $n = d \cdot d'$. Combining these results, we have explicitly determined $(A^{-1}M)^{\mathfrak{g}}$, and proven the following theorem.

Theorem 4.1.1. *Let K be a finite Galois extension over $\mathbb{Q}$ of degree n. Consider the system $(\mathfrak{g}, (G, M))$, where $\mathfrak{g} = Gal(K, \mathbb{Q})$, $G \cong \mathcal{O}_K$, and $M = (\mathcal{O}_K)^2$. Let $\xi \in \Xi_K/\mathbb{Z} \subseteq \mathcal{O}_K/\mathbb{Z}$, with $\xi = \alpha_2\omega_2 + \cdots + \alpha_n\omega_n$. Let $d = gcd(\alpha_2, \ldots, \alpha_n, n)$, and let d' be the number such that $n = d \cdot d'$. Then,*

$$(A^{-1}M)^{\mathfrak{g}} = \begin{pmatrix} \mathbb{Z} \\ d'\mathbb{Z} \end{pmatrix}.$$

Remark 4.1.2. We will now look at the quadratic extension case, determining $i_\gamma(\mathfrak{g}, M)$ explicitly. We note that due to Proposition 3.1.1, we need only consider $\xi \in \Xi_K/\mathbb{Z}$; that is, we can choose ξ with no integer part. We shall see that this will not be true in the three-dimensional case.

4.2 The Quadratic Extension Case

We will now determine i_γ when $n = 2$.

Consider a degree 2 extension, $K = \mathbb{Q}(\sqrt{m})$ over $\mathbb{Q}$, m a squarefree integer. We can write $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$, where

$$\omega = \begin{cases} \frac{1+\sqrt{m}}{2} & \text{if } m \equiv 1 \bmod 4, \\ \sqrt{m} & \text{if } m \equiv 2, 3 \bmod 4. \end{cases}$$

Note here that $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$ is simply the group $\mathfrak{g} = \{1, \sigma\}$. We can therefore discuss how σ acts on the generator ω :

$$^{\sigma}\omega = \begin{cases} \frac{1-\sqrt{m}}{2} = 1 - \omega & \text{if } m \equiv 1 \bmod 4, \\ -\sqrt{m} = -\omega & \text{if } m \equiv 2, 3 \bmod 4. \end{cases}$$

Knowing this, we can determine i_γ.

We will look at the case $m \equiv 1 \bmod 4$ and the case $m \equiv 2, 3 \bmod 4$ separately.

4.2.1 $m \equiv 1$ mod 4

We first look at the case where $m \equiv 1 \bmod 4$. This makes $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$, where $\omega = \frac{1+\sqrt{m}}{2}$, and $^{\sigma}\omega = 1 - \omega$.

We begin by determining Ξ_K. Here, $n = 2$, so plugging into (3.1), we obtain:

$$\Xi_K = \{\xi \in \mathcal{O}_K : {}^{\sigma}\xi \equiv \xi \bmod 2\}$$

Let $\xi = \alpha + \beta\omega, \alpha, \beta \in \mathbb{Z}$. We see we must satisfy the equation $\alpha + \beta\omega \equiv \alpha + \beta(1 - \omega) \bmod 2$. This reduces to $\beta \equiv 0 \bmod 2$. We can then write $\xi = \alpha + 2\gamma\omega, \alpha, \gamma, \in \mathbb{Z}$, i.e.

$$\Xi_K = \mathbb{Z} + 2\mathbb{Z}\omega. \tag{4.7}$$

Recalling Proposition 3.1 and Remark 3.1.4, we can choose $\xi \in \Xi_K/\mathbb{Z}$. Let $\xi = 2\alpha\omega$ for $\alpha, \in \mathbb{Z}$. Now, we shall use Theorem 3.2.2 and 4.1.1 to determine i_γ. That is, we will determine $(A^{-1}M)^{\mathfrak{g}}$ and $T(A^{-1}M.)$ We first discover $(A^{-1}M)^{\mathfrak{g}}$.

According to Theorem 4.1.1, we need only concern ourselves with $\xi \in \Xi_K/\mathbb{Z}$. From above, we know that $\xi = 2\alpha\omega$. Therefore, $d = gcd(2\alpha, 2) = 2$. We obtain that $d' = 1$, and therefore by our theorem:

$$(A^{-1}M)^{\mathfrak{g}} = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}. \tag{4.8}$$

Turning our attention to $T(A^{-1}M)$, we can see that:

$$T(A^{-1}M) = \left\{ \begin{pmatrix} \sum_{\tau\in\mathfrak{g}} x_1 - \eta x_2 \\ \sum_{\tau\in\mathfrak{g}} x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} Tx_1 - T(\eta x_2) \\ T(x_2) \end{pmatrix} \right\}.$$

We note that, if $x = a + b\omega \in \mathcal{O}_K$, then,

$$Tx = a + b\omega + {}^{\sigma}(a + b\omega) = a + b\omega + a + b - b\omega = 2a + b.$$

This implies that

$$T(\mathcal{O}_K) = \mathbb{Z}. \tag{4.9}$$

Using (4.9), we simply get:

$$T(A^{-1}M) = \left\{ \mathbf{y} \in M : \mathbf{y} = \begin{pmatrix} T(x_1) - T(x_2\eta) \\ Tx_2 \end{pmatrix}, x_1, x_2 \in \mathcal{O}_K \right\} = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}. \tag{4.10}$$

Combining (4.8) and (4.10), we have proven:

Theorem 4.2.1. *Let $\mathfrak{g} = Gal(\mathbb{Q}(\sqrt{m})/\mathbb{Q}), G = \mathcal{O}_K, M = \mathcal{O}_K^2$ and $m \equiv 1$ mod 4. Let $\gamma = [\alpha\omega]$ be a cohomology class in $H^1(\mathfrak{g}, G)$. Then:*

$$i_\gamma = 1, \textit{ for any choice of } \gamma.$$

4.2.2 $m \equiv 2, 3$ mod 4

We now study the case where $m \equiv 2, 3$ mod 4. As we shall see, the parity of m will play a role in the determination of i_γ.

In this case, $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$, where $\omega = \sqrt{m}$, and ${}^\sigma\omega = -\omega$.

We again start by examining Ξ_K. As in the previous case, $n = 2$, which means equation (3.1) is the same as before:

$$\Xi_K = \{\xi \in \mathcal{O}_K : {}^\sigma\xi \equiv \xi \text{ mod } 2\}$$

However, since our action is different, our Ξ_K will be different. Let $\xi = \alpha + \beta\omega, \alpha, \beta \in \mathbb{Z}$. We need to satisfy the equation $\alpha + \beta\omega \equiv \alpha - \beta\omega$ mod 2. This identity is always true, so we have:

$$\Xi_K = \mathcal{O}_K. \tag{4.11}$$

Recalling Remark 3.1.4 again, we choose $\xi \in \Xi_K/\mathbb{Z}$. Let $\xi = \alpha\omega$ for $\alpha \in \mathbb{Z}$.

Again, we use Theorem 3.2.2 to determine i_γ. That is, we must determine $(A^{-1}M)^{\mathfrak{g}}$ and $T(A^{-1}M.)$

As in the previous case, we use Theorem 4.1.1. Since in this case $\xi = \alpha\omega$, we see that $d = g.c.d.(\alpha, 2)$ will depend on α. That is, $d = 2$ if α is even, and $d = 1$ if α is odd. Then, the opposite is true for d', i.e. $d' = 1$ if α is even, and $d' = 2$ if α is odd. This gives us the form of $(A^{-1}M)^{\mathfrak{g}}$:

$$(A^{-1}M)^{\mathfrak{g}} = \begin{cases} \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} & \text{if } \alpha \equiv 0 \bmod 2, \\ \\ \begin{pmatrix} \mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix} & \text{if } \alpha \equiv 1 \bmod 2. \end{cases} \tag{4.12}$$

Turning our attention to $T(A^{-1}M)$, we can see that:

$$T(A^{-1}M) = \left\{ \begin{pmatrix} \sum_{\tau \in \mathfrak{g}} x_1 - \frac{x_2\xi}{2} \\ \sum_{\tau \in \mathfrak{g}} x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} Tx_1 - \frac{1}{2}T(x_2\xi) \\ T(x_2) \end{pmatrix} \right\}.$$

We first note that, if $x = a + b\omega \in \mathcal{O}_K$, then,

$$Tx = a + b\omega + {}^{\sigma}(a + b\omega) = a + b\omega + a - b\omega = 2a.$$

This implies that

$$T(\mathcal{O}_K) = 2\mathbb{Z}. \tag{4.13}$$

Let $x_2 = a_2 + b_2\omega$, and recall $\xi = \alpha\omega$. We see then that $x_2\xi = (a_2 + b_2\omega)(\alpha\omega) = \alpha b_2 m + \alpha a_2\omega$. Due to (4.13), we have that $\frac{1}{2}T(x_2\xi) = \frac{1}{2}(2\alpha b_2 m) = \alpha b_2 m$. We again must examine this by cases.

First, we note that if $\alpha \equiv 0 \bmod 2$, we have determined $T(A^{-1}M)$:

$$T(A^{-1}M) = \left\{ \begin{pmatrix} Tx_1 - \alpha b_2 m \\ Tx_2 \end{pmatrix}, b_2 \in \mathbb{Z} \right\} = \begin{pmatrix} 2\mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix}. \tag{4.14}$$

If $\alpha \equiv 1 \bmod 2$, then we see that the parity of $x_2\xi$ is dependent on the parity of m. So, this will give us two more cases, when $m \equiv 2 \bmod 4$ and $m \equiv 3 \bmod 4$:

If $m \equiv 2 \bmod 4$, we see the same result for P_C as above, namely:

$$T(A^{-1}M) = \left\{ \begin{pmatrix} Tx_1 - \alpha b_2 m \\ Tx_2 \end{pmatrix}, b_2 \in \mathbb{Z} \right\} = \begin{pmatrix} 2\mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix}. \tag{4.15}$$

If $m \equiv 3 \bmod 4$, we see that we can obtain any integer in the top, giving us:

$$T(A^{-1}M) = \left\{ \begin{pmatrix} Tx_1 - 2\alpha b_2 m \\ Tx_2 \end{pmatrix}, b_2 \in \mathbb{Z} \right\} = \begin{pmatrix} \mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix}. \tag{4.16}$$

Combining (4.14), (4.15), and (4.16), we explicitly write $T(A^{-1}M)$:

$$T(A^{-1}M) = \begin{cases} \begin{pmatrix} 2\mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix} & \text{if } \alpha \equiv 0 \bmod 2, \\ \begin{pmatrix} 2\mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix} & \text{if } \alpha \equiv 1 \bmod 2, m \equiv 2 \bmod 4, \\ \begin{pmatrix} \mathbb{Z} \\ 2\mathbb{Z} \end{pmatrix} & \text{if } \alpha \equiv 1 \bmod 2, m \equiv 3 \bmod 4. \end{cases} \tag{4.17}$$

Finally, we can combine (4.12) and (4.17) to express i_γ, where $\gamma = [\alpha\omega]$.

Theorem 4.2.2. *Let* $\mathfrak{g} = Gal(\mathbb{Q}(\sqrt{m})/\mathbb{Q}), G = \mathcal{O}_K, M = \mathcal{O}_K^2$ *and* $m \equiv 2, 3$ *mod* 4. *Let* $\gamma = [\alpha\omega]$ *be a cohomology class in* $H^1(\mathfrak{g}, G)$. *Then:*

$$i_\gamma = |M_C/P_C| = \begin{cases} 4 & \textit{if } \alpha \equiv 0 \textit{ mod } 2, \\ 2 & \textit{if } \alpha \equiv 1 \textit{ mod } 2, m \equiv 2 \textit{ mod } 4, \\ 1 & \textit{if } \alpha \equiv 1 \textit{ mod } 2, m \equiv 3 \textit{ mod } 4. \end{cases}$$

Chapter 5

The Heisenberg Group of Dimension Three

5.1 Introduction

In the previous chapters, we studied the general setting $(\mathfrak{g}, (G, M))$ in the case where $\mathfrak{g}$ was the Galois group of an extension $K/\mathbb{Q}$, G was the ring of integers of this extension $\mathcal{O}_K$ viewed as an additive group, and M was $\mathcal{O}_K^2$. To study this case, we identified an element $t \in \mathcal{O}_K$ with the 2×2 matrix:

$$T_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This allowed us to take the additive structure of $\mathcal{O}_K$ but view it in the context of matrix multiplication.

We recognize that this matrix has a similar structure to that of the Heisenberg group $H_{2n+1}(\mathcal{O}_K)$, with 1's along the diagonal, 0's below the diagonal, and entries running along the top row and the rightmost column. We now take on the task of generalizing the results of the previous chapters to the three dimensional Heisenberg group. Therefore, we wish to study $(\mathfrak{g}, (G, M))$, again where $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$, but now with $G = H_3(\mathcal{O}_K)$ and $M = \mathcal{O}_K^3$. We will again be examining M_C, P_C, and the index of M_C/P_C.

5.2 The Modules Ξ_K and Θ_η

Let $K/\mathbb{Q}$ be a finite Galois extension of degree n, and let $\mathfrak{g}$ be the Galois group of this extension, i.e. $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$. Clearly, $|\mathfrak{g}| = n$.

As in the previous case, we wish to use Ξ_K from equation (3.1) in the construction of our multiplicitive cocycle which will act on M. However, we must first introduce another module which will be just as instrumental in our cocycle construction.

Let $\eta \in \Xi_K$. We now define:

$$\Theta_\eta := \{(\xi, \theta) \in \Xi_K \times \mathcal{O}_K : n\theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)^\sigma \eta \bmod n^2, \forall \sigma \in \mathfrak{g}\}. \tag{5.1}$$

Clearly, $(0, 0) \in \Theta_\eta$, and if $(\xi, \theta) \in \Theta_\eta$, then clearly $(-\xi, -\theta) \in \Theta_\eta$. This makes Θ_η an additive abelian group, meaning it is a $\mathbb{Z}$-module. The conditions on Θ_η allow us to construct a cocycle.

5.3 Setup

We wish to consider the system $(\mathfrak{g}, (G, M))$. Let $K/\mathbb{Q}$ be a Galois extension with $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$, and $|\mathfrak{g}| = n$. Let:

$$G = \left\{ \begin{pmatrix} 1 & u & t \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathcal{O}_K \right\}$$

and

$$M = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathcal{O}_K \right\}.$$

The actions in this system are natural. For G acting on M:

$$\begin{pmatrix} 1 & u & t \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + ux_2 + tx_3 \\ x_2 + vx_3 \\ x_3 \end{pmatrix}. \tag{5.2}$$

For $\sigma \in \mathfrak{g}$, the Galois action is:

$$^{\sigma}(A\mathbf{x}) = {}^{\sigma}A^{\sigma}\mathbf{x},$$

where $^{\sigma}A$ applies the Galois action to each element of the matrix.

We will now concern ourselves with the cocycle needed to define our M_C and P_C.

5.4 Cocycle Structure

Our goal in this section is to show that, for any cocycle $C \in Z^1(\mathfrak{g}, G)$, where $G = H_3(\mathcal{O}_K)$, we can choose $\eta \in \Xi_K, (\xi, \theta) \in \Theta_\eta$ such that we can explicitly determine the cocycle C.

Recall from equation (3.3), we created the additive cocycle: $t_\sigma(\xi) = \frac{1}{n}(\xi - {}^\sigma\xi)$. In that case, our 2×2 matrix C_σ preserved the cocycle relation, but unfortunately this is not the case in the 3×3 setting, since our upper right corner has an extra term due to the matrix multiplication. Therefore, we must choose carefully our entries in the matrix.

Given a fixed $\eta \in \Xi_K$, let $(\xi, \theta) \in \Theta_\eta$. Then, we define:

$$v_\sigma := \frac{1}{n}(\eta - {}^\sigma\eta), \tag{5.3}$$

$$u_\sigma := \frac{1}{n}(\xi - {}^\sigma\xi), \tag{5.4}$$

and

$$t_\sigma := \frac{1}{n}(\theta - {}^\sigma\theta) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta. \tag{5.5}$$

We will now prove that every cocycle in $Z^1(\mathfrak{g}, H_3(\mathcal{O}_K))$ is explicitly determined by these three elements.

Proposition 5.4.1. *Let u_σ, v_σ, and t_σ be as above. Then, every cocycle $C \in Z^1(\mathfrak{g}, G)$*

is of the form:

$$C_\sigma = \begin{pmatrix} 1 & u_\sigma & t_\sigma \\ 0 & 1 & v_\sigma \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.6}$$

for some admissible choice of ξ, η, *and* θ.

Proof

Our goal is to show that every cocycle in $Z^1(\mathfrak{g}, G)$ is of the above form. More specifically, let:

$$C_\sigma = \begin{pmatrix} 1 & a_\sigma & b_\sigma \\ 0 & 1 & c_\sigma \\ 0 & 0 & 1 \end{pmatrix} \in Z^1(\mathfrak{g}, G). \tag{5.7}$$

We begin by noting that this C_σ satisfies the equation $C_\sigma {}^\sigma C_\tau = C_{\sigma\tau}$. Examining this relation more closely we see:

$$C_\sigma {}^\sigma C_\tau = \begin{pmatrix} 1 & a_\sigma & b_\sigma \\ 0 & 1 & c_\sigma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & {}^\sigma a_\tau & {}^\sigma b_\tau \\ 0 & 1 & {}^\sigma c_\tau \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_\sigma + {}^\sigma a_\tau & b_\sigma + {}^\sigma b_\tau + a_\sigma {}^\sigma c_\tau \\ 0 & 1 & c_\sigma + {}^\sigma c_\tau \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_{\sigma\tau} & b_{\sigma\tau} \\ 0 & 1 & c_{\sigma\tau} \\ 0 & 0 & 1 \end{pmatrix},$$

where this last equality comes from the cocycle nature of C. This gives us three equations:

$$a_\sigma + {}^\sigma a_\tau = a_{\sigma\tau}, \tag{5.8}$$

$$c_\sigma + {}^\sigma c_\tau = c_{\sigma\tau}, \tag{5.9}$$

and

$$b_\sigma + {}^\sigma b_\tau + a_\sigma {}^\sigma c_\tau = b_{\sigma\tau}. \tag{5.10}$$

We now let

$$\xi = \sum_\tau a_\tau,$$

$$\eta = \sum_\tau c_\tau,$$

and

$$\theta = \sum_\tau b_\tau.$$

We will show that for these choices, our three equalities hold, and that this choice is admissible, i.e. $\eta \in \Xi_K, (\xi, \theta) \in \Theta_\eta$.

First, we consider $\xi = \sum_\tau a_\tau$. Since we know from the cocycle nature of C that $a_\sigma + {}^\sigma a_\tau = a_{\sigma\tau}$, we see that:

$$\begin{aligned} u_\sigma &= \frac{1}{n}\Big(\xi - {}^\sigma\xi\Big) \\ &= \frac{1}{n}\Big(\sum_\tau a_\tau - {}^\sigma(\sum_\tau a_\tau)\Big) \\ &= \frac{1}{n}\Big(\sum_\tau a_\tau - \sum_\tau {}^\sigma a_\tau\Big). \end{aligned}$$

Note that since the sum is running over all $\tau \in \mathfrak{g}$, this means that $\sum_\tau a_\tau = \sum_\tau a_{\sigma\tau}$. Also, from the cocycle relation of a, we get that $a_{\sigma\tau} - {}^\sigma a_\tau = a_\sigma$. Substituting in, we see:

$$u_\sigma = \frac{1}{n}\Big(\sum_\tau (a_{\sigma\tau} - {}^\sigma a_\tau)\Big) = \frac{1}{n}\Big(\sum_\tau a_\sigma\Big) = \frac{1}{n}(na_\sigma) = a_\sigma. \tag{5.11}$$

From equation (5.11), we get both that $\xi = \sum_\tau a_\tau \in \Xi_K$ and that for this choice of $\xi, a_\sigma = u_\sigma$.

The same calculations hold for $\eta = \sum_\tau c_\tau$, showing that $\eta = \sum_\tau c_\tau \in \Xi_K$ and that for this choice of $\eta, c_\sigma = v_\sigma$.

We now turn our attention to the final piece, b_σ. Let $\theta = \sum_\tau b_\tau$. We must show that $b_\sigma = t_\sigma$ and that θ is admissible, i.e. $(\xi, \theta) \in \Theta_\eta$. We start with showing that $b_\sigma = t_\sigma$, recalling the relation $b_\sigma + {}^\sigma b_\tau + a_\sigma {}^\sigma c_\tau = b_{\sigma\tau}$.

$$\begin{aligned} t_\sigma &= \frac{1}{n}(\theta - {}^\sigma\theta) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta \\ &= \frac{1}{n}\Big(\sum_\tau b_\tau - {}^\sigma\big(\sum_\tau b_\tau\big)\Big) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta \\ &= \frac{1}{n}\Big(\sum_\tau b_\tau - \big(\sum_\tau {}^\sigma b_\tau\big)\Big) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta. \end{aligned}$$

As before, we see that a sum over all $\tau \in \mathfrak{g}$ give us $\sum_\tau b_\tau = \sum_\tau b_{\sigma\tau}$. We will use the relationship of b_σ, i.e. $b_{\sigma\tau} - {}^\sigma b_\tau = b_\sigma + a_\sigma {}^\sigma c_\tau$.

$$\begin{aligned} t_\sigma &= \frac{1}{n}\Big(\sum_\tau (b_{\sigma\tau} - {}^\sigma b_\tau)\Big) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta \\ &= \frac{1}{n}\Big(\sum_\tau (b_\sigma + a_\sigma {}^\sigma c_\tau)\Big) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta \\ &= \frac{1}{n}\Big(nb_\sigma + a_\sigma \sum_\tau {}^\sigma c_\tau\Big) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta. \end{aligned}$$

Now, we now that $\xi = \sum_\tau a_\tau$ and $\eta = \sum_\tau c_\tau$. We substitute in for ξ and η to get:

$$\begin{aligned} t_\sigma &= \frac{1}{n}\Big(nb_\sigma + a_\sigma \sum_\tau {}^\sigma c_\tau\Big) - \frac{1}{n^2}\Big(\sum_\tau a_\tau - {}^\sigma\big(\sum_\tau a_\tau\big)\Big)^\sigma\big(\sum_\tau c_\tau\big) \\ &= b_\sigma + \frac{a_\sigma}{n}\sum_\tau {}^\sigma c_\tau - \frac{1}{n^2}\Big(\sum_\tau (a_{\sigma\tau} - {}^\sigma a_\tau)\Big)\big(\sum_\tau {}^\sigma c_\tau\big) \\ &= b_\sigma + \frac{a_\sigma}{n}\sum_\tau {}^\sigma c_\tau - \frac{1}{n^2}\big(\sum_\tau a_\sigma\big)\big(\sum_\tau {}^\sigma c_\tau\big) \\ &= b_\sigma + \frac{a_\sigma}{n}\sum_\tau {}^\sigma c_\tau - \frac{a_\sigma}{n}\sum_\tau {}^\sigma c_\tau = b_\sigma. \end{aligned} \tag{5.12}$$

Finally, we have to show $(\xi, \theta) \in \Theta_\eta$. We consider $n(\theta - {}^\sigma\theta)$. Our goal is to show this is congruent to $(\xi - {}^\sigma\xi)^\sigma\eta$ mod n^2.

$$\begin{aligned} n(\theta - {}^\sigma\theta) &= n\Big(\sum_\tau b_\tau - {}^\sigma(\sum_\tau b_\tau)\Big) \\ &= n\Big(\sum_\tau b_{\sigma\tau} - \sum_\tau {}^\sigma b_\tau\Big) \\ &= n\Big(\sum_\tau b_{\sigma\tau} - {}^\sigma b_\tau\Big) \\ &= n\Big(\sum_\tau b_\sigma + a_\sigma \sum_\tau {}^\sigma c_\tau\Big). \end{aligned}$$

Here, we have again used the relation of b_σ. Now, we recall that $\xi = \sum_\tau a_\tau$ and $\eta = \sum_\tau c_\tau$.

$$\begin{aligned} n(\theta - {}^\sigma\theta) &= n(nb_\sigma + a_\sigma \sum_\tau ({}^\sigma c_\tau)) \\ &= n(nb_\sigma + \frac{1}{n}(\sum_\tau a_\sigma) \sum_\tau ({}^\sigma c_\tau)) \\ &= n^2 b_\sigma + (\sum_\tau a_\sigma) \sum_\tau ({}^\sigma c_\tau)) \\ &= n^2 b_\sigma + (\sum_\tau a_{\sigma\tau} - {}^\sigma a_\tau)^\sigma (\sum_\tau c_\tau) \\ &= n^2 b_\sigma + (\sum_\tau a_\tau - \sum_\tau {}^\sigma a_\tau)^\sigma \eta \\ &= n^2 b_\sigma + (\xi - {}^\sigma\xi)^\sigma\eta. \end{aligned}$$

We know $b_\sigma \in \mathcal{O}_K$, so this implies $n\theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)^\sigma\eta$ mod n^2. And, since we made no assumption about σ, this argument applies for all $\sigma \in \mathfrak{g}$, which means $(\xi, \theta) \in \Theta_{\eta.}$ This completes the proof. □

5.5 M_C, P_C, and the Matrix $A(\xi, \eta, \theta)$

We will now use this cocycle to define our groups M_C and P_C. Recall that $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$, $G = H_3(\mathcal{O}_K)$, and $M = \mathcal{O}_K^3$. We define our M_C and P_C, which come from the general setting:

$$M_C = \{\mathbf{x} \in M : C_\sigma{}^\sigma\mathbf{x} = \mathbf{x}, \forall\sigma \in \mathfrak{g}\}. \tag{5.13}$$

$$P_C = \{\mathbf{y} = p_C(\mathbf{x}) = \sum_{\tau\in\mathfrak{g}} C_\tau{}^\tau\mathbf{x}, \mathbf{x} \in M\}. \tag{5.14}$$

We first show that M_C/P_C depends only on the cohomology class $[C]$. Let $C \sim C'$. This means that $C'_\sigma = U^{-1}C_\sigma{}^\sigma U$, where $U \in H_3(\mathcal{O}_K)$. We will now have the following two properties from Remark 2.3.1

Property 5.5.1. (a)$UM_{C'} = M_C$.
(b)$UP_{C'} = P_C$.

Combining the two properties, we obtain:

$$M_C/P_C = M_{C'}/P_{C'}. \tag{5.15}$$

Next, we wish to express C_σ as a product of matrices. In the following proposition we prove that we can do this.

Proposition 5.5.2. *For any $C \in Z^1(\mathfrak{g}, G)$, there exist $\eta \in \Xi_K, (\xi, \theta) \in \Theta_\eta$ such that*

$C_\sigma = A(\xi,\eta,\theta)\cdot {}^\sigma(A(\xi,\eta,\theta)^{-1})$, *where the* 3×3 *matrix* A *is defined by:*

$$A(\xi,\eta,\theta) = \begin{pmatrix} 1 & \frac{\xi}{n} & \frac{\theta}{n} \\ 0 & 1 & \frac{\eta}{n} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.16}$$

Proof

We know from Proposition 5.4.1 that there exist $\eta \in \Xi_K, (\xi,\theta) \in \Theta_\eta$ that determine explicitly C_σ. We use these same choices to show that $A(\xi,\eta,\theta)\cdot {}^\sigma(A(\xi,\eta,\theta)^{-1}) = C_\sigma$.

First, based on the choice for A, we get that A^{-1} is:

$$A^{-1} = \begin{pmatrix} 1 & -\frac{\xi}{n} & -\frac{\theta}{n} + \frac{\xi\eta}{n^2} \\ 0 & 1 & -\frac{\eta}{n} \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying the Galois action of σ to both sides, we get:

$${}^\sigma A^{-1} = \begin{pmatrix} 1 & -\frac{{}^\sigma\xi}{n} & -\frac{{}^\sigma\theta}{n} + \frac{{}^\sigma\xi{}^\sigma\eta}{n^2} \\ 0 & 1 & -\frac{{}^\sigma\eta}{n} \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, multiplying together, we obtain:

$$A\cdot {}^\sigma(A^{-1}) = \begin{pmatrix} 1 & \frac{\xi}{n} & \frac{\theta}{n} \\ 0 & 1 & \frac{\eta}{n} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{{}^\sigma\xi}{n} & -\frac{{}^\sigma\theta}{n} + \frac{{}^\sigma\xi{}^\sigma\eta}{n^2} \\ 0 & 1 & -\frac{{}^\sigma\eta}{n} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1}{n}(\xi - {}^\sigma\xi) & \frac{1}{n}(\theta - {}^\sigma\theta) - \frac{1}{n^2}(\xi - {}^\sigma\xi){}^\sigma\eta \\ 0 & 1 & \frac{1}{n}(\eta - {}^\sigma\eta) \\ 0 & 0 & 1 \end{pmatrix}.$$

This is exactly our C_σ, completing our proof. $\square$

The last important property we need is the finiteness of $H^1(\mathfrak{g}, G)$. I now prove it.

Proposition 5.5.3. *The set* $H^1(\mathfrak{g}, G) = Z^1(\mathfrak{g}, G)/\sim$ *is finite, and is bounded by:* $|H^1(\mathfrak{g}, G)| < n^n \cdot (n^{n-1})^2$.

Proof

To prove this, we will find an upper bound on our set. Combining equations (5.3), (5.4), and (5.5), and Proposition 5.4.1, we can explicit write our cocycle as:

$$C_\sigma = \begin{pmatrix} 1 & \frac{1}{n}(\xi - {}^\sigma\xi) & \frac{1}{n}(\theta - {}^\sigma\theta) - \frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta \\ 0 & 1 & \frac{1}{n}(\eta - {}^\sigma\eta) \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.17}$$

for an admissible choice of (η, ξ, θ). Let $C, C' \in Z^1(\mathfrak{g}, G)$. Recall from Definition 2.1.4 and Equation (2.7) that C and C' are cohomologous if there exists a $U \in G$ such that $C'_\sigma = U^{-1}C_\sigma{}^\sigma U$. Let:

$$C_\sigma = \begin{pmatrix} 1 & u_\sigma & t_\sigma \\ 0 & 1 & v_\sigma \\ 0 & 0 & 1 \end{pmatrix}, C'_\sigma = \begin{pmatrix} 1 & u'_\sigma & t'_\sigma \\ 0 & 1 & v'_\sigma \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } U = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where (η, ξ, θ) are an admissible choice of parameters that corresponds to u_σ, v_σ, and t_σ, and $a, b, c \in \mathcal{O}_K$.

Then, (2.7) becomes:

$$\begin{pmatrix} 1 & u'_\sigma & t'_\sigma \\ 0 & 1 & v'_\sigma \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_\sigma & t_\sigma \\ 0 & 1 & v_\sigma \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & {}^\sigma a & {}^\sigma b \\ 0 & 1 & {}^\sigma c \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & u_\sigma - a + {}^\sigma a & t_\sigma - b + {}^\sigma b - av_\sigma + ac + {}^\sigma c u_\sigma - a^\sigma c \\ 0 & 1 & v_\sigma - c + {}^\sigma c \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that, in order for a fixed C, C' to be in the same cohomology class, the following three equations must be satisfied:

$$u_\sigma - a + {}^\sigma a = u'_\sigma. \tag{5.18}$$

$$v_\sigma - c + {}^\sigma c = v'_\sigma. \tag{5.19}$$

$$t_\sigma - b + {}^\sigma b - av_\sigma + ac + {}^\sigma c u_\sigma - a^\sigma c = t'_\sigma. \tag{5.20}$$

Starting with (5.18), we see:

$$\begin{aligned} \frac{1}{n}(\xi - {}^\sigma\xi) - a + {}^\sigma a &= \frac{1}{n}(\xi' - {}^\sigma\xi') \\ &\iff \xi - \xi' - na = {}^\sigma(\xi - \xi' - na) \\ &\iff \xi - \xi' - na \in \mathbb{Z} \\ &\iff \xi = \xi' + na + \alpha, \alpha \in \mathbb{Z}. \end{aligned} \tag{5.21}$$

The dimension of $\mathcal{O}_K$ as a vector space over $\mathbb{Z}$ is n, and we know that we can write

$\mathcal{O}_K = \langle 1, \omega_2, \ldots, \omega_n \rangle_{\mathbb{Z}}$. From this, and the fact that our choice of $na \in n\mathcal{O}_K$ can be dependent on C, C', we see that the number of different cohomology classes is bounded by n^{n-1}, since the integer term need not be a multiple of n.

Similarly, for (5.19), we have:

$$\begin{aligned}
\frac{1}{n}(\eta - {}^{\sigma}\eta) - c + {}^{\sigma}c &= \frac{1}{n}(\eta' - {}^{\sigma}\eta') \\
&\Longleftrightarrow \eta - \eta' - nc = {}^{\sigma}(\eta - \eta' - nc) \\
&\Longleftrightarrow \eta - \eta' - nc \in \mathbb{Z} \\
&\Longleftrightarrow \eta = \eta' + nc + \gamma, \gamma \in \mathbb{Z}. \qquad (5.22)
\end{aligned}$$

Again, $\mathcal{O}_K = \langle 1, \omega_2, \ldots, \omega_n \rangle_{\mathbb{Z}}$ and our choice of $nc \in n\mathcal{O}_K$ can be dependent on C, C'. However, for this case, it will be useful to require our integer term to also be multiple of n, i.e. we require that $\gamma \in n\mathbb{Z}$, and this means for this case, the number of separate cohomology classes is bounded by n^n.

Finally, we turn our attention to (5.20). We start with:

$$t_{\sigma} - b + {}^{\sigma}b - av_{\sigma} + ac + {}^{\sigma}cu_{\sigma} - a^{\sigma}c = t'_{\sigma}$$

$$\Longleftrightarrow$$

$$\begin{aligned}
&\frac{1}{n}(\theta - {}^{\sigma}\theta) - \frac{1}{n^2}(\xi - {}^{\sigma}\xi)^{\sigma}\eta - b + {}^{\sigma}b - \frac{a}{n}(\eta - {}^{\sigma}\eta) + ac + \frac{{}^{\sigma}c}{n}(\xi - {}^{\sigma}\xi) - a^{\sigma}c \\
&\qquad = \frac{1}{n}(\theta' - {}^{\sigma}\theta') - \frac{1}{n^2}(\xi' - {}^{\sigma}\xi')^{\sigma}\eta'. \qquad (5.23)
\end{aligned}$$

We first start by studying $\frac{1}{n^2}(\xi - {}^{\sigma}\xi)^{\sigma}\eta$. In order for C, C' to be in the same cohomology class, we must require that Equations (5.18) and (5.19) have already been satisfied. That means we can use (5.21) and (5.22) to get:

$$
\begin{aligned}
\frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta =& \frac{1}{n^2}([\xi' + na + \alpha] - {}^\sigma[\xi' + na + \alpha])^\sigma[\eta' + nc + \gamma] \\
=& \frac{1}{n^2}(\xi' + na + \alpha - {}^\sigma\xi' - n^\sigma a - \alpha)({}^\sigma\eta' + n^\sigma c + \gamma) \\
=& \frac{1}{n^2}([\xi' - {}^\sigma\xi'] + n[a - {}^\sigma a])({}^\sigma\eta' + n^\sigma c + \gamma) \\
=& \frac{1}{n^2}([\xi' - {}^\sigma\xi']^\sigma\eta' + [\xi' - {}^\sigma\xi']n^\sigma c + [\xi' - {}^\sigma\xi']\gamma) \\
&+ \frac{1}{n^2}(n[a - {}^\sigma a]^\sigma\eta' + n^2[a - {}^\sigma a]^\sigma c + n[a - {}^\sigma a]\gamma).
\end{aligned}
$$

Distributing the $\frac{1}{n^2}$, we obtain:

$$
\begin{aligned}
\frac{1}{n^2}(\xi - {}^\sigma\xi)^\sigma\eta =& \frac{1}{n^2}[\xi' - {}^\sigma\xi']^\sigma\eta' + \frac{1}{n}[\xi' - {}^\sigma\xi']^\sigma c + \frac{1}{n^2}[\xi' - {}^\sigma\xi']\gamma \\
&+ \frac{1}{n}[a - {}^\sigma a]^\sigma\eta' + \frac{1}{n}[na - n^\sigma a]^\sigma c + \frac{1}{n}[a - {}^\sigma a]\gamma.
\end{aligned} \tag{5.24}
$$

We take (5.24), multiply by -1 and insert into (5.23). Now, looking only at the left-hand side (LHS), we get:

$$
\begin{aligned}
LHS =& \frac{1}{n}(\theta - {}^\sigma\theta) - b + {}^\sigma b - \frac{1}{n^2}[\xi' - {}^\sigma\xi']^\sigma\eta' \\
&- \frac{1}{n}[\xi' - {}^\sigma\xi']^\sigma c - \frac{1}{n^2}[\xi' - {}^\sigma\xi']\gamma \\
&- \frac{1}{n}[a - {}^\sigma a]^\sigma\eta' - \frac{1}{n}[na - n^\sigma a]^\sigma c - \frac{1}{n}[a - {}^\sigma a]\gamma \\
&- \frac{a}{n}(\eta - {}^\sigma\eta) + ac + \frac{{}^\sigma c}{n}(\xi - {}^\sigma\xi) - a^\sigma c.
\end{aligned}
$$

Grouping terms:

$$
\begin{aligned}
LHS =& \frac{1}{n}(\theta - {}^\sigma\theta) - b + {}^\sigma b - \frac{1}{n^2}[\xi' - {}^\sigma\xi']^\sigma\eta' \\
& - \frac{1}{n^2}[\xi' - {}^\sigma\xi']\gamma + ac - a^\sigma c \\
& - \frac{1}{n}[a - {}^\sigma a]^\sigma\eta' - \frac{1}{n}[a - {}^\sigma a]\gamma - \frac{a}{n}(\eta - {}^\sigma\eta) \\
& + \frac{{}^\sigma c}{n}([\xi - \xi' - na] - {}^\sigma[\xi - \xi' - na]).
\end{aligned} \tag{5.25}
$$

The last line of (5.25) is of particular interest. Recall from (5.21) that $\xi - \xi' + na = \alpha \in \mathbb{Z}$. This implies ${}^\sigma\alpha = \alpha$, and as a result the last line vanishes. Now, note that from (5.22), $\eta - \gamma = \eta' + nc$. This implies that $\eta - {}^\sigma\eta = \eta' + nc - {}^\sigma(\eta + nc)$. Similarly, from (5.21), $\xi - {}^\sigma\xi = \xi' + na - {}^\sigma(\xi' + na)$. Returning to the rest of (5.25) we plug in and group terms:

$$
\begin{aligned}
LHS =& \frac{1}{n}(\theta - {}^\sigma\theta) - b + {}^\sigma b - \frac{1}{n^2}[\xi' - {}^\sigma\xi']^\sigma\eta' \\
& - \frac{\gamma}{n^2}[(\xi' + na) - {}^\sigma(\xi' + na)] + \frac{a}{n}(nc - n^\sigma c) \\
& - \frac{1}{n^2}[na - n^\sigma a]^\sigma\eta' - \frac{a}{n}([\eta' + nc] - {}^\sigma[\eta' + nc]) \\
=& \frac{1}{n}(\theta - {}^\sigma\theta) - b + {}^\sigma b - \frac{1}{n^2}[\xi' - {}^\sigma\xi']^\sigma\eta' \\
& - \frac{\gamma}{n^2}[\xi - {}^\sigma\xi] - \frac{a}{n}(\eta' - {}^\sigma\eta') - \frac{1}{n^2}[na - n^\sigma a]^\sigma\eta'.
\end{aligned}
$$

Setting this equal to the righthand side of (5.23), we notice immediately that the $\frac{1}{n^2}[\xi' - {}^\sigma\xi']^\sigma\eta'$ term cancels out. This leaves us with:

$$
\begin{aligned}
& \frac{1}{n}(\theta - {}^\sigma\theta) - b + {}^\sigma b - \frac{\gamma}{n^2}[\xi - {}^\sigma\xi] - \frac{a}{n}(\eta' - {}^\sigma\eta') - \frac{1}{n^2}[na - n^\sigma a]^\sigma\eta' \\
& \quad = \frac{1}{n}(\theta' - {}^\sigma\theta').
\end{aligned}
$$

Multiplying out and regrouping:

$$n\theta - n^\sigma\theta - n^2b + n^{2\sigma}b - \gamma\xi + \gamma^\sigma\xi - an\eta' + an^\sigma\eta' - an^\sigma\eta' + n^\sigma a^\sigma\eta' = n\theta' - n^\sigma\theta'.$$

$$\iff$$

$$n\theta - n\theta' - n^2b - \gamma\xi - an\eta' = {}^\sigma[n\theta - n\theta' - n^2b - \gamma\xi - na\eta']$$

$$\iff n\theta - n\theta' - n^2b - \gamma\xi - an\eta' \in \mathbb{Z}$$

$$\iff n\theta = n\theta' + n^2b + \gamma\xi + an\eta' + \beta, \beta \in \mathbb{Z}. \tag{5.26}$$

Now, recall that we chose $\gamma \in n\mathbb{Z}$, i.e., $\gamma = n\delta, \delta \in \mathbb{Z}$ Hence, we can divide (5.26) through by n to get:

$$\theta = \theta' + \delta\xi + a\eta' + nb + \beta, \beta \in \mathbb{Z}. \tag{5.27}$$

As in the other two cases, since the dimension of $\mathcal{O}_K$ as a vector space over $\mathbb{Z}$ is n, and since we can write $\mathcal{O}_K = \langle 1, \omega_2, \ldots, \omega_n\rangle_\mathbb{Z}$, we see that the number of different cohomology classes is bounded by n^{n-1}, again due to the fact that the integer term need not be a multiple of n. Combining the three parts needed to have $C \sim C'$, we get the bound required: $H^1(g, G) < n^n \cdot (n^{n-1})^2 < \infty$.□

Corollary 5.5.4. *Let $C, C' \in Z^1(\mathfrak{g}, G)$ be cocycles, with C determined by $\eta \in \Xi_K, (\xi, \theta) \in \Theta_\eta$, and C' determined by $\eta \in \Xi_K, (\xi - a, \theta - b) \in \Theta_\eta$, with $a, b \in \mathbb{Z}$. Then, $C \sim C'$, i.e. $[C] = [C'] \in H^1(\mathfrak{g}, G)$.*

Corollary 5.5.4 implies that when considering elements in $H^1(\mathfrak{g}, G)$, we do not need to consider all of Θ_η, but rather the following smaller group:

$$\overline{\Theta}_\eta := \{(\xi,\theta) \in \Xi_K/\mathbb{Z} \times \mathcal{O}_K/\mathbb{Z} : n\theta - {}^{\sigma}(n\theta) \equiv (\xi - {}^{\sigma}\xi)^{\sigma}\eta \bmod n^2, \forall \sigma \in \mathfrak{g}\}. \tag{5.28}$$

Propositions 5.5.2 and 5.5.3 will be useful in the next section.

Finally we set $i_\gamma(\mathfrak{g}, M) = [M_C : P_C]$. The theorem of the next section gives us a relation between i_γ and the 0-th Tate Homology.

5.6 The Main Theorem

We now state and prove our main result.

Theorem 5.6.1. *Let $K/\mathbb{Q}$ be a finite Galois extension of degree n, and let $\mathfrak{g}$ be its Galois group. Let $G = H_3(\mathcal{O}_K)$, and $M = \mathcal{O}_K^3$. Then, for a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, G)$, corresponding to elements $\eta \in \Xi_K$ and $(\xi, \theta) \in \overline{\Theta}_\eta$ by Proposition 5.4.1 and Corollary 5.28, we have $i_\gamma(\mathfrak{g}, M) = |\widehat{H}^0(\mathfrak{g}, A(\xi,\eta,\theta)^{-1}M)|$.*

Proof

First, we recall that, for a group A and an A-module B, $\widehat{H}^0(A, B)$ is defined to be: $\widehat{H}^0(A, B) = B^A/T(B)$.

We know from Section 5 Proposition 2 that we can express our cocycle C_σ as a product of matrices: $C_\sigma = A^{\sigma}(A^{-1})$. We will use this to find equivalent ways of expressing M_C and P_C.

Let $\mathbf{x} \in M$. We see:

$$\begin{aligned} x \in M_C &\iff C_\sigma{}^{\sigma}\mathbf{x} = \mathbf{x} \\ &\iff A^{\sigma}(A^{-1})^{\sigma}\mathbf{x} = \mathbf{x} \\ &\iff A^{-1}\mathbf{x} = {}^{\sigma}(A^{-1}\mathbf{x}). \end{aligned}$$

Since $A^{-1}\mathbf{x}$ is fixed by every element of the Galois group, this means $A^{-1}\mathbf{x} \in \mathbb{Q}^3$. However, by definition $A^{-1}\mathbf{x} \in A^{-1}M$. This means that $A^{-1}M_C = (A^{-1}M)^{\mathfrak{g}}$.

Similarly, we consider $A^{-1}P_C$.

$$\begin{aligned} A^{-1}P_C &= \{A^{-1}\sum_{\tau\in\mathfrak{g}} C_\tau{}^\tau\mathbf{x}\} \\ &= \{\sum_{\tau\in\mathfrak{g}} A^{-1}A^\tau(A^{-1})^\tau\mathbf{x}\} \\ &= \{\sum_{\tau\in\mathfrak{g}}{}^\tau(A^{-1}\mathbf{x})\}. \end{aligned}$$

So, this means that $A^{-1}P_C = Tr(A^{-1}M)$. Combining the results, we obtain our result:

$$i_\gamma(\mathfrak{g}, M) = |\widehat{H}^0(\mathfrak{g}, A(\xi,\eta,\theta)^{-1}M)|. \tag{5.29}$$

Chapter 6

The Quadratic Case and a Theorem when $\Xi_K = \mathbb{Z} + n\mathcal{O}_K$

We will now use Theorem 5.6.1 to determine i_γ in the quadratic extension case; i.e. where $K = \mathbb{Q}(\sqrt{m})$, m squarefree, $M = \mathcal{O}_K^3, G = H_3(\mathcal{O}_K)$, and $\mathfrak{g} = \mathrm{Gal}(K/\mathbb{Q})$. As before, we will break this down into cases.

6.1 $m \equiv 3$ mod 4

We start with the case where m is congruent to 3 modulo 4. This means that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z} \cdot \omega$ where $\omega = \sqrt{m}$.

In order to use Theorem 5.6.1, we need to know both Ξ_K and $\overline{\Theta}_\eta$. We recall from Equation (4.11) that $\Xi_K = \mathcal{O}_K$ when $m \equiv 3 \bmod 4$. Recalling the definition of Θ_η :

$$\Theta_\eta = \{(\xi, \theta) \in \mathcal{O}_K \times \Xi_K : 2(\theta - {}^\sigma\theta) \equiv (\xi - {}^\sigma\xi)^\sigma \eta \bmod 4\}.$$

We know from the proof of Proposition 5.5.3 that we only need concern ourselves with $2^2 = 4$ choices of η, namely $\eta = 0, 1, \omega$, and $1 + \omega$.

We start by determining Θ_0, i.e. we must find all pairs (ξ, θ) such that $2(\theta - {}^{\sigma}\theta) \equiv (\xi - {}^{\sigma}\xi)^{\sigma}\eta \bmod 4$. We first note that since η is 0, the right-hand side is 0. However, since $\Xi_K = \mathcal{O}_K$, the left-hand side is identically 0. Therefore, we see that $\Theta_\eta = \mathcal{O}_K \times \mathcal{O}_K$.

Now, we consider the case where $\eta = A + B\omega$, where not both $A, B = 0$. Also, let $\xi = C + D\omega$. Again we will look at the condition on Θ_η. The left-hand side of the equation is again 0. Since ${}^{\sigma}(A + B\omega) = A - B\omega$ and ${}^{\sigma}(C + D\omega) = C - D\omega$, we get:

$0 \equiv (\xi - {}^{\sigma}\xi)(A - B\omega) \equiv (2D)(A - B\omega) \bmod 4$. Since A and B are not both 0, in order for this relation to be true, D must be even. This gives us $\Theta_\eta = (\mathbb{Z} + 2\mathbb{Z}\omega) \times \mathcal{O}_K$.

Finally, due to Corollary 5.5.4, in order to apply our theorem we need only consider $\overline{\Theta}_\eta$, which due to the above calculations gives us:

$$\overline{\Theta}_\eta = \begin{cases} \mathbb{Z}\omega \times \mathbb{Z}\omega & \text{if } \eta = 0, \\ 2\mathbb{Z}\omega \times \mathbb{Z}\omega & \text{else.} \end{cases} \tag{6.1}$$

Using Theorem 5.6.1, in order to determine the index of M_C/P_C, we can instead study $(A^{-1}M)^{\mathfrak{g}}$ and $T(A^{-1}M)$, where:

$$A^{-1} = \begin{pmatrix} 1 & -\xi/2 & -\theta/2 + \xi\eta/4 \\ 0 & 1 & -\eta/2 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}$$

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\xi/2)x_2 + [-(\theta/2) + (\xi\eta/4)]x_3 \\ x_2 - (\eta/2)x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}. \tag{6.2}$$

We now continue to evaluate case by case. We begin with the $\overline{\Theta}_0$ case.

6.1.1 $\eta = 0$

Let $\eta = 0$. We then must choose $(\xi, \theta) \in \overline{\Theta}_\eta$. From (6.1), we know $\overline{\Theta}_\eta = \mathbb{Z}\omega \times \mathbb{Z}\omega$, and we know from Proposition 5.5.3 that we need only consider the parity of the coefficients. Therefore, let $\xi = \alpha\omega$ and $\theta = \beta\omega$, where $(\alpha, \beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We first determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$. By (6.2), we have:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega/2)x_2 - (\beta\omega/2)x_3 \\ x_2 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

In order for an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements:

$$x_1 - (\alpha\omega/2)x_2 - (\beta\omega/2)x_3 \in \mathbb{Q}. \tag{6.3}$$

$$x_2 \in \mathbb{Q}. \tag{6.4}$$

$$x_3 \in \mathbb{Q}. \tag{6.5}$$

Let $x_i = a_i + b_i\omega$, where $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, this means $x_1, x_2, x_3 \in \mathcal{O}_K$. Combining this with (6.4) and (6.5), we get that $x_2, x_3 \in \mathbb{Z}$. Turning our attention to (6.3), we must satisfy:

$$a_1 + [b_1 - (\alpha/2)x_2 - (\beta/2)x_3]\omega \in \mathbb{Q}.$$

In order for this to be true, the ω term must be 0. This means that $b_1 = (\alpha/2)x_2 + (\beta/2)x_3$. Now, depending on our choice of (α, β) we get different results. Expressing

$(A^{-1}M)^{\mathfrak{g}}$ in terms of bases, we get:

$$\text{(6.6)} \qquad \text{If } (\alpha,\beta)=(0,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.7)} \qquad \text{If } (\alpha,\beta)=(1,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.8)} \qquad \text{If } (\alpha,\beta)=(0,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.9)} \qquad \text{If } (\alpha,\beta)=(1,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

Now, we now turn our attention to $T(A^{-1}M)$. We note that $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - \alpha m b_2 - \beta m b_3 \\ 2a_2 \\ 2a_3 \end{pmatrix}.$$

Again, we will give our answer in basis form in terms of (α, β):

$$\text{If } (\alpha,\beta) = (0,0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.10}$$

$$\text{If } (\alpha,\beta) = (1,0), T(A^{-1}M) \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.11}$$

$$\text{If } (\alpha,\beta) = (0,1), T(A^{-1}M) \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.12}$$

$$\text{If } (\alpha,\beta) = (1,1), T(A^{-1}M) \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.13}$$

Now, to determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$, we look at the determinants of the bases of each group. Therefore, using Equations (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), and (6.13), we get the following result:

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = \begin{cases} 8 & \text{if } (\alpha,\beta)=(0,0), \\ 2 & \text{if } (\alpha,\beta)=(1,0), \\ 2 & \text{if } (\alpha,\beta)=(0,1), \\ 2 & \text{if } (\alpha,\beta)=(1,1). \end{cases} \tag{6.14}$$

6.1.2 $\eta = 1$

We now consider the case where $\eta = 1$. We choose $(\xi, \theta) \in \overline{\Theta}_\eta$.

From (6.1), we know $\overline{\Theta}_\eta = 2\mathbb{Z}\omega \times \mathbb{Z}\omega$, and again we need only consider the parity of the coefficients. Therefore, let $\xi = 2\alpha\omega$ and $\theta = \beta\omega$, where $(\alpha, \beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We first determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$. By (6.2), we have:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega)x_2 + ((\alpha-\beta)\omega/2)x_3 \\ x_2 - 1/2x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

Again, for an element to be in $(A^{-1}M)^{\mathfrak{g}}$, the following three statements must be satisfied:

$$x_1 - (\alpha\omega)x_2 + ((\alpha-\beta)\omega/2)x_3 \in \mathbb{Q}. \tag{6.15}$$

$$x_2 - 1/2x_3 \in \mathbb{Q}. \tag{6.16}$$

$$x_3 \in \mathbb{Q}. \tag{6.17}$$

Let $x_i = a_i + b_i\omega$, where $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, this means $x_1, x_2, x_3 \in \mathcal{O}_K$. Combined with (6.16) and (6.17), we have $x_2, x_3 \in \mathbb{Z}$. Now, studying (6.15), we must satisfy:

$$a_1 + [b_1 - \alpha x_2 + ((\alpha - \beta)/2)x_3]\omega \in \mathbb{Q}.$$

In order for this to be true, the ω term must be 0. This means that $b_1 = \alpha x_2 + ((\beta - \alpha)/2)x_3$. Again, our results are dependent on our choice of (α, β). Expressing $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases, we get:

$$\text{(6.18)} \quad \text{If } (\alpha,\beta) = (0,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.19)} \quad \text{If } (\alpha,\beta) = (1,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.20)} \quad \text{If } (\alpha,\beta) = (0,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.21)} \quad \text{If } (\alpha,\beta) = (1,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

Turning our attention to $T(A^{-1}M)$, we note that $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - 2\alpha mb_2 + \alpha mb_3 - \beta mb_3 \\ 2a_2 - a_3 \\ 2a_3 \end{pmatrix}.$$

We again give our answer in basis form in terms of (α, β):

$$\text{(6.22)} \quad \text{If } (\alpha,\beta) = (0,0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.23)} \quad \text{If } (\alpha,\beta) = (1,0), T(A^{-1}M) \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.24)} \quad \text{If } (\alpha,\beta) = (0,1), T(A^{-1}M) \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.25)} \quad \text{If } (\alpha,\beta) = (1,1), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

To determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$, we use the determinants of the bases of each group. Therefore, using Equations (6.18), (6.19), (6.20), (6.21), (6.22), (6.23), (6.24), and (6.25), we obtain:

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = \begin{cases} 8 & \text{if } (\alpha,\beta)=(0,0), \\ 2 & \text{if } (\alpha,\beta)=(1,0), \\ 2 & \text{if } (\alpha,\beta)=(0,1), \\ 8 & \text{if } (\alpha,\beta)=(1,1). \end{cases} \tag{6.26}$$

6.1.3 $\eta = \omega$

Next, we look at the case where $\eta = \omega$, again choosing $(\xi, \theta) \in \overline{\Theta}_\eta$.

As before, $\overline{\Theta}_\eta = 2\mathbb{Z}\omega \times \mathbb{Z}\omega$. Let $\xi = 2\alpha\omega$ and $\theta = \beta\omega$, where $(\alpha,\beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Using Equation (6.2), we can start to determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega)x_2 - (\beta\omega/2)x_3 + (\alpha m)/2x_3 \\ x_2 - (\omega/2)x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

Again, for an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements:

$$x_1 - (\alpha\omega)x_2 - (\beta\omega/2)x_3 + (\alpha m)/2x_3 \in \mathbb{Q}. \tag{6.27}$$

$$x_2 - (\omega/2)x_3 \in \mathbb{Q}. \tag{6.28}$$

$$x_3 \in \mathbb{Q}. \tag{6.29}$$

Once again, let $x_i = a_i + b_i\omega$, with $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, we know $x_1, x_2, x_3 \in \mathcal{O}_K$. Looking at (6.28) and (6.29), $a_2 + [b_2 - x_3/2]\omega \in \mathbb{Q}$, which means that $x_3 \in 2\mathbb{Z}$, and $x_2 = a_2 + (x_3/2)\omega$.

Looking at (6.27), we need to satisfy:

$$a_1 + m\alpha(x_3/2) + [b_1 - \alpha x_2 - \beta(x_3/2)]\omega \in \mathbb{Q}.$$

To satisfy this statement, we again need the ω term must be 0. This means $b_1 = \alpha x_2 + \beta(x_3/2)$. In this particular case, our results are the same, regardless of our choice of (α, β). We write $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases to get:

$$(A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.30}$$

Looking now at $T(A^{-1}M)$, we know $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - 2\alpha m b_2 + \alpha m a_3 - \beta m b_3 \\ 2a_2 - mb_3 \\ 2a_3 \end{pmatrix}.$$

We again give our answers in basis form in terms of (α, β):

$$\text{(6.31)} \qquad \text{If } (\alpha,\beta)=(0,0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.32)} \qquad \text{If } (\alpha,\beta)=(1,0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} m \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.33)} \qquad \text{If } (\alpha,\beta)=(0,1), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.34)} \qquad \text{If } (\alpha,\beta)=(1,1), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} m \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

In order to determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$, we use the determinants of the bases of each group. Using Equations (6.30), (6.31), (6.32), (6.33), and (6.34), we again get an answer that is independent of our choice of (α, β):

$$\text{(6.35)} \qquad |(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = 2.$$

6.1.4 $\eta = 1 + \omega$

Finally, we look at the case where $\eta = 1 + \omega$, and again choose $(\xi, \theta) \in \overline{\Theta}_\eta$.

We are still in the case where $\overline{\Theta}_\eta = 2\mathbb{Z}\omega \times \mathbb{Z}\omega$. Let $\xi = 2\alpha\omega$ and $\theta = \beta\omega$, with $(\alpha, \beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We will again determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$. Using (6.2), we have:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega)x_2 - (\beta\omega/2)x_3 + (\alpha m)/2x_3 + (\alpha\omega)/2x_3 \\ x_2 - 1/2x_3 - (\omega/2)x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

For an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements:

$$x_1 - (\alpha\omega)x_2 - (\beta\omega/2)x_3 + (\alpha m)/2x_3 + (\alpha\omega/2)x_3 \in \mathbb{Q}. \tag{6.36}$$

$$x_2 - 1/2x_3 - (\omega/2)x_3 \in \mathbb{Q}. \tag{6.37}$$

$$x_3 \in \mathbb{Q}. \tag{6.38}$$

Let $x_i = a_i + b_i\omega$, with $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, we again have $x_i, x_2, x_3 \in \mathcal{O}_K$. Using (6.37) and (6.38), we see $a_2 - 1/2x_3 + [b_2 - x_3/2]\omega \in \mathbb{Q}$, which means that $x_3 \in 2\mathbb{Z}$, and $x_2 = a_2 + (x_3/2)\omega$.

Turning our attention to (6.37), we need to satisfy:

$$a_1 + m\alpha(x_3/2) + [b_1 - \alpha x_2 - \beta(x_3/2) + \alpha(x_3/2)]\omega \in \mathbb{Q}.$$

To satisfy this statement, we again need the ω term must be 0. This means $b_1 =$

$\alpha x_2 + \beta(x_3/2) - \alpha(x_3/2)$. In this particular case, our results are the same, regardless of our choice of (α, β). We write $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases to get:

$$(A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.39}$$

Finally, turning our attention to $T(A^{-1}M)$, recall that $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - 2\alpha mb_2 - \beta mb_2 + \alpha ma_3 - \alpha mb_3 \\ 2a_2 - a_3 - mb_3 \\ 2a_3 \end{pmatrix}.$$

We give our answer in basis form in terms of (α, β):

$$\text{If } (\alpha, \beta) = (0, 0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.40}$$

$$\text{If } (\alpha, \beta) = (1, 0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} m \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.41}$$

$$\text{If } (\alpha,\beta)=(0,1), T(A^{-1}M)\cong \begin{pmatrix}1\\0\\0\end{pmatrix} r_1+\begin{pmatrix}0\\1\\0\end{pmatrix} r_2+\begin{pmatrix}0\\-1\\2\end{pmatrix} r_3, r_i\in\mathbb{Z}. \tag{6.42}$$

$$\text{If } (\alpha,\beta)=(1,1), T(A^{-1}M)\cong \begin{pmatrix}1\\0\\0\end{pmatrix} r_1+\begin{pmatrix}1\\1\\0\end{pmatrix} r_2+\begin{pmatrix}0\\-1\\2\end{pmatrix} r_3, r_i\in\mathbb{Z}. \tag{6.43}$$

We determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$ in the same way as before. Using Equations (6.39), (6.40), (6.41), (6.42), and (6.43), we get:

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = \begin{cases} 2 & \text{if } (\alpha,\beta)=(0,0),\\ 2 & \text{if } (\alpha,\beta)=(1,0),\\ 1 & \text{if } (\alpha,\beta)=(0,1),\\ 1 & \text{if } (\alpha,\beta)=(1,1).\end{cases} \tag{6.44}$$

Finally, combining Equations (6.14), (6.26), (6.35), and (6.44), we get the following result:

Theorem 6.1.1. *Let* $K=\mathbb{Q}(\sqrt{m}), m\equiv 3$ *mod* 4. *Let* $\mathfrak{g}=Gal(K/\mathbb{Q}), G=H_3(\mathcal{O}_K), M=\mathcal{O}_K^3$. *Then, for a cocycle class* $\gamma=[C]$ *corresponding to some choice of* $\eta\in\Xi_K$, *and*

$(\xi,\theta) = \overline{\Theta}_\eta$, *we have the following explicit determination of* i_γ :

(6.45)
$$i_\gamma = \begin{cases} 1 & \text{if } \eta \equiv 1+\omega \bmod 2\mathcal{O}_K, \theta \equiv \omega \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 0 \bmod 2\mathcal{O}_K, \theta \equiv \xi \equiv 0 \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 1 \bmod 2\mathcal{O}_K, \theta \equiv \xi \bmod 2\mathbb{Z}\omega, \\ 2 & \text{otherwise.} \end{cases}$$

6.2 $m \equiv 2$ mod 4

Next, we turn to the case where m is congruent to 2 modulo 4. This again means that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\cdot\omega$ where $\omega = \sqrt{m}$.

In order for us to use Theorem 5.6.1, we must know both Ξ_K and $\overline{\Theta}_\eta$. We recall from Equation (4.11) that $\Xi_K = \mathcal{O}_K$ when $m \equiv 2$ mod 4. Recall the definition of Θ_η :

$$\Theta_\eta = \{(\xi,\theta) \in \mathcal{O}_K \times \Xi_K : 2(\theta - {}^\sigma\theta) \equiv (\xi - {}^\sigma\xi)^\sigma\eta \bmod 4\}.$$

From the proof of Proposition 5.5.3, we know we only need concern ourselves with $2^2 = 4$ choices of η, namely $\eta = 0, 1, \omega$, and $1+\omega$.

We will start by determining the case where $\eta = B\omega$, $B = 0, 1$. We must find all pairs (ξ,θ) such that $2(\theta - {}^\sigma\theta) \equiv (\xi - {}^\sigma\xi)^\sigma\eta$ mod 4. Let $\xi = C + D\omega$. First, note that since ${}^\sigma(C+D\omega) = C - D\omega$, $\omega^2 = m$, and $\eta = B\omega$, the right-hand side becomes $-2DBm$. However, since m is even, this side becomes identically zero. And, since $\Xi_K = \mathcal{O}_K$, the left-hand side is also 0. Therefore, we see that $\Theta_\eta = \mathcal{O}_K \times \mathcal{O}_K$ in this case.

Now, consider the case where $\eta = 1 + B\omega$, where $B = 0, 1$. Again, let $\xi = C + D\omega$. We look at the condition on Θ_η. The left-hand side of the equation is again 0. Since ${}^\sigma(1+B\omega) = 1 - B\omega$ and ${}^\sigma(C+D\omega) = C - D\omega$, we get:

$0 \equiv (\xi - {}^\sigma\xi)(1 - B\omega) \equiv (2D)(1 - B\omega)$ mod 4. In order for this relation to be true,

D must be even. This gives us $\Theta_\eta = (\mathbb{Z} + 2\mathbb{Z}\omega) \times \mathcal{O}_K$.

Finally, due to Corollary 5.5.4, in order to apply our theorem we need only consider $\overline{\Theta}_\eta$, which, due to the above calculations, gives us:

$$\overline{\Theta}_\eta = \begin{cases} \mathbb{Z}\omega \times \mathbb{Z}\omega & \text{if } \eta = 0 \text{ or } \omega, \\ 2\mathbb{Z}\omega \times \mathbb{Z}\omega & \text{else.} \end{cases} \tag{6.46}$$

Using Theorem 5.6.1, in order to determine the index of M_C/P_C, we can instead study $(A^{-1}M)^{\mathfrak{g}}$ and $T(A^{-1}M)$, where: $A^{-1} = \begin{pmatrix} 1 & -\xi/2 & -\theta/2 + \xi\eta/4 \\ 0 & 1 & -\eta/2 \\ 0 & 0 & 1 \end{pmatrix}$, and

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\xi/2)x_2 + [-(\theta/2) + (\xi\eta/4)]x_3 \\ x_2 - (\eta/2)x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}. \tag{6.47}$$

We now continue to evaluate case by case. We begin with the $\overline{\Theta}_0$ case.

6.2.1 $\eta = 0$

Let $\eta = 0$. We then must choose $(\xi, \theta) \in \overline{\Theta}_\eta$. From (6.46), we know $\overline{\Theta}_\eta = \mathbb{Z}\omega \times \mathbb{Z}\omega$, and we know from Proposition 5.5.3 that we only need consider the parity of the coefficients. Therefore, let $\xi = \alpha\omega$ and $\theta = \beta\omega$, where $(\alpha, \beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. First, we determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$. By (6.47), we have:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega/2)x_2 - (\beta\omega/2)x_3 \\ x_2 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

In order for an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements:

$$x_1 - (\alpha\omega/2)x_2 - (\beta\omega/2)x_3 \in \mathbb{Q}. \tag{6.48}$$

$$x_2 \in \mathbb{Q}. \tag{6.49}$$

$$x_3 \in \mathbb{Q}. \tag{6.50}$$

Let $x_i = a_i + b_i\omega$, where $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, this means $x_1, x_2, x_3 \in \mathcal{O}_K$. Combining this with (6.49) and (6.50), we get that $x_2, x_3 \in \mathbb{Z}$. Now, turning our attention to (6.48), we must satisfy:

$$a_1 + [b_1 - (\alpha/2)x_2 - (\beta/2)x_3]\omega \in \mathbb{Q}.$$

In order for this to be true, the ω term must be 0. This means that $b_1 = (\alpha/2)x_2 + (\beta/2)x_3$. Now, depending on our choice of (α, β), we will get different results. Expressing $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases, we have:

$$\text{If } (\alpha,\beta) = (0,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.51}$$

$$\text{(6.52)} \qquad \text{If } (\alpha,\beta)=(1,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.53)} \qquad \text{If } (\alpha,\beta)=(0,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

$$\text{(6.54)} \qquad \text{If } (\alpha,\beta)=(1,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

Turning our attention to $T(A^{-1}M)$, we note that $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - \alpha m b_2 - \beta m b_3 \\ 2a_2 \\ 2a_3 \end{pmatrix}.$$

Here, we see that regardless of (α, β), our $T(A^{-1}M)$ is the same:

$$\text{(6.55)} \qquad T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

To determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$, we look at the determinants of the bases of each group. Therefore, using Equations (6.51), (6.52), (6.53), (6.54), and (6.55), we get the following result:

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = \begin{cases} 8 & \text{if } (\alpha,\beta) = (0,0), \\ 4 & \text{if } (\alpha,\beta) = (1,0), \\ 4 & \text{if } (\alpha,\beta) = (0,1), \\ 4 & \text{if } (\alpha,\beta) = (1,1). \end{cases} \tag{6.56}$$

6.2.2 $\eta = 1$

We now consider the case where $\eta = 1$. We choose $(\xi, \theta) \in \overline{\Theta}_\eta$.

From (6.46), we know $\overline{\Theta}_\eta = 2\mathbb{Z}\omega \times \mathbb{Z}\omega$, and again we need only consider the parity of the coefficients. Therefore, let $\xi = 2\alpha\omega$ and $\theta = \beta\omega$, where $(\alpha,\beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We first determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$. By (6.47), we have:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega)x_2 + ((\alpha-\beta)\omega/2)x_3 \\ x_2 - 1/2x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

Again, for an element to be in $(A^{-1}M)^{\mathfrak{g}}$, the following three statements must be satisfied:

$$x_1 - (\alpha\omega)x_2 + ((\alpha-\beta)\omega/2)x_3 \in \mathbb{Q}. \tag{6.57}$$

$$x_2 - 1/2x_3 \in \mathbb{Q}. \tag{6.58}$$

$$x_3 \in \mathbb{Q}. \tag{6.59}$$

Let $x_i = a_i + b_i\omega$, where $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, this means $x_1, x_2, x_3 \in \mathcal{O}_K$. Combined with (6.58) and (6.59), we have $x_2, x_3 \in \mathbb{Z}$. Now, studying (6.57), we must satisfy:

$$a_1 + [b_1 - \alpha x_2 + ((\alpha - \beta)/2)x_3]\omega \in \mathbb{Q}.$$

In order for this to be true, the ω term must be 0. This means that $b_1 = \alpha x_2 + ((\beta - \alpha)/2)x_3$. Again, our results are dependent on our choice of (α, β). Expressing $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases, we get:

$$\text{If } (\alpha,\beta) = (0,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.60}$$

$$\text{If } (\alpha,\beta) = (1,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.61}$$

$$\text{If } (\alpha,\beta) = (0,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.62}$$

$$(6.63)\quad \text{If } (\alpha,\beta)=(1,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix}1\\0\\0\end{pmatrix} r_1 + \begin{pmatrix}0\\1\\0\end{pmatrix} r_2 + \begin{pmatrix}0\\-1/2\\1\end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

Turning to $T(A^{-1}M)$, we note that $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - 2\alpha m b_2 + \alpha m b_3 - \beta m b_3 \\ 2a_2 - a_3 \\ 2a_3 \end{pmatrix}.$$

We again give our answer in basis form, noting that our answer is independent of (α, β):

$$(6.64)\qquad T(A^{-1}M) \cong \begin{pmatrix}2\\0\\0\end{pmatrix} r_1 + \begin{pmatrix}0\\2\\0\end{pmatrix} r_2 + \begin{pmatrix}0\\-1\\2\end{pmatrix} r_3, r_i \in \mathbb{Z}.$$

To determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$, we use the determinants of the bases of each group. Therefore, using Equations (6.60), (6.61), (6.62), (6.63), and (6.64), we obtain:

$$(6.65)\qquad |(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = \begin{cases} 8 & \text{if } (\alpha,\beta)=(0,0), \\ 4 & \text{if } (\alpha,\beta)=(1,0), \\ 4 & \text{if } (\alpha,\beta)=(0,1), \\ 8 & \text{if } (\alpha,\beta)=(1,1). \end{cases}$$

6.2.3 $\eta = \omega$

Next, we look at the case where $\eta = \omega$, again choosing $(\xi, \theta) \in \overline{\Theta}_\eta$.

We have that $\overline{\Theta}_\eta = \mathbb{Z}\omega \times \mathbb{Z}\omega$. Let $\xi = \alpha\omega$ and $\theta = \beta\omega$, where $(\alpha, \beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Using Equation (6.47), we can start to determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega/2)x_2 - (\beta\omega/2)x_3 + (\alpha m/4)x_3 \\ x_2 - (\omega/2)x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

Again, for an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements:

$$x_1 - (\alpha\omega/2)x_2 - (\beta\omega/2)x_3 + (\alpha m/4)x_3 \in \mathbb{Q}. \tag{6.66}$$

$$x_2 - (\omega/2)x_3 \in \mathbb{Q}. \tag{6.67}$$

$$x_3 \in \mathbb{Q}. \tag{6.68}$$

Once again, let $x_i = a_i + b_i\omega$, with $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, we know $x_1, x_2, x_3 \in \mathcal{O}_K$. Looking at (6.67) and (6.68), $a_2 + [b_2 - x_3/2]\omega \in \mathbb{Q}$, which means that $x_3 \in 2\mathbb{Z}$, and $x_2 = a_2 + (x_3/2)\omega$.

Looking at (6.66), we need to satisfy:

$$a_1 + [b_1 - \beta(x_3/2) - (\alpha/2)(x_2 - (\omega/2)x_3)]\omega \in \mathbb{Q}.$$

First, notice that $(x_2 - (\omega/2)x_3) \in \mathbb{Z}$ by (6.67), so our equation becomes:

$$a_1 + [b_1 - \beta(x_3/2) - (\alpha/2)(a_2)]\omega \in \mathbb{Q}.$$

To satisfy this statement, we need the ω term must be 0. This means if $\alpha = 1, a_2 \in 2\mathbb{Z}$. Our $(A^{-1}M)^{\mathfrak{g}}$ is dependent on our choice of (α, β), and we get:

$$\text{If } (\alpha,\beta) = (0,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.69}$$

$$\text{If } (\alpha,\beta) = (1,0), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.70}$$

$$\text{If } (\alpha,\beta) = (0,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.71}$$

$$\text{If } (\alpha,\beta) = (1,1), (A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.72}$$

Looking now at $T(A^{-1}M)$, we know $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) =$

$T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - \alpha mb_2 + \alpha ma_3/2 - \beta mb_3 \\ 2a_2 - mb_3 \\ 2a_3 \end{pmatrix}.$$

We again give our answers in basis form in terms of (α, β):

$$\text{If } (\alpha,\beta) = (0,0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.73}$$

$$\text{If } (\alpha,\beta) = (1,0), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} m/2 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.74}$$

$$\text{If } (\alpha,\beta) = (0,1), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.75}$$

$$\text{If } (\alpha,\beta) = (1,1), T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} m/2 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.76}$$

In order to determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$, we use the determinants of the bases of each group. Using Equations (6.69), (6.70), (6.71), (6.72), (6.73), (6.74), (6.75), and (6.76), we get:

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = \begin{cases} 4 & \text{if } (\alpha,\beta) = (0,0), \\ 2 & \text{if } (\alpha,\beta) = (1,0), \\ 4 & \text{if } (\alpha,\beta) = (0,1), \\ 2 & \text{if } (\alpha,\beta) = (1,1). \end{cases} \tag{6.77}$$

6.2.4 $\eta = 1 + \omega$

Finally, we look at the case where $\eta = 1 + \omega$, and again choose $(\xi, \theta) \in \overline{\Theta}_\eta$.

We are back in the case where $\overline{\Theta}_\eta = 2\mathbb{Z}\omega \times \mathbb{Z}\omega$. Let $\xi = 2\alpha\omega$ and $\theta = \beta\omega$, with $(\alpha, \beta) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We will again determine $(A^{-1}M)^{\mathfrak{g}} \subset \mathbb{Q}^3$. Using (6.47), we have:

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - (\alpha\omega)x_2 - (\beta\omega/2)x_3 + (\alpha m)/2x_3 + (\alpha\omega)/2x_3 \\ x_2 - 1/2x_3 - (\omega/2)x_3 \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}.$$

For an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements:

$$x_1 - (\alpha\omega)x_2 - (\beta\omega/2)x_3 + (\alpha m)/2x_3 + (\alpha\omega/2)x_3 \in \mathbb{Q}. \tag{6.78}$$

$$x_2 - 1/2x_3 - (\omega/2)x_3 \in \mathbb{Q}. \tag{6.79}$$

$$x_3 \in \mathbb{Q}. \tag{6.80}$$

Let $x_i = a_i + b_i\omega$, with $a_i, b_i \in \mathbb{Z}$. Since $\mathbf{x} \in M$, we again have $x_1, x_2, x_3 \in \mathcal{O}_K$. Using (6.79) and (6.80), we see $a_2 - 1/2x_3 + [b_2 - x_3/2]\omega \in \mathbb{Q}$, which means that $x_3 \in 2\mathbb{Z}$, and $x_2 = a_2 + (x_3/2)\omega$.

Turning our attention to (6.79), we need to satisfy:

$$a_1 + m\alpha(x_3/2) + [b_1 - \alpha x_2 - \beta(x_3/2) + \alpha(x_3/2)]\omega \in \mathbb{Q}.$$

To satisfy this statement, we again need the ω term must be 0. This means $b_1 = \alpha x_2 + \beta(x_3/2) - \alpha(x_3/2)$. In this particular case, our results are the same, regardless of our choice of (α, β). We write $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases to get:

$$(A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.81}$$

Finally, turning our attention to $T(A^{-1}M)$, recall that $T(x_i) = T(a_i + b_i\omega) = 2a_i$, and $T(\omega x_i) = T(mb_i + a_i\omega) = 2mb_i$. Therefore, taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} 2a_1 - 2\alpha m b_2 - \beta m b_2 + \alpha m a_3 - \alpha m b_3 \\ 2a_2 - a_3 - mb_3 \\ 2a_3 \end{pmatrix}.$$

Since m is even, our answer is independent of (α, β):

$$T(A^{-1}M) \cong \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.82}$$

We determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$ in the same way as before. Using Equations (6.81) and (6.82), we have:

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = 4. \tag{6.83}$$

Finally, combining Equations (6.56), (6.65), (6.77), and (6.83), we get the following result:

Theorem 6.2.1. *Let $K = \mathbb{Q}(\sqrt{m}), m \equiv 2 \bmod 4$. Let $\mathfrak{g} = Gal(K/\mathbb{Q}), G = H_3(\mathcal{O}_K), M = \mathcal{O}_K^3$. Then, for a cocycle class $\gamma = [C]$ corresponding to some choice of $\eta \in \Xi_K$, and $(\xi, \theta) = \overline{\Theta}_\eta$, we have the following explicit determination of i_γ :*

$$i_\gamma = \begin{cases} 2 & \text{if } \eta \equiv \omega \bmod 2\mathcal{O}_K, \xi \equiv \omega \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 0 \bmod 2\mathcal{O}_K, \theta \equiv \xi \equiv 0 \bmod 2\mathbb{Z}\omega, \\ 8 & \text{if } \eta \equiv 1 \bmod 2\mathcal{O}_K, \theta \equiv \xi \bmod 2\mathbb{Z}\omega, \\ 4 & \text{otherwise.} \end{cases} \tag{6.84}$$

6.3 $m \equiv 1$ mod 4

6.3.1 A Theorem when $\Xi_K = \mathbb{Z} + n\mathcal{O}_K$

Before we look at the quadratic extension case where $m \equiv 1$ mod 4, we consider the following theorem, which will make our case a corollary.

Theorem 6.3.1. *Let K be a finite Galois extension over $\mathbb{Q}$ of dimension n. Let $\mathfrak{g} = Gal(K/\mathbb{Q}), G = H_3(\mathcal{O}_K), M = \mathcal{O}_K^3$. Then, if $\Xi_K = \mathbb{Z} + n\mathcal{O}_K$, we have:*

$$|M_C/P_C| = |\mathbb{Z}/T(\mathcal{O}_K)|^3.$$

6.3.2 Proof of the Theorem

We start by letting $\eta \in \Xi_K$. We know from our assumption that $\eta = a + n\mu, \mu \in \mathcal{O}_K$. First, we note there will only be n cases we must examine, since if $a \geqslant n$, $a + n\mu = (a-n) + n(\mu' - 1)$. Therefore, we only concern ourselves with $0 \leqslant a \leqslant n$. We now determine Θ_η.

$$\begin{aligned}\Theta_\eta : &= \{(\xi,\theta) \in \Xi_K \times \mathcal{O}_K : n\theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)^\sigma\eta \bmod n^2, \forall\sigma \in \mathfrak{g}\} \\ &= \{(\xi,\theta) : n\theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)(a + n^\sigma\mu) \bmod n^2, \forall\sigma \in \mathfrak{g}\}.\end{aligned}$$

We start with the first case, namely $a = 0$:

$$\Theta_{0+n\mu} := \{(\xi,\theta) \in \Xi_K \times \mathcal{O}_K : n\theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)^\sigma(n\mu) \bmod n^2, \forall\sigma \in \mathfrak{g}\}.$$

We see here that, since $\xi \in \Xi_K, \xi - {}^\sigma\xi \in n\mathcal{O}_K$, the righthand side is 0. This in turn makes the condition on θ equivalent to saying $\theta \equiv {}^\sigma\theta \bmod n$, i.e. $\theta \in \Xi_K \Rightarrow \theta = c + n\zeta, \zeta \in \mathcal{O}_K$. So, we obtain:

$$\Theta_{0+n\mu} = \Xi_K \times \Xi_K.$$

Now, turning our attention to the case where $a \neq 0$:

$$\Theta_\eta := \{(\xi,\theta) \in \Xi_K \times \mathcal{O}_K : n\theta - {}^\sigma(n\theta) \equiv (\xi - {}^\sigma\xi)(a + n^\sigma\mu) \bmod n^2, \forall\sigma \in \mathfrak{g}\}.$$

Again, since $\xi \in \Xi_K, \xi - {}^\sigma\xi \in n\mathcal{O}_K$, we are left with:

$$\Theta_\eta := \{(\xi,\theta) \in \Xi_K \times \mathcal{O}_K : n\theta - {}^\sigma(n\theta) \equiv a(\xi - {}^\sigma\xi) \bmod n^2, \forall \sigma \in \mathfrak{g}\}.$$

We also see that $\xi \in \Xi_K, \xi = b + n\nu$, and plugging in we get:

$$n\theta - {}^\sigma(n\theta) \equiv na\nu - n^\sigma a\nu \bmod n^2.$$

This condition is equivalent to: $\theta - a\nu = {}^\sigma(\theta - a\nu) \bmod n$. We therefore get $\theta - a\nu \in \Xi_K$, meaning $\theta = c + n\zeta + a\nu$ where $\zeta \in \Xi_K$. We then have:

$$\Theta_{a+n\mu} = \{(\xi,\theta) \in \Xi_K \times \mathcal{O}_K : \theta = c + n\zeta + a\nu, c \in \mathbb{Z}, \zeta \in \mathcal{O}_K\}.$$

We will determine M_C/P_C using Theorem 5.6.1; that is, we will determine $(A^{-1}M)^{\mathfrak{g}}$ and $T(A^{-1}M)$.

In using our theorem, we need to look only at $\overline{\Theta}_\eta$. In other words, we take $\eta = a + n\mu \in \Xi_K$ and $(\xi,\theta) = (n\nu, n\zeta + a\nu) \in \overline{\Theta}_\eta$. We now look at $A^{-1}M$. Recall from (5.16):

$$A = \begin{pmatrix} 1 & \frac{\xi}{n} & \frac{\theta}{n} \\ 0 & 1 & \frac{\eta}{n} \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \nu & \zeta + \frac{a\nu}{n} \\ 0 & 1 & \mu + \frac{a}{n} \\ 0 & 1 & 1 \end{pmatrix},$$

which gives us:

$$A^{-1} = \begin{pmatrix} 1 & -\frac{\xi}{n} & -\frac{\theta}{n} + \frac{\xi\eta}{n^2} \\ 0 & 1 & -\frac{\eta}{n} \\ 0 & 1 & 1 \end{pmatrix}, = \begin{pmatrix} 1 & -\nu & -\zeta - \frac{a\nu}{n} + \mu\nu + \frac{a\nu}{n} \\ 0 & 1 & -\frac{a}{n} - \mu \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\nu & -\zeta + \mu\nu \\ 0 & 1 & -\frac{a}{n} - \mu \\ 0 & 1 & 1 \end{pmatrix},$$

and

$$A^{-1}M = \left\{ \begin{pmatrix} x_1 - \nu x_2 - \zeta x_3 + \mu\nu x_3 \\ x_2 - \mu x_3 - \frac{a x_3}{n} \\ x_3 \end{pmatrix}, \text{ where } x_i \in \mathcal{O}_K \right\}. \tag{6.85}$$

For an element to be in $(A^{-1}M)^{\mathfrak{g}}$, we need the following three statements to hold:

$$x_1 - \nu(x_2 - \mu x_3) - \zeta x_3 \in \mathbb{Q}. \tag{6.86}$$

$$x_2 - \mu x_3 - \frac{a}{n}(x_3) \in \mathbb{Q}. \tag{6.87}$$

$$x_3 \in \mathbb{Q}. \tag{6.88}$$

Clearly, (6.88) gives us $x_3 \in \mathbb{Z}$. Knowing this, we have from (6.87) that $x_2 - \mu x_3$ must be rational, but since this is also an algebraic integer, we know $x_2 - \mu x_3 \in \mathbb{Z}$. And, using this fact implies immediately that (6.86) is satisfied, and in fact $x_1 - \nu(x_2 - \mu x_3) - \zeta x_3 \in \mathbb{Z}$.

We write $(A^{-1}M)^{\mathfrak{g}}$ in terms of bases to get:

$$(A^{-1}M)^{\mathfrak{g}} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -\frac{a}{n} \\ 1 \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.89}$$

We now look at $T(A^{-1}M)$. Taking the trace of $A^{-1}M$ gives us:

$$T(A^{-1}M) = \begin{pmatrix} T(x_1) - T(x_2 - \mu x_3) - T(\zeta x_3) \\ T(x_2 - \mu x_3) - \frac{a}{n}T(x_3) \\ T(x_3) \end{pmatrix}.$$

Let $T(\mathcal{O}_K) = d\mathbb{Z}$. We then see that $T(A^{-1}M)$ can be expressed as:

$$T(A^{-1}M) \cong \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix} r_1 + \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} 0 \\ -\frac{a}{n} \\ d \end{pmatrix} r_3, r_i \in \mathbb{Z}. \tag{6.90}$$

We determine the order of $(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)$ using Equations (6.89) and (6.90):

$$|(A^{-1}M)^{\mathfrak{g}}/T(A^{-1}M)| = d^3. \tag{6.91}$$

Since $|\mathbb{Z}/T(\mathcal{O}_K)| = d$, we have proved our theorem.□

6.3.3 Our Corollary

Finally, to complete our determination of the quadratic case, we use the above theorem to prove this corollary.

Corollary 6.3.2. *Let* $K = \mathbb{Q}(\sqrt{m}), m \equiv 1$ *mod* 4. *Let* $\mathfrak{g} = Gal(K/\mathbb{Q}), G = H_3(\mathcal{O}_K), M = \mathcal{O}_K^3$. *Then, for a cocycle class* $\gamma = [C]$ *corresponding to some choice of* $\eta \in \Xi_K$, *and* $(\xi, \theta) = \overline{\Theta}_\eta$, *we have the following determination of* i_γ :

$$i_\gamma = 1 \text{ for all choices of } (\eta, \xi, \theta). \tag{6.92}$$

Proof:

First, we note that from equation (4.7) that $\Xi_K = \mathbb{Z} + 2\mathcal{O}_K$, meaning we have satisfied the condition of Theorem 6.3.1. Secondly, by equation (4.9), we know $T(\mathcal{O}_K) = \mathbb{Z}$. Combining these two equations and using Theorem 6.3.1, we get the result.$\square$

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Vitae

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