SMOOTH WEAK FANO THREEFOLDS

by

Nicholas A. Marshburn

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

April, 2011

©Nicholas Marshburn 2011

All rights reserved

Abstract

In this dissertation, we investigate smooth weak Fano varieties with Picard number 2. We only consider those varieties for which both the unique extremal contractions before and after a flop are divisorial, one of which is of type E1, the other of type E2, E3, E4, or E5. We construct a list of possible numerical invariants for these varieties and show which cases are geometrically realizable. If one contraction is of type E2 we find three possible sets of numerical invariants all of which are geometrically realizable. In case E3/E4, we find seven possibilities, four of which are geometrically realizable. In case E5, we again find seven possibilities, five of which are realizable.

READERS: Dr. V. Shokurov (Advisor), Dr. S. Zucker, Dr. B. Shiffman, Dr. B. Barnett, and Dr. C. Herman.

Acknowledgments

I would like to thank my advisor, V. V. Shokurov, without whose guidance this thesis could not have been written. I must also thank my undergraduate advisor, Patrick Brewer, at Lebanon Valley College for suggesting I pursue a Ph.D. in the first place. Many thanks to the math department staff at JHU, especially Sabrina Raymond and Morris Hunt, for taking care of me during my stay.

I would also like to thank my friend and collaborator, Joe Cutrone, for keeping me company. Thanks to Joe as well as Caleb Hussey, Sinan Ariturk, and Jingzhou Sun for distracting me with various sports. Lastly, I would like to thank the fine workers of Chipotle for their friendly service and for nearly always remembering that my relatively small chicken burritos are $1.80 cheaper (before tax.)

This thesis is dedicated to my Mom Lisa, who may have been better served by kicking me out on the streets, and to my brother Zach, who was a willing outlet for my aggression, both on and off the tennis courts, especially on Saturdays.

This thesis was partially supported by NSF grants DMS-0701465 and DMS-1001427

Contents

Chapter 1

Introduction

In this dissertation we aim to further the classification of threefolds with nef and big anticanonical divisor. These varieties are called *weak Fano* varieties. We additionally assume our weak Fano varieties to be smooth and have Picard number two.

In dimension two over the complex numbers, there is only one smooth weak Fano variety X with Picard number two which is not Fano: the ruled surface $\mathbb{F}_2$. Indeed, by Mori theory in dimension two, there exists an extremal contraction $X \to Y$ which is either (i) the blow up of a smooth point on a del Pezzo surface Y with Picard number one or (ii) a $\mathbb{P}^1$-bundle over $Y \cong \mathbb{P}^1$. In case (i), Y is $\mathbb{P}^2$ by the classification of del Pezzo surfaces and X is $\mathbb{F}_1$, a Fano variety. In case (ii), a simple calculation shows that the $\mathbb{P}^1$-bundle $\mathbb{F}_n$ is weak Fano (with at least one K-trivial curve) if and only if $n = 2$.

In dimension three, the classification problem is more difficult but follows the same basic approach. By the MMP in dimension three, a smooth weak Fano variety X with Picard number two is the blow up of a Fano variety with Picard number one or has the structure of Mori fibration. It can be shown that X has a K_X-trivial birational contraction as well. The case where this second contraction is divisorial is considered in [JPR05]. We will focus on the case where this contraction is small. It is shown in [Ko89] that this contraction induces a flop of X to another smooth weak Fano variety X^+, which itself by the MMP is the blow up of a Fano variety or has the structure of Mori fibration. The resulting diagram is known as a Sarkisov link with center X (or X^+).

A Sarkisov link is one of four types of simple birational transformations between Mori fibrations. Sarkisov conjectured that any birational map of Mori fibrations could be decomposed into a composition of Sarkisov links. Corti [Cor95] verified this claim in dimension three, using a triple of invariants of a birational map of Mori fibrations introduced by Sarkisov called the Sarkisov degree. Corti proved the result by factoring an arbitrary birational map into an elementary link followed by another birational map with lesser Sarkisov degree and verifying that Sarkisov degrees satisfy the descending chain condition. Each elementary link is centered at a (relative) weak Fano variety, not necessarily smooth, so the problem of classifying weak Fano varieties is related to the problem of classifying Sarkisov links.

Jahnke, Peternell, and Radloff [JPR07] numerically classified smooth weak Fano threefolds X with Picard number two for which one side of the corresponding Sarkisov link is a Mori fibration. They found 10 possibilities for X with a conic bundle structure, eight of which they geometrically realized. If X has a del Pezzo fibration, they find 43 numerical possibilities, with 29 shown to exist. Takeuchi [Tak09] went further and completely classified varieties X with the structure of del Pezzo fibration of degree not equal to six. He found 34 deformation families of such varieties. Joseph Cutrone in his Ph.D. thesis [Cu11] numerically classifies varieties X for which both sides of the link are birational contractions of the same type and comments on their geometric realization. We do the same in the case where one contraction takes a divisor to a curve and the other takes a divisor to a point.

Chapter 2

Background

2.1 Basic Definitions

All varieties are assumed to be projective and defined over $\mathbb{C}$.

Let X be a normal variety. A *divisor* on X is a $\mathbb{Z}$-linear combination $D = \Sigma b_i E_i$ of irreducible reduced codimension one subvarieties E_i of X. If all $b_i > 0$, D is said to be *effective*. The *support* $\operatorname{Supp}(D)$ of a nonzero divisor D is $\bigcup E_i$. The abelian group of divisors on X is denoted by $Div(X)$. We may also consider divisors with coefficients in other rings. We consider the $\mathbb{Q}$-linear space $Div_{\mathbb{Q}}(X) = Div(X) \otimes \mathbb{Q}$ of divisors with rational coefficients. The group of invertible sheaves on X is denoted by $\operatorname{Pic} X$.

Two divisors D_1 and D_2 are said to be *linearly equivalent* or reside in the same linear equivalence class (in short, class) if $D_1 - D_2$ is the divisor of a rational function f on X. The *canonical class* K_X of X is an important invariant divisor class of X. On the nonsingular part X^0 of X the canonical divisor K_{X^0} is defined as the divisor of a top rational differential form. The divisor K_X is the natural extension of K_{X^0} to X.

The free $\mathbb{Z}$-module generated by the irreducible curves on X is denoted by $Z_1(X)$.

Assume X is smooth. Then there is a bilinear *intersection form* $Div(X) \times Z_1(X) \to \mathbb{Z}$, where intersection is denoted by $D.C$, such that if a prime divisor D and an irreducible curve C are in general position (i.e. C is not contained in $\mathrm{Supp}(D)$) then $D.C$ is the number of points of intersection of D and C taken with appropriate multiplicities. In fact, $D.C$ is the degree of the pullback of D to the normalization of C. We say two divisors D_1 and D_2 are numerically equivalent, denoted $D_1 \equiv D_2$, if $D_1.C = D_2.C$ for any curve C. The set of numerical equivalence classes of divisors is denoted $N^1(X)$. The *Picard number* ρ_X of X is the rank of $N^1(X)$. Likewise, the set of numerical equivalence classes of curves is denoted $N_1(X)$. The closure of the cone in $N_1(X)$ spanned by effective curves is called the *Mori cone* $\overline{NE}(X)$. A divisor D is *nef* if the intersection of D with any curve is nonnegative. A nef divisor D is *big* if $D^n > 0$, where n $= \dim(X)$. Some multiple of a big divisor induces a birational map.

A morphism $\pi : X \to Y$ is called a *contraction* if the induced sheaf morphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism. A contraction is called $K-trivial$ (resp. K-negative) if $K_X.C = 0$ (resp. $K_X.C < 0$) for all curves C contracted by π. A contraction π is called a *Mori extremal contraction* if π contracts a single K-negative extremal ray R of $\overline{NE}(X)$. That is, a curve C is contracted if and only if the numerical class $[C]$ of C in $N_1(X)$ lies in R.

A variety X is said to have *canonical singularities* if for any proper birational morphism $f : Y \to X$ with exceptional divisors E_i and Y nonsingular (a resolution of singularities) such that $K_Y = f^*K_X + \Sigma a_i E_i$, it follows that $a_i \geqslant 0$ for all i. If the inequality is strict, we say X has *terminal singularities*.

2.1.1 Fano and Weak Fano Threefolds

In this section we introduce the basic objects of study.

Definition 2.1.1. A variety X is *Fano* if $-K_X$ is ample. We suppose that K_X is

$\mathbb{Q}$-Gorenstein.

Definition 2.1.2. The *Fano index* r_X of a Fano variety X is the greatest positive integer n such that a Cartier divisor H exists with $nH \sim -K_X$. The divisor H is called a *fundamental divisor* and $|H|$ is the *fundamental system* on X.

If X is a smooth Fano threefold, then it was shown [Isk78] that Pic X is torsion free. In this case, the fundamental divisor class H is unique.

Smooth Fano threefolds have been classified by Iskovskikh [Isk78] and Mori-Mukai [MM82]. There are 104 deformation families of smooth Fano threefolds. A smooth Fano threefold with Picard number greater than five is a product of a projective line and a smooth del Pezzo surface (2-dimensional Fano variety), and thus has $\rho \leqslant 10$. In this thesis we mainly utilize Fano threefolds with Picard number one.

Theorem 2.1.3 (Isk78)**.** *There are 17 families of smooth Fano threefolds with Picard number one, with index $r \leqslant 4$. These include:*

- *Index 1: X_k, where k is a positive even integer $\leqslant 22$, $k \neq 20$.*
- *Index 2: V_k, where k is a positive integer $\leqslant 5$.*
- *Index 3: $Q \subset \mathbb{P}^4$, a smooth quadric with $-K^3 = 54$.*
- *Index 4: $\mathbb{P}^3$, projective 3-space with $-K^3 = 64$.*

In the index 1 or 2 cases, $-K^3 = r^3k$.

Some smooth Fano threefolds with $\rho \geqslant 2$ are blow ups of smooth Fano threefolds with Picard number one less centered at a point or smooth curve. These threefolds are called *imprimitive*. We investigate the case where the blow up of a smooth Fano threefold on Iskovskikh's list, or more generally the blow up of a singular Fano threefold with Picard number one, is a smooth threefold X with $-K_X$ nef and big, but not ample.

Definition 2.1.4. A smooth variety X is *weak Fano* if $-K_X$ is nef and big.

If X is a smooth weak Fano variety, some multiple $-nK_X$ of the anticanonical class induces a birational morphism $\psi : X \to X'$.

2.1.2 MMP and Mori Fibrations

Let X be a smooth threefold. The Minimal Model Program (MMP) for threefolds concerns the following question: is there a threefold Y, with mild singularities and birational to X, such that K_Y is nef? The answer turns out to be no, but it is instructive to first look at the case of surfaces.

MMP in Dimension 2: Let X be a nonsingular projective surface over $\mathbb{C}$. Then there is a chain $X = X_0 \to X_1 \to X_2 \to ... \to X_n = Y$ of contractions of (-1)-curves (i.e. curves $C \cong \mathbb{P}^1$ with C^2 = -1), where Y has no (-1)-curves. There is a number $\kappa(X) \in \{-\infty, 0, 1, 2\}$, called the *Kodaira dimension* of X, which measures the positivity of the canonical class. One of the following hold:

- $\kappa(X) = -\infty$: K_Y is not nef. Either Y is a del Pezzo surface or there exists a morphism $\pi : Y \to C$ from Y to a smooth curve C such that each fibre is a smooth conic in $\mathbb{P}^2$. Such a morphism is called a *conic bundle*.

- $\kappa(X) \geqslant 0$: K_Y is nef. Some multiple of K_Y induces a morphism $\pi : Y \to Y'$, where the dimension of Y' is $\kappa(X)$.

Using the language of Mori theory allows us to generalize this process to higher dimensions. A (-1)-curve on a nonsingular surface spans an extremal ray, so contracting a (-1)-curve is a Mori extremal contraction. The Castelnuovo Theorem ensures that a (-1)-curve may be contracted, but the contractibility of an extremal ray on a higher-dimensional variety is a trickier question. For threefolds this question is settled by the *contraction theorem* of the MMP. One result of contracting an extremal ray is a Mori fibration, the generalization of del Pezzo surfaces and $\mathbb{P}^1$-bundles to all dimensions.

Definition 2.1.5. A contraction $f : Y \to S$ of Y to a normal variety S is a *Mori fibration* if the following hold:

- Y has at most $\mathbb{Q}$-factorial terminal singularities.
- $\rho(Y/S) = 1$.
- f is K_Y-negative.

The algorithm of the MMP consists of contracting K-negative extremal rays until the canonical class is nef or a Mori fibration is obtained. But in dimension three, we encounter two problems not present in dimension two. First, even if we start with a smooth threefold, the image after an extremal contraction may contain singularities. So the category of varieties must be enlarged to include those with $\mathbb{Q}$-factorial terminal singularities. In fact, this is the smallest category in which the MMP works. Secondly, an extremal contraction $f : X \to Z$ in dimension three can be *small*. That is, the contracted locus is one-dimensional, a union of rational curves. In that case Z is not even $\mathbb{Q}$-factorial so the MMP cannot continue. Indeed, if Z were $\mathbb{Q}$-factorial, then for a contracted curve C, we would have $0 > K_X.C = K_Z.f_*(C) = 0$. To continue the MMP, a small birational modification is made to X.

A flip thus replaces a K-negative extremal ray with a K-positive ray. The MMP can then be continued with X^+. Shokurov (1985) proved that a sequence of flips must terminate and Mori (1988) later proved that flips always exist. Since a divisorial contraction decreases the Picard number by one, these facts imply that the MMP terminates. In what follows, we are only concerned with extremal divisorial contractions of smooth threefolds. Such contractions are not flipping and have been classified.

Theorem 2.1.6. *(Mori (1982)) Let X be a smooth three dimensional projective variety. Let R be an extremal ray on X, and let $\phi : X \to Y$ be the corresponding extremal contraction. Then only the following cases are possible:*

1. *R is not numerically effective. Then $\phi : X \to Y$ is a divisorial contraction of an irreducible exceptional divisor $E \subset X$ onto a curve or a point. In addition, ϕ is the blow-up of the subvariety $\varphi(E)$ (with the reduced structure). All the possible types of extremal rays R which can occur in this situation are listed in the following table, where $\mu(R)$ is the length of the extremal ray R (that is, the number $min\{(-K_X) \cdot C \,|\, C \in R$ is a rational curve$\}$) and l_R is a rational curve*

such that $-K_X \cdot l_R = \mu(R)$ *and* $[l_R] = R$.

Type of R	ϕ *and* E	$\mu(R)$	l_R
E1	$\phi(E)$ *is a smooth curve, and* Y *is a smooth variety*	*1*	*a fiber of a ruled surface* E
E2	$\phi(E)$ *is a point,* Y *is a smooth variety,* $E \simeq \mathbb{P}^2$*, and* $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$	*2*	*a line on* $E \simeq \mathbb{P}^2$
E3	$\phi(E)$ *is an ordinary double point,* $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *and* $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$	*1*	$s \times \mathbb{P}^1$ *or* $\mathbb{P}^1 \times t$ *in* E
E4	$\phi(E)$ *is a double (cDV)-point,* E *is a quadric cone in* $\mathbb{P}^3$*, and* $\mathcal{O}_E(E) \simeq \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$	*1*	*a ruling of cone* E
E5	$\phi(E)$ *is a quadruple non Gorenstein point on* Y*,* $E \simeq \mathbb{P}^2$*, and* $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$	*1*	*a line on* $E \simeq \mathbb{P}^2$

2. *R is numerically effective. Then* $\phi : X \to Y$ *is a Mori fibration,* Y *is nonsingular, dim* $Y \leqslant 2$*, and all the possible situations are the following:*

   (a) *dim* $Y = 2$*: Then* $\phi : X \to Y$ *is a standard conic bundle.*

   (b) *dim* $Y = 1$*: Then* $\phi : X \to Y$ *is a del Pezzo fibration.*

   (c) *dim* $Y = 0$*: Then* X *is a smooth Fano variety with Picard number one.*

See [IP99] for more details regarding the non divisorial cases.

A related birational construction to the flip is the flop. A *flop* is a diagram as above with both f and f^+ K-trivial morphisms. Two simple types of flop are the following:

- The diagram (**??**) is a *simple Atiyah flop* if the exceptional locus of f is a rational curve with normal bundle of type $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. An Atiyah flop is resolved by first blowing up the flopping curve on X. The resulting exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle of type $(-1,-1)$, and thus may be blown down in a different direction. Performing this blow down results in X^+. The flopping curve is contracted to an ordinary double point on Z. If the exceptional locus of a flop consists of disjoint rational curves, each with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, it is called an *Atiyah flop*.

- *Reid's Pagoda* is a type of flop with a rational flopping curve with normal bundle of type $\mathcal{O}(-2) \oplus \mathcal{O}$. The flopping curve C is contracted by f to a singular point of type $xy = t^2 + z^{2k}$. The resolution of Reid's Pagoda is attained by blowing up C and its strict transforms k times, and then blowing down the k exceptional divisors in reverse order.

Every flop appearing in this dissertation is an Atiyah flop. A general flop may be much more complicated.

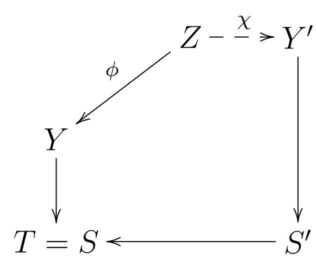
2.1.3 Sarkisov Links

Suppose Y/S and Y'/S' are threefold Mori fibrations connected by a birational map χ:

(2.1)
$$\begin{array}{ccc} Y & \overset{\chi}{\dashrightarrow} & Y' \\ \downarrow & & \downarrow \\ S & & S' \end{array}$$

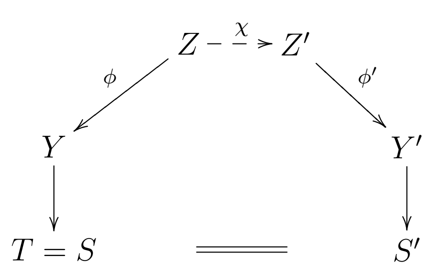
We wish to decompose χ into elementary birational maps called *links*. Sarkisov conjectured and Corti [Cor95] verified that χ can be decomposed into four types of elementary links listed below:

Type I:



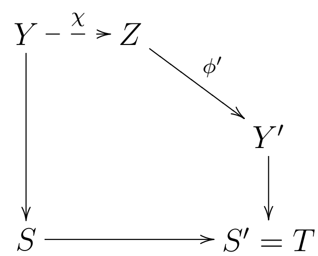
ϕ is an extremal divisorial contraction and χ is a composition of anti-flips, a flop, and flips, each of which may not occur in a particular link. The relative Picard number $\rho(S/S')$ is one.

Type II:



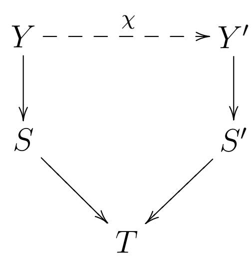
ϕ and ϕ' are extremal divisorial contractions and χ is a composition of anti-flips, a flop, and flips, each of which may not occur in a particular link. Note that S and S' are isomorphic.

Type III (inverse to type I):



ϕ' is an extremal divisorial contraction and χ is a composition of anti-flips, a flop, and flips. The relative Picard number $\rho(S'/S)$ is one.

Type IV:



χ is a composition of flips, a flop, and anti-flips. T is a normal variety and $\rho(S/T) = \rho(S'/T) = 1$.

Chapter 3

E1-E* Case

3.1 Setup

Our aim is to classify smooth weak Fano threefolds with Picard number 2. We first recall the classification of Fano threefolds X with Picard number two by Mori and Mukai [MM82]. There are 36 families of such threefolds. The Mori cone of X is two-dimensional and hence polyhedral with two K_X-negative extremal rays. Each ray gives rise to an extremal contraction, each of which is a birational contraction, a conic bundle, or a del Pezzo fibration by Mori's classification of extremal contractions on smooth threefolds. The classification of smooth Fano threefolds with Picard number two follows by studying how the two contractions interact. We aim to do the same with regards to weak Fano threefolds.

Throughout this dissertation, we study smooth complex projective 3-folds X satisfying the conditions below. Assumptions (i), (iii), (iv), (v), and (vi) are nonstandard for weak Fano varieties and will be explained later.

(i) X is smooth.

(ii) $-K_X$ is nef and big (i.e. X is *weak Fano*).

(iii) X has a finite non-zero number of K-trivial curves (i.e. $-K_X$ is big in codim 1).

(iv) $\rho(X) = 2$.

(v) The linear system $|-K_X|$ is basepoint free.

(vi) The Fano index r_X of X is 1.

Assumptions (ii) and (iv) above imply that X has two extremal contractions: a K_X-negative contraction ϕ and a K_X-trivial contraction ψ. Assumptions (iii) and (v) imply that ψ is a small nontrivial birational contraction induced by the linear system $|-K_X|$ (the *anticanonical contraction*). By [Ko89], ψ induces a flop χ, and we obtain the following diagram:

(3.1)

$$\begin{array}{ccccc} X & & \overset{\chi}{\dashrightarrow} & & X^+ \\ \downarrow{\scriptstyle \phi} & \searrow^{\psi} & & {}^{\psi^+}\swarrow & \downarrow{\scriptstyle \phi^+} \\ Y & & X' & & Y^+ \end{array}$$

In the above diagram, χ is a flop which is an isomorphism outside of $\mathrm{Exc}(\psi)$ and X^+ satisfies the conditions (i)-(vi) above. The morphism ϕ^+ is a K_{X^+}-negative extremal contraction and ψ^+ is the anticanonical morphism. The variety X' is a terminal Gorenstein Fano threefold with Picard number one, but is not $\mathbb{Q}$-factorial since ψ is small. Note that, in contrast with the case of Fano varieties with Picard number two, we must perform a flop before obtaining a second extremal contraction.

Jahnke, Peternell, and Radloff [JPR05] considered the case where the anticanonical contraction ψ is divisorial. These cases do not result in Sarkisov links. We therefore make assumption (iii) above.

By the classification of extremal contractions on smooth threefolds by Mori, ϕ and ϕ^+ are birational contractions, conic bundles, or del Pezzo fibrations. Jahnke, Peternell, and Radloff [JPR07] numerically classified weak Fano threefolds with a conic bundle or del Pezzo fibration structure on at least one side of the corresponding Sarkisov link. They found ten possibilities for X with a conic bundle structure, eight of which they geometrically realized. If X has a del Pezzo fibration, they find 43 numerical possibilities, with 29 shown to exist. Takeuchi [Tak09] went further and completely classified varieties X with the structure of del Pezzo fibration of degree not equal to six. He found 34 deformation families of such varieties.

The list of all possibilities for Sarkisov links centered at smooth weak Fano threefolds that can numerically exist was determined for all combinations of extremal contractions ϕ and ϕ^+ except for the case when both rays are divisorial contractions. We complete the numerical classification of weak Fano threefolds X with Picard number two in the case where both extremal contractions ϕ and ϕ^+ are divisorial, assuming ϕ contracts a divisor to a smooth curve (an E1 contraction) and ϕ^+ contracts a divisor to a point (an E2,E3,E4, or E5 contraction). In addition, we establish the existence of these cases, providing specific constructions for existence or proving non-existence. Our main result is the following:

Theorem 3.1.1. *For a geometrically realizable link (3.1), if ϕ contracts a divisor to a smooth curve and ϕ^+ contracts a divisor to a (possibly singular) point, then the numerical invariants associated to the link are found in one of the tables (3.6.1), (3.6.2), and (3.6.3). Twelve families of these links exist.*

3.2 Assumptions

To classify links (3.1), we need to find relations between the invariants associated to X and Y on the left side of the flop and X^+ and Y^+ on the right side of the flop. The goal is to write a computer program to find a finite list of values of these invariants satisfying the relations. A more detailed case-by-case check can then reveal which sets of invariants are associated to geometrically realizable links. To this end we start with a lemma:

Lemma 3.2.1. *Let D be any divisor which is not ψ-nef, i.e $-D$ is ψ-ample. Then the D-flop of ψ exists, i.e. a small birational map $\psi^+ : X^+ \to X'$ such that the strict transform $\widetilde{D} \subset X^+$ is ψ^+-ample. Moreover, X^+ is smooth with $-K_{X^+}$ big and nef and*

$$\rho(X^+) = 2$$
$$(-K_X)^3 = (-K_{X^+})^3$$
$$h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_{X^+}(\widetilde{D}))$$

Proof. See Proposition 2.2 in [JPR07]. □

By [JPR07], $|-K_X|$ is base point free unless X' is a deformation of the Fano threefold V_2. In this case X has the structure of del Pezzo fibration, so does not give rise to a link between Fano threefolds. We henceforth assume $|-K_X|$ to be base point free. This is assumption (v) above.

Let r_X be the Fano index of X. Recall that r_X is the largest positive integer such that an integral divisor H_X exists such that $K_X = r_X H_X$. It is also the Fano index of X', which has Picard number one. By [Shi89], X' has a smoothing which preserves the Fano index. Thus by Iskovskikh's classification, $r_X \leqslant 4$. If $r_X = 4$, then in fact $X' \cong \mathbb{P}^3$, an impossibility since the images of the flopping curves on X' are singular points. If $r_X = 3$, then X' is a quadric in $\mathbb{P}^4$. It is shown in [JPR07] that X' is the cone over a smooth quadric and X/Y and X^+/Y^+ are del Pezzo fibrations. The case $r_X = 2$ is handled in [JP06]. In this case X and X^+ are both $\mathbb{P}^1$-bundles, quadric bundles, or E2 contractions. So we may assume $r_X = 1$. This is assumption (vi) above.

By [Isk78], the anticanonical contraction $\sigma : X' \to W$ is generically finite and has degree one or two. In the latter case, X' (and X) is called *hyperelliptic*. In [JPR07] it is shown that if X is hyperelliptic, then $-K_X^3 \leqslant 8$. In fact, if $-K_X^3 = 2$, then X is always hyperelliptic. Indeed, in this case X' is a double cover of $\mathbb{P}^3$. If X is hyperelliptic, then the flop χ takes a particularly simple form. It is the birational involution on X induced by σ and in particular, $X \cong X^+$. The contractions ϕ and ϕ^+ then have the same type.

3.3 Equations and Bounds

In this section, we begin the classification of weak Fano threefolds X in the diagram (3.1) where ϕ is of type E1 and ϕ^+ is of type E2 - E5.

We define the following notation:

- r, the Fano index of Y.
- $E \subset X$, the exceptional divisor $\phi^{-1}(C)$.
- $E^+ \subset X^+$, the exceptional divisor $(\phi^+)^{-1}(P)$.
- $H \subset Y$, a fundamental divisor (i.e. $rH = -K_Y$).
- H will also denote the pullback of $H \subset Y$ to X.
- $C \subset Y$, the smooth curve blown up by ϕ.
- g, the genus of C.

- d, the degree of C with respect to H.

- $\widetilde{D}$, the strict transform of a divisor D across the flop χ.

Note that since χ is small, $\widetilde{K_X} = K_{X^+}$.

The Picard group $\mathrm{Pic}(Y)$ is generated by H, so $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by $H \subset X$ and E. In the following we will instead consider the divisors $-K_X$ and E, which do not form a basis of $\mathrm{Pic}(X)$ over $\mathbb{Z}$ if $r \neq 1$. Thus it will be necessary to express some divisors on X as rational combinations of $-K_X$ and E. Since $-K_X = \phi^*(-K_Y) - E = rH - E$, the denominators of the rational coefficients will divide r.

We define

$$\widetilde{E^+} = \alpha(-K_X) + \beta E \tag{3.2}$$

for some $\alpha, \beta \in \mathbb{Z}/r$.

Similarly, we let

$$\widetilde{E} = \alpha^+(-K_{X^+}) + \beta^+ E^+ \tag{3.3}$$

for some α^+, β^+.

We will see below that in fact $\alpha^+, \beta^+ \in \mathbb{Z}$.

From (3.2) and (3.3), we easily obtain the following relations:

$$\beta\beta^+ = 1, \;\; \alpha + \beta\alpha^+ = \alpha^+ + \beta^+\alpha = 0 \tag{3.4}$$

It is easy to see that $\alpha, \alpha^+ > 0$ and $\beta, \beta^+ < 0$. It is actually possible to get an exact formula for β and β^+ if ϕ^+ is of type E2. First note that the Fano index r^+ of Y^+ is 1. Indeed, if $r^+ = 2$, then K_{X^+} (and thus K_X) would be imprimitive, and if r^+

is 3 or 4, then X^+ would be Fano. Thus $\mathrm{Pic}(X)$ is generated by $-K_{X^+}$ and E^+. This fact is used in the following lemma:

Lemma 3.3.1. *Using the notation as above, if $X^+ \xrightarrow{\phi^+} Y^+$ is an $E2$ contraction, then $\beta^+ = -r$ and from (3.4), $\beta = \frac{-1}{r}$.*

Proof. Since ϕ is the blow up of a smooth curve, we have $-K_X = rH - E$. Combining this with (3.3), we can rewrite $\widetilde{E}$ as follows:

$$\begin{aligned} \widetilde{E^+} &\cong \alpha(rH - E) + \beta E \\ &\cong \alpha r H + (\beta - \alpha)E. \end{aligned}$$

Thus

$$\begin{aligned} &\mathrm{Pic}(X^+)/\langle -K_{X^+}, E^+\rangle \\ &\cong \mathrm{Pic}(X)/\langle -K_X, \widetilde{E^+}\rangle \\ &\cong \mathrm{Pic}(X)/\langle rH - E, \alpha r H + (\beta-\alpha)E\rangle \end{aligned}$$

By the remark before the lemma, the top group is trivial. Since $\widetilde{E^+}$ is integral, it follows that αr and $\beta - \alpha$ are integers. Comparing orders yields:

$$1 = \begin{vmatrix} \alpha r & r \\ \beta - \alpha & -1 \end{vmatrix} = -\alpha r - r\beta + \alpha r = -r\beta$$

and thus $\beta = \frac{-1}{r}$ as desired. □

In general, for divisors D_1, D_2, D_3 on X, the intersection numbers $D_1.D_2.D_3$ on X and $\widetilde{D_1}.\widetilde{D_2}.\widetilde{D_3}$ on X^+ will be different. The following theorem enumerates the cases where intersection numbers are preserved by the flop χ.

Lemma 3.3.2. *For any divisors D and D' on X:*

1. $K_X.D.D' = K_{X^+}.\widetilde{D}.\widetilde{D'}$.

2. *Let C be a curve in X such that C is disjoint from any flopping curves in X. Then $\widetilde{C}$ is disjoint from any flopping curves in X^+ and $D.C = \widetilde{D}.\widetilde{C}$.*

Proof. Since $-K_X$ is assumed to be basepoint free, a general anticanonical divisor on X will not intersect any flopping curves. In fact, the anticanonical divisors on X are pullbacks of anticanonical divisors on X', where the flopping curves are contracted to finitely many points. The theorem then follows from the projection formula. □

Using 3.3.2, we may compare intersection number on both sides of the flop. We use the following well known formulas (see e.g. [IP99]).

$$\begin{aligned} E^3 &= -rd + 2 - 2g. \\ K_X^2.E &= rd + 2 - 2g. \\ K_X.E^2 &= 2 - 2g. \\ -K_Y^3 &= -K_X^3 + 2rd + 2 - 2g. \end{aligned} \tag{3.5}$$

Define $\sigma := K_X^2.E$. By assumption (iii), K_X is big in codimension 1, so it follows that $\sigma > 0$.

The goal is to use the relations above to find Diophantine equations involving invariants of X and X^+, find all solutions using a computer program, and then check their geometric realization. The first observation to note is that regardless of $\phi^+ : X^+ \to Y^+$ is of type E2, E3, E4, or E5, we always have the following relation:

$$K_X.(\widetilde{E^+})^2 = 2. \tag{3.6}$$

Using the known values in each case for $E^+|_{E^+}$ in Theorem 2.1.6, we can easily see the above equality using adjunction as follows:

$$\begin{aligned} K_X.(\widetilde{E^+})^2 &= K_{X^+}.(E^+)^2 \\ &= K_{X^+}|_{E^+}.E^+|_{E^+} \\ &= (K_{E^+} - E^+|_{E^+}).E^+|_{E^+} \\ &= 2. \end{aligned}$$

For each non E1 contraction, it is easy to show the following intersection numbers using similar reasoning as above (see Lemma 4.1.6 in [IP99]):

$$K^2_{X^+}.E^+ = \begin{cases} 4 & \text{if } E2. \\ 2 & \text{if } E3/E4. \\ 1 & \text{if } E5. \end{cases}$$

Now from the following equalities:

$$\begin{gathered} K_X.(\widetilde{E^+})^2 = 2; \\ K^2_X.\widetilde{E^+} = K^2_{X^+}.E^+; \\ K_X.E^2 = K_{X^+}.(\widetilde{E})^2; \\ K^2_X.E = K^2_{X^+}.\widetilde{E}; \end{gathered} \tag{3.7}$$

using (3.2), (3.3), and (3.5), we have the following system of Diophantine equations:

$$\alpha^2(K_X)^3 - 2\alpha\beta rd + (2-2g)(-2\alpha\beta + \beta^2) = 2.$$

$$\alpha(-K_X)^3 + \beta(rd + 2 - 2g) = \begin{cases} 4 & \text{if } E2. \\ 2 & \text{if } E3/E4. \\ 1 & \text{if } E5. \end{cases}$$

(3.8)
$$2 - 2g = (\alpha^+)^2(K_X)^3 - 2\alpha^+\beta^+ \cdot \begin{cases} 4 & \text{if } E2. \\ 2 & \text{if } E3/E4. \\ 1 & \text{if } E5. \end{cases} + 2(\beta^+)^2$$

$$rd + 2 - 2g = \alpha^+(-K_X)^3 + \beta^+ \cdot \begin{cases} 4 & \text{if } E2. \\ 2 & \text{if } E3/E4. \\ 1 & \text{if } E5. \end{cases}$$

To run a computer program, we still have to compute the necessary bounds on α^+ and β^+ since Lemma 3.3.1 only applies if $\phi^+ : X^+ \to Y^+$ is an $E2$ contraction.

We first bound β^+. We know that $r\beta \in \mathbb{Z}$. Using (3.4), this becomes $\frac{r}{\beta^+} \in \mathbb{Z}$, so $|\beta^+| \leqslant r$.

Now we bound both d and g. From $22 \geqslant (-K_X)^3 = (-K_Y)^3 - \sigma - rd \geqslant 2$, we get

$$d \leqslant \frac{(-K_Y)^3 - 3}{r} \leqslant 19 \text{ and } \sigma \leqslant (-K_Y)^3 - rd - 2 \leqslant \begin{cases} 19, & r = 1 \\ 36, & r = 2 \\ 49, & r = 3 \\ 58, & r = 4 \end{cases}$$

Lastly, $\sigma = rd - 2g + 2$ and $\sigma > 0$ imply that $g \leqslant \frac{19r}{2} + 1$.

Using the last Diophantine equation above and the bounds for d and g computed above, we can now bound α^+. Since $\sigma > 0$, we have

$$rd + 2 - 2g = 2rd - \sigma + 4 - 4g \leqslant 2rd + 4 \leqslant 2 \cdot 4 \cdot 19 + 4 = 156$$

Therefore

$$\alpha^+(-K_X)^3 + \beta^+ \cdot \begin{cases} 4 & \text{if } E2 \\ 2 & \text{if } E3/E4 \\ 1 & \text{if } E5 \end{cases} \leqslant 140$$

Rearranging the terms, and using the fact that $|\beta^+| \leqslant r$, we have that

$$\alpha^+(-K_X)^3 \leqslant 156 + 16 = 172,$$

so using the lower bound on $(-K_X)^3$, we have now shown the following:

Proposition 3.3.3. *For an E2, E3, E4 or E5 contraction,* $0 < \alpha^+ \leqslant 86$ *and* $|\beta^+| \leqslant r$. *Since* $\beta^+ < 0$, $-r \leqslant \beta^+ \leqslant -1$

We need one more proposition:

Proposition 3.3.4. *If* $\phi^+ : X^+ \to Y^+$ *is an* $E2, E3, E4$ *or* $E5$ *type contraction, then* α^+ *and* β^+ *are integers.*

Proof. When ϕ^+ *is E2*: If ϕ^+ is E2, then Y^+ is a smooth Fano threefold with Fano index 1 and Picard number 1. Shokurov has shown that such a variety contains a line l^+. Then $-K_{X^+}.(\phi^+)^*(l^+) = -K_{Y^+}.l^+ = 1$ and $E^+.(\phi^+)^*(l^+) = 0$. Therefore $\mathbb{Z} \ni \widetilde{E}.(\phi^+)^*(l^+) = (\alpha^+(-K_{X^+}) + \beta^+E^+).(\phi^+)^*(l^+) = \alpha^+$. Therefore $\widetilde{E} + \alpha^+K_{X^+} = \beta^+E^+$ is Cartier, so $\beta^+ \in \mathbb{Z}$.

When ϕ^+ *is E3,E4,or E5*: Let ψ^+ be the flopping contraction. Then ψ^+ restricted to

E^+ is a finite birational morphism, so $H^0(X^+, -K_{X^+}) \to H^0(X^+, -K_{X^+}|_{E^+})$ is surjective. Indeed, it can be checked that any proper subsystem of $|-K_{X^+}|_{E^+}|$ does not induce a birational morphism. Therefore $\psi^+|_{E^+}$ is an embedding since $-K_{X^+}|_{E^+}$ is very ample. Thus the intersection of E^+ with any flopping curve is 1. Let Γ be a flopping curve. Then $\mathbb{Z} \ni \widetilde{E}.\Gamma = (\alpha^+(-K_{X^+}) - \beta^+ E^+).\Gamma = \beta^+$. So $\widetilde{E} - \beta^+ E^+ = \alpha^+(-K_{X^+})$ is Cartier and the index of X^+ is 1, and thus $\alpha^+ \in \mathbb{Z}$. □

3.4 Numerical Checks

In this section we prove or mention more numerical constraints. First we must check that $-K_Y^3$ is a possible degree of a smooth Fano threefold with Picard number 1 found on Iskovskikh's list. Using the formulas in (3.2) and (3.3), we also need to make sure both $\widetilde{E}^3$ and $\widetilde{E^+}^3 \in \mathbb{Z}$.

Define

$$e = E^3 - \widetilde{E}^3 \tag{3.9}$$

The integer e is related to the number of flopping curves, and in fact is equal to this number if the flop is a simple Atiyah flop and if for each flopping curve Γ we have $H.\Gamma = 1$. At present, there is no known upper bound. In our tables, we still have open examples with very large values of e. If a strict upper bound could be found, then this bound could be used to eliminate the geometric realization of some open cases.

We have the following lemmas:

Lemma 3.4.1. *[Tak02] The correction term e in (3.9) is a strictly positive integer*

Proof. See the proof of Lemma 3.1 in [Kal09]. □

Lemma 3.4.2. *The number r^3 divides e.*

Proof. From the equality $E = rH + K_X$ and (3.3.2), we obtain $e = r^3(H^3 - \widetilde{H}^3)$. $\square$

Lemma 3.4.3. *If χ is an Atiyah flop, then $e = \Sigma_\Gamma (E.\Gamma)^3$, where the sum is taken over all flopping curves $\Gamma \subset X$.*

Proof. We prove the lemma in the case of a simple Atiyah flop. The general case is similar. The flop χ is resolved by blowing up the flopping curve $\Gamma \subset X$ and contracting the resulting exceptional divisor $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in a different direction.

(3.10)
$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & & \searrow^{g} & \\ X & & \overset{\chi}{\dashrightarrow} & & X^+ \end{array}$$

Let E_Z denote the strict transform of E on Z. Let $a = E.\Gamma$. Then $f^*(E) = E_Z$ and $g^*(\widetilde{E}) = E_Z + aD$. We have

$$\begin{aligned} e &= E^3 - \widetilde{E}^3 \\ &= f^*(E)^3 - g^*(\widetilde{E})^3 \\ &= E_Z^3 - (E_Z + aD)^3 \\ &= -3aE_Z^2 D - 3a^2 E_Z D^2 - a^3 D^3. \end{aligned} \tag{3.11}$$

An easy check verifies that $E_Z^2 D = 0$, $E_Z D^2 = -a$, and $D^3 = 2$. Thus $e = a^3$, as claimed. $\square$

Since $\widetilde{E^+} = \alpha(-K_X) + \beta E$, by replacing $-K_Y$ with rH and $-K_X$ with $-K_Y - E =$

$rH - E$, we have that

$$\widetilde{E^+} = \alpha r H + (\beta - \alpha)E. \tag{3.12}$$

Since $\widetilde{E^+} \in \mathrm{Pic}(X)$ is primitive, we have the following numerical checks:

Proposition 3.4.4. *The numbers $\alpha r, \alpha^+ r^+, \alpha - \beta, \alpha^+ - \beta^+ \in \mathbb{Z}$ and GCD($\alpha r, \beta - \alpha$) = 1. Similarly, GCD($\alpha^+ r^+, \beta^+ - \alpha^+$)=1.*

The base X' of the flop χ is a terminal threefold with Picard number one and since the divisors on X' exactly correspond to the divisors on X, X' has Weil Picard number two. In particular, X' is not $\mathbb{Q}$-factorial. If $-K_X^3 = 4$, then X' is a quartic hypersurface in $\mathbb{P}^4$. In that case, Cheltsov [Chel06] gives a condition on the singular points of X' for it to be $\mathbb{Q}$-factorial.

Theorem 3.4.5. *[Chel06] If $X' \subset \mathbb{P}^4$ is a nodal quartic hypersurface with fewer than 9 singular points, then X' is $\mathbb{Q}$-factorial.*

3.5 Existence of Cases

With the equations and bounds above we may write computer programs to numerically classify links (3.1) with ϕ an E1 contraction and ϕ^+ of type E2,E3/E4, or E5. To exhibit some of these cases geometrically, we use nonsingular del Pezzo surfaces S_k of degree $k = 3, 4, 5$. These surfaces are blow ups of $\mathbb{P}^2$ at $9-k$ general points $P_1, ..., P_{9-k}$. Denote by l the class of the pullback of a line on $\mathbb{P}^2$ to S_k. Denote by e_i the exceptional curves on S_k of the blowup $S_k \to \mathbb{P}^2$. Then l and the e_i generate Pic(S) and there is a nonsingular curve linearly equivalent to a divisor D if and only if $D.L \geqslant 0$ for every line L on S_k and $D^2 > 0$ (See e.g. [Ha77] for the case of the cubic in $\mathbb{P}^3$). The lines on S_k are the e_i, the strict transforms $f_{i,j} \equiv l - e_i - e_j$ of lines in $\mathbb{P}^2$ passing through two of the blown up points P_i and P_j, and the strict transforms of conics in $\mathbb{P}^2$ passing through five of the P_i. In the case of S_4, the strict transform of the unique conic through $P_1, ..., P_5$ will be denoted by $g \equiv 2l - \Sigma_{i=1}^{5} e_i$; on S_3, $g_j \equiv 2l - \Sigma_{i \neq j} e_i$ will denote the strict transform of the conic passing through all P_i with $i \neq j$.

3.5.1 E1-E2

Since X, X^+, Y, and Y^+ are smooth in this case, we may perform one additional check. The Hodge numbers $h^{1,2}$ are preserved by flops and blow ups of points. Blowing up a curve of genus g on Y increases $h^{1,2}$ by g. We now have to check that

$$h^{1,2}(Y) + g = h^{1,2}(Y^+).$$

The Hodge numbers of Y and Y^+ are found in Iskovskikh's list. After running the program and performing the aforementioned checks, there are only three solutions in the $E1-E2$ case. These can be found in (3.6.1) in the Tables section of this paper. All three of these solutions are well known examples of smooth weak Fano threefolds with Picard number two and their existence is described in [Tak89]. While their existence may have been previously known, the new result here is that these are the only three examples of contractions of type $E1 - E2$.

3.5.2 E1-E3/E4

Numerically, the cases $E1 - E3$ and $E1 - E4$ are equivalent and so we will treat them as such in this paper. In the examples below, we always get a a link of type $E1 - E3$. We have not found any examples of a link of type $E1 - E4$. Since the Fano threefold Y^+ now has a singular point, we can not use any of the Hodge number checks as we did in the previous examples. Running a computer program gives seven possible numerical cases which are listed on Table 3.6.2. We proceed case by case:

Cases 1 and 2: We eliminate case 1; case 2 is handled similarly. Since E^+ is an exceptional divisor and since χ preserves linear equivalence, the linear system of $\widetilde{E^+}$ on X is zero-dimensional. Rewriting 3.2 with $\alpha = 1.5$ and $\beta = -0.5$ in terms of H and E, we get

$$\widetilde{E^+} = 3H - E.$$

Thus the linear system $|\widetilde{E^+}|$ corresponds to the linear system $|3H-C|$ on Y of effective divisors linearly equivalent to $3H$ containing C. We have

$$|3H - C| \supset |2H - C| + |H|$$

and $\dim|3H - C| = 0$ and $\dim|H| > 0$, so $|2H - C|$ must be empty. We will see below that this is not the case. Twisting the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$$

by $2H$ results in

$$0 \to \mathcal{I}_C(2H) \to \mathcal{O}_Y(2H) \to \mathcal{O}_C(2H) \to 0.$$

By the long exact sequence of cohomology,

$$h^0(\mathcal{O}_Y(2H - C)) = h^0(\mathcal{I}_C(2H)) \geqslant h^0(\mathcal{O}_Y(2H)) - h^0(\mathcal{O}_C(2H)).$$

By Riemann-Roch on C, $h^0(\mathcal{O}_C(2H)) = 10$. And $h^0(\mathcal{O}_Y(2H)) = 11$ by the following formula found in [Isk78]:

$$h^0(\mathcal{O}_Y(jH)) = \frac{j(r+j)(r+2j)}{12}H^3 + \frac{2j}{r} + 1.$$

Thus $h^0(\mathcal{O}_Y(2H-C)) > 0$, so $|2H-C|$ is nonempty and this case does not exist.

Case 3: To eliminate this case, consider the short exact sequence:

$$0 \to \mathcal{O}_X(-K_X - E) \to \mathcal{O}_X(-K_X) \to \mathcal{O}_X(-K_X|_E) \to 0$$

Then since $\alpha = -\beta = 1$, $\widetilde{E^+} = -K_X - E$. So the long exact sequence of cohomology gives:

$$0 \to H^0(X, \mathcal{O}_X(-K_X - E)) \to H^0(X, \mathcal{O}_X(-K_X)) \to H^0(X, \mathcal{O}_X(-K_X|_E)) \to \dots$$

Since we are in the $E3/E4$ case, $\mathcal{O}_X(-K_X|_E) \cong \mathcal{O}_E(1)$ thus $h^0(X, \mathcal{O}_X(-K_X|_E)) = 4$. Since $h^0(X, \mathcal{O}_X(-K_X)) = \frac{-K_X^3}{2} + 3 = 6$, by exactness we must have

$$h^0(X, \mathcal{O}_X(-K_X - E)) = h^0(X, \mathcal{O}_X(\widetilde{E^+})) \geqslant 2$$

which is a contradiction.

Case 4: On any $S_3 \subset \mathbb{P}^3$ there is a nonsingular curve C linearly equivalent to $10l - 4e_1 - 4e_2 - 4e_3 - 4e_4 - 3e_5 - 3e_6$. The curve C has genus 6 and degree 8. Let X be the blowup of $C \subset Y$ and let $\widetilde{S}$ denote the strict transform of S. Suppose a curve $\Gamma \subset X$ satisfies $-K_X.\Gamma \leqslant 0$. Since $-K_X = \widetilde{S} + H$, we see that $\widetilde{S}.\Gamma < 0$, so $\Gamma \subset \widetilde{S} \cong S$. Writing $-K_X = 4H + E$ and $\Gamma \sim al - \sum b_i e_i$ in $\widetilde{S} \cong S$, we obtain $4 \cdot \deg_{\mathbb{P}^3}(\Gamma) \leqslant \Gamma.C$, which simplifies to

$$2a \leqslant b_5 + b_6$$

The only curves which satisfy this inequality are e_1, e_2, e_3, e_4 and $f_{5,6}$, where $f_{5,6}$ is the strict transform of the line through P_5 and P_6. For each of these curves, equality is attained, so X is a weak Fano threefold with five flopping curves e_1, e_2, e_3, e_4 and $f_{5,6}$. Using

$$0 \to \mathcal{N}_{\Gamma/\widetilde{S}} \to \mathcal{N}_{\Gamma/X} \to \mathcal{N}_{\widetilde{S}/X}|_\Gamma \to 0$$

we see that the normal bundle of each flopping curve is $\mathcal{O}(1) \oplus \mathcal{O}(1)$, so the flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop. $\chi(\widetilde{S})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ since it is the result of contracting the flopping curves on $\widetilde{S}$. The sheaf associated to $(\chi(\widetilde{S}))^2$, or equivalently the normal bundle of $\chi(\widetilde{S})$, is then $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$ for some integers a and b. Choosing a fiber f in $\chi(\widetilde{S})$ not passing though any of the points of intersection of the flopping curves with $\chi(\widetilde{S})$, we see that $a = f.\chi(\widetilde{S})$. Since the flop χ restricted to $\widetilde{S}$ is just the blow-down of the five flopping curves, $f.\chi(\widetilde{S})$ is preserved under the flop so to compute a we will instead compute $\widetilde{f}.\widetilde{S}$. Using the fact that $\widetilde{S} = 3H - E$ and $\widetilde{f} \sim l - e_5$, we can compute the intersection inside of the cubic surface $\widetilde{S}$. Since $H|_{\widetilde{S}} \sim 3l - \sum e_i$ and $E|_{\widetilde{S}} = C \sim 10l - 4e_1 - 4e_2 - 4e_3 - 4e_4 - 3e_5 - 3e_6$, we obtain $a = f.\chi(\widetilde{S}) = \widetilde{f}.\widetilde{S} = -1$. A similar calculation shows that $b = -1$ and thus the normal bundle of $\chi(\widetilde{S})$ is $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$, so X^+ has an $E3$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 5: Since $-K_Y^3 = 54$, $Y \cong Q$, a smooth quadric threefold in $\mathbb{P}^4$. Let $S \in |2H| \subset Y$ be a smooth surface. S is the complete intersection of two quadrics in $\mathbb{P}^4$ so is del Pezzo of degree $K_S^2 = 4$. There is a nonsingular curve C with degree 8 and genus 4 linearly equivalent to $5l - 2e_1 - 2e_2 - e_3 - e_4 - e_5$ on S. We proceed using the techniques and notation of case 4 above. We obtain

$$2a \leqslant b_4 + b_5$$

so X is a weak Fano threefold with four flopping curves e_1, e_2, e_3 and $f_{4,5}$. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E3$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 6: Since $-K_Y^3 = 32$, $Y \cong Q_1 \cap Q_2 \subset \mathbb{P}^5$, the smooth intersection of two quadrics. Let $S \in |H| \subset Y$ be a smooth surface. S is the complete intersection of two quadrics in $\mathbb{P}^4$ so is del Pezzo of degree $K_S^2 = 4$. There is a nonsingular curve C with degree 4 and genus 0 linearly equivalent to $4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5$ on S. We proceed using the techniques and notation of case 4 above. We obtain

$$2a \leqslant b_4 + b_5$$

so X is a weak Fano threefold with four flopping curves e_1, e_2, e_3 and $f_{4,5}$. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E3$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 7: Since $-K_Y^3 = 40$, $Y \subset \mathbb{P}^6$ is a section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a subspace of codimension 3. Let $S \in |H| \subset Y$ be a smooth surface. S is del Pezzo of degree $K_S^2 = 5$. There is a nonsingular curve C with degree 6 and genus 1 linearly equivalent to $4l - 2e_1 - 2e_2 - e_3 - e_4$ on S. We proceed using the techniques and notation of case 4 above. We obtain

$$2a \leqslant b_3 + b_4$$

so X is a weak Fano threefold with three flopping curves e_1, e_2 and $f_{3,4}$. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E3$-contraction with exceptional divisor $\chi(\widetilde{S})$.

3.5.3 E1-E5

For the $E1-E5$ case, the table for all the numerical possibilities are listed on Table (3.6.3). We proceed case by case as before:

Case 1: On X, $|-K_X|$ is base point free if and only if the scheme-theoretic base locus of $|-K_Y - C| = |H - C|$ on Y is exactly the curve C. But C is a plane elliptic curve and Y is an intersection of quadrics, so this is impossible.

Case 2: Since $-K_X^3 = 4$ and $e < 9$, by (3.10) and (3.4.5), if this case exists, then X' must have a singular point which is not a simple node. In other words, the flop χ cannot be an Atiyah flop. It is not known if this case exists.

Case 3: On a nonsingular cubic surface $S \subset Y \cong \mathbb{P}^3$, there is a nonsingular curve C in the class $7l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6$ with degree 9 and genus 9. Let X be the blowup of $C \subset Y$ and let $\widetilde{S}$ denote the strict transform of S. Suppose $\Gamma \subset X$ satisfies $-K_X.\Gamma \leqslant 0$. As in E1-E3/E4 Case 4, we see that $\Gamma \subset \widetilde{S} \cong S$. Writing $-K_X = 4H + E$ and $\Gamma \sim al - \sum b_i e_i$ in $\widetilde{S} \cong S$, we obtain $4 \cdot \deg_{\mathbb{P}^3}(\Gamma) \leqslant \Gamma.C$, which simplifies to

$$5a \leqslant 2 \sum b_i$$

The only curves which satisfy this inequality are $g_1, \ldots, g_6$ which are the strict transform of conics in $\mathbb{P}^2$ passing through 5 of $P_1, \ldots, P_6$. For each of these curves, equality is attained, so X is a weak Fano threefold with six flopping curves $g_1, \ldots, g_6$. Using

$$0 \to \mathcal{N}_{\Gamma/\widetilde{S}} \to \mathcal{N}_{\Gamma/X} \to \mathcal{N}_{\widetilde{S}/X}|_\Gamma \to 0$$

we see that the normal bundle of each flopping curve is $\mathcal{O}(1) \oplus \mathcal{O}(1)$, so the flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop. $\chi(\widetilde{S})$ is isomorphic to $\mathbb{P}^2$ since it is the result of contracting the flopping curves on $\widetilde{S}$. The normal bundle of $\chi(\widetilde{S})$ is $\mathcal{O}_{\mathbb{P}^2}(a)$ for some integer a. Choosing a line f in $\chi(\widetilde{S}) \cong \mathbb{P}^2$ not passing though any of the points of intersection of the flopping curves with $\chi(\widetilde{S})$, we see that $a = f.\chi(\widetilde{S}) = \widetilde{f}.\widetilde{S}$. Using the fact that $\widetilde{S} = 3H - E$ and $\widetilde{f} \sim 5l - 2\sum e_i$, we can compute the intersection inside of the cubic surface $\widetilde{S}$. Since $H|_{\widetilde{S}} \sim 3l - \sum e_i$ and $E|_{\widetilde{S}} = C \sim 7l - 2\sum e_i$, we obtain $a = f.\chi(\widetilde{S}) = \widetilde{f}.\widetilde{S} = -2$. The normal bundle of $\chi(\widetilde{S})$ is $\mathcal{O}_{\mathbb{P}^2}(-2)$, so X^+ has an $E5$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 4: Since $-K_Y^3 = 24$, $Y \subset \mathbb{P}^4$ is a smooth cubic. Let $S \in |H| \subset Y$ be a smooth surface. S is del Pezzo of degree $K_S^2 = 3$. On S there is a nonsingular curve C in the class $5l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6$ with degree 3 and genus 0. We proceed using the techniques and notation of case 3 above. We obtain

$$a \leqslant 0$$

so X is a weak Fano threefold with six flopping curves $e_1, \ldots, e_6$. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E5$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 5: Since $-K_Y^3 = 54$, $Y \cong Q$, a smooth quadric threefold in $\mathbb{P}^4$. Let $S \in |2H| \subset Y$ be a smooth surface. S is the complete intersection of two quadrics in $\mathbb{P}^4$ so is del Pezzo of degree $K_S^2 = 4$. On S there is a nonsingular curve C in the class

$7l-3e_1-3e_2-2e_3-2e_4-2e_5$ with degree 9 and genus 6. We proceed using the techniques and notation of case 3 above. We obtain

$$2a \leqslant b_3+b_4+b_5$$

so X is a weak Fano threefold with five flopping curves $e_1, e_2, f_{3,4}, f_{3,5}$ and $f_{4,5}$, where $f_{i,j}$ denotes the pullback of a line through the points P_i and P_j. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E5$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 6: Since $-K_Y^3 = 32$, $Y \cong Q_1 \cap Q_2 \subset \mathbb{P}^5$, the smooth intersection of two quadrics. Let $S \in |H| \subset Y$ be a smooth surface. S is the complete intersection of two quadrics in $\mathbb{P}^4$ so is del Pezzo of degree $K_S^2 = 4$. On S there is a nonsingular curve C in the class $5l-2e_1-2e_2-2e_3-2e_4-2e_5$ with degree 5 and genus 1. We proceed using the techniques and notation of case 3 above. We obtain

$$a \leqslant 0$$

so X is a weak Fano threefold with five flopping curves $e_1, \dots e_5$. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E5$-contraction with exceptional divisor $\chi(\widetilde{S})$.

Case 7: Since $-K_Y^3 = 40$, $Y \subset \mathbb{P}^6$ is a section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a subspace of codimension 3. Let $S \in |H| \subset Y$ be a smooth surface. S is del Pezzo of degree $K_S^2 = 5$. On S there is a nonsingular curve C in the class $5l-2e_1-2e_2-2e_3-2e_4$ with degree 7 and genus 2. We proceed using the techniques and notation of case 3 above. We obtain

$$a \leqslant 0$$

so X is a weak Fano threefold with four flopping curves $e_1, \dots, e_4$. The flop $X \overset{\chi}{\dashrightarrow} X^+$ is an Atiyah flop and X^+ has an $E5$-contraction with exceptional divisor $\chi(\widetilde{S})$.

3.6 Tables

3.6.1 E1 - E2

The following table is a list of all the numerical possibilities for the $E1 - E2$ case, where r is the index of Y in (3.1). The index of Y^+ is always 1. All other notation can be found in (3.2) and (3.3). Due to space constraints, missing from the table are the values of α^+ and β^+. Those can be determined from the given values of α and β using (3.4). There are 3 entries on the table, all known to exist. Their geometric realization is shown in Takeuchi's paper [Tak89].

Table 3.1: E1-E2

No.	$-K_X^3$	$-K_Y^3$	$-K_{Y^+}^3$	α	β	r	d	g	e/r^3	Exist?	Ref
1.	4	40	12	5/2	-1/2	2	12	7	24	:)	[Tak89]
2.	6	24	14	3/2	-1/2	2	4	0	16	:)	[Tak89]
3.	14	64	22	3/4	-1/4	4	6	0	6	:)	[Tak89]

3.6.2 E1 - E3/E4

The following table is a list of all the numerical possibilities for the $E1 - E3$ and equivalently the $E1 - E4$ case. Here r is the index of Y in (3.1). All other notation can be found in (3.2) and (3.3). Due to space constraints, missing from the table are the values of α^+ and β^+. Those can be determined from the given values of α and β using (3.4). There are 7 entries on the table, 3 which do not exist and 4 which are geometrically realizable.

Table 3.2: E1-E3/E4

No.	$-K_X^3$	$-K_Y^3$	$-K_{Y^+}^3$	α	β	r	d	g	e/r^3	Exist?	Ref
1.	4	24	6	1.5	-0.5	2	6	3	9	x	3.5.2
2.	4	54	6	5/3	-1/3	3	12	12	8	x	3.5.2
3.	6	14	8	1	-1	1	4	1	4	x	3.5.2
4.	10	64	12	0.75	-0.25	4	8	6	5	:)	3.5.2
5.	12	54	14	2/3	-1/3	3	8	4	4	:)	3.5.2
6.	14	32	16	0.5	-0.5	2	4	0	4	:)	3.5.2
7.	16	40	18	0.5	-0.5	2	6	1	3	:)	3.5.2

3.6.3 E1 - E5

The following table is a list of all the numerical possibilities for the $E1 - E5$ case, where r is the index of Y in (3.1). All other notation can be found in (3.2) and (3.3). Due to space constraints, missing from the table are the values of α^+ and β^+. Those can be determined from the given values of α and β using (3.4). There are 7 entries on the table, 2 known not to exist and 5 which are geometrically realizable.

Table 3.3: E1-E5

No.	$-K_X^3$	$-K_Y^3$	$-K_{Y^+}^3$	α	β	r	d	g	e/r^3	Exist?	Ref
1.	4	10	9/2	1	-1	1	3	1	6	x	3.5.3
2.	4	40	9/2	1.5	-0.5	2	13	9	1	?	3.5.3
3.	8	64	17/2	0.75	-0.25	4	9	9	6	:)	3.5.3
4.	10	24	21/2	0.5	-0.5	2	3	0	6	:)	3.5.3
5.	10	54	21/2	2/3	-1/3	3	9	6	5	:)	3.5.3
6.	12	32	25/2	0.5	-0.5	2	5	1	5	:)	3.5.3
7.	14	40	29/2	0.5	-0.5	2	7	2	4	:)	3.5.3

Appendix: E1-E2 Program Source Code

The C++ program below numerically classifies weak Fano varieties occurring in an E1-E2 link diagram.

```
#include <iostream>
#include <fstream>
#include <vector>

using namespace std;

int gcdint(int a, int b);

int gcdint(int a, int b) {
    if(b == 0)
        return a;
    else
        return gcdint(b,a%b);
}

//Omitted: Definition of rational class

bool possKY3(int KY3, int r, int d);
int gcd(rational a, rational b);

int main()
{
```

```
vector<int> vminusKX3, vsigma, vminusKY3, vminusKYPlus3, vr, vg, vd;
vector<rational> valpha, valphaPlus, vbeta, vbetaPlus, ve, vePlus;

ofstream out;
out.open("E1E2.txt");

out << "KX3 \t KY3 \t KY+3 \t alpha \t beta \t alpha+ \t beta+ \t r \t d \t
    g \t sigma \t sigma+ \t e \t e+" << endl;

int sigma, sigmaPlus;
rational alpha, alphaPlus, beta, betaPlus, e, ePlus;
rational KXE2, E3, EPlus3, ES3, ESPlus3;

for(int minusKX3 = 4; minusKX3 <= 22; minusKX3 += 2) {
for(int r=1; r <= 4; r++) {
for(int g = 0; g <= 19*r/2 + 1; g++) {
for(int d = 1; d <= 19; d++) {

int minusKY3 = minusKX3 + 2*r*d + 2 - 2*g;
int minusKYPlus3 = minusKX3 + 8;

if(!possKY3(minusKY3, r, d) || !possKY3(minusKYPlus3, 1, -100))
    continue;

KXE2 = 2 - 2*g;
sigma = r*d + 2 - 2*g;
```

```
sigmaPlus = 4;

if(sigma <= 0)
    continue;

switch(r) {
case 1:
    if(sigma > 19)
        continue;
    break;
case 2:
    if(sigma > 36)
        continue;
    break;
case 3:
    if(sigma > 49)
        continue;
    break;
case 4:
    if(sigma > 58)
        continue;
}

beta = rational(-1,r);
betaPlus = -r;
alphaPlus = (sigma - betaPlus*sigmaPlus)/minusKX3;
```

```
alpha = -alphaPlus*beta;

E3 = -r*d + 2 - 2*g;
EPlus3 = 1;
ES3 = alphaPlus*alphaPlus*alphaPlus*minusKX3
    + 3*alphaPlus*alphaPlus*betaPlus*sigmaPlus
    - 3*alphaPlus*betaPlus*betaPlus*2
    + betaPlus*betaPlus*betaPlus*EPlus3;
ESPlus3 = alpha*alpha*alpha*minusKX3
    + 3*alpha*alpha*beta*sigma
    - 3*alpha*beta*beta*KXE2
    + beta*beta*beta*E3;

e = E3 - ES3;
ePlus = EPlus3 - ESPlus3;

if(e <= 0 || ePlus <= 0)
    continue;

if(-alpha*alpha*minusKX3 - 2*alpha*beta*sigma + beta*beta*KXE2 != 2
    || KXE2 != -alphaPlus*alphaPlus*minusKX3
    - 2*alphaPlus*betaPlus*sigmaPlus + betaPlus*betaPlus*2)
        continue;

rational temp = alpha*r;
if(!temp.isInt())
```

```
    continue;

temp = alphaPlus;
if(!temp.isInt())
    continue;

temp = alpha - beta;
if(!temp.isInt())
    continue;

temp = alphaPlus - betaPlus;
if(!temp.isInt())
    continue;

temp = ES3;
if(!temp.isInt())
    continue;

temp = ESPlus3;
if(!temp.isInt())
    continue;

if(gcd(alpha*r, alpha - beta) != 1
    || gcd(alphaPlus, alphaPlus - betaPlus) != 1)
        continue;
```

```
vminusKX3.push_back(minusKX3);
vminusKY3.push_back(minusKY3);
vminusKYPlus3.push_back(minusKYPlus3);
valpha.push_back(alpha);
vbeta.push_back(beta);
valphaPlus.push_back(alphaPlus);
vbetaPlus.push_back(betaPlus);
vr.push_back(r);
vd.push_back(d);
vg.push_back(g);
vsigma.push_back(sigma);
ve.push_back(e);
vePlus.push_back(ePlus);


}
}
}
}


for(int i=0; i<vr.size();i++)
    out << vminusKX3[i] << "\t" << vminusKY3[i] << "\t" <<
        vminusKYPlus3[i] << "\t" << valpha[i] << "\t" << vbeta[i] << "\t"
        << valphaPlus[i] << "\t" << vbetaPlus[i] << "\t" << vr[i] << "\t" <<
        vd[i] << "\t" << vg[i] << "\t" << vsigma[i] << "\t" << 4 << "\t" <<
        ve[i] << "\t" << vePlus[i] << endl;
```

```
    out << endl << vr.size() << endl;
    out.close();
    return 0;
}

bool possKY3(int KY3, int r, int d)
{
    switch(r) {

    case 1:
        if (KY3 < 2 || KY3 > 22 || KY3 % 2 != 0 || KY3 == 20)
            return false;
        break;
    case 2:
        if (KY3 < 8 —— KY3 > 40 —— KY3 % 8 != 0)
            return false;
        break;
    case 3:
        if (KY3 != 54)
            return false;
        break;
    case 4:
        if (KY3 != 64)
            return false;
    }
```

```
    if(KY3 < r*d + 5)
        return false;

    return true;
}

int gcd(rational a, rational b)
{
    int c = a.num/a.denom;
    int d = b.num/b.denom;

    return gcdint(c,d);
}
```

Bibliography

[Ba] V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ (1998), 1-32.

[Chel06] I. Cheltsov. Nonrational Nodal Quartic Threefolds, *Pacific J. Math.* 226:1 (2006), 65-82

[Cor95] A. Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.* 4(2): 223-254, 1995.

[Cu11] J. Cutrone. Symmetric Sarkisov Links of Fano Threefolds. (Thesis), 2011.

[Ha77] R. Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics. No 52. Springer-Verlag. New York-Heidelberg, 1977.

[Isk78] V.A. Iskovskikh. Fano 3-folds I, II. Math USSR, Izv. **11**, 485-527 (1977); 12, 469-506 (1978).

[Isk79] V.A. Iskovskikh. Birational automorphisms of three-dimensional algebraic varieties. Current problems in mathematics, VINITI, Moscow, 12, 159-236 (Russian). [English transl.: J. Soviet Math. 13 (1980) 815-868], Zbl. 428.14017.

[IP99] V.A. Iskovskikh, Yu.G. Prokhorov. Algebraic Geometry V: Fano varieties. Springer 1999.

[JP06] P. Jahnke, T. Peternell. Almost del Pezzo manifolds. arXiv:math/0612516v1 [math.AG].

[JPR05] P. Jahnke, T. Peternell, I. Radloff. Threefolds with Big and Nef AntiCanonical Bundles I, Math. Ann. 333, No.3, 569-631 (2005).

[JPR07] P. Jahnke, T. Peternell, I. Radloff. Threefolds with Big and Nef AntiCanonical Bundles II, arXiv:0710.2763v1 [math.AG].

[Kal09] A.-S. Kaloghiros. A Classification of Terminal Quartic 3-Folds and Applications to Rationality Questions. arXiv:0908.0289v1 [math.AG].

[Ko89] J. Kollár. Flops. Nagoya Math. J. 113, 15-36 (1989).

[MM82] S. Mori, S. Mukai. Classification of Fano 3-folds with $B_2 \geqslant 2$. *Manuscripta Math.*, 36(2): 147-162, 1981/82.

[Shi89] Kil-Ho Shin. 3-dimensional Fano Varieties with Canonical Singularities. *Tokyo J. Math.*, 12(2): 375-385, 1989.

[Tak89] Kiyohiko Takeuchi. Some birational maps of Fano 3-Folds *Compositio Math.*,71(3): 265-283, 1989.

[Tak09] Kiyohiko Takeuchi. Weak Fano Threefolds with del Pezzo Fibration, arXiv:0910.2188v1 [math.AG].

[Tak02] Hiromichi Takagi. On classification of $\mathbb{Q}$-Fano 3-folds of Gorenstein Index 2. I, II. *Nagoya Math. J.*, 167:117-155,157-216, 2002.

Vitae

Nicholas Marshburn was born on August 18, 1986 in Jacksonville, North Carolina. He graduated from Lebanon Valley College in 2006 with a degree in Math and Actuarial Science. He entered the Ph.D. program at Johns Hopkins University in fall 2006 and completed his thesis under Dr. Vyacheslav Shokurov. His thesis was defended on March 15, 2011.