FANO FIBRATIONS OVER A DISCRETE VALUATION RING

by

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ABSTRACT

Using the Log minimal model program we construct nice models of del Pezzo fibrations which are relevant to the inductive study of del Pezzo fibrations.

To investigate birational maps of Fano fibrations, we study elements in anticanonical linear systems of Fano varieties, which have the "worst" singularities. We also investigate certain special points on smooth hypersurfaces of degree $n \geq 3$ in $\mathbb{P}^n$, which are generalizations of Eckardt points on smooth cubic surfaces.

Finally, we show that if two del Pezzo fibrations of degree $d \leq 4$ over a discrete valuation ring with residue field of characteristic zero have smooth special fibers, then there is no birational map between them, which is biregular on generic fibers. Also, similar statements will be proved for the cases of hypersurfaces of degree n in $\mathbb{P}^n$.

Advisor : Vyacheslav Shokurov.

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Contents

1. Introduction.

1.1. Introduction.

Since Mori's Minimal model program, which is the higher dimensional analogue of classification of algebraic surfaces, was launched, Mori fiber spaces, one family of varieties in the program, have been getting more attention from many algebraic geometers. The present dissertation will study Mori fiber spaces using a broader point of view. Specifically, we will study the following varieties.

Let $\pi : X \longrightarrow T$ be a projective morphism between normal $\mathbb{Q}$-factorial varieties such that

- $\dim(X) > \dim(T)$,
- $\pi_* \mathcal{O}_X = \mathcal{O}_T$,
- $-K_X$ is π-ample.

Then, we say that X is a *Fano fibration* over T. In addition, if T is a point, it is called a *Fano variety*. In particular, if the relative Picard number $\rho(X/T)$ of $\pi : X \longrightarrow T$ is one, then it is called a *Mori fiber space*. If $\dim(T) > 0$, then $\pi : X \longrightarrow T$ will frequently be called a Fano fibration. Note that we usually permit "mild" singularities on them. For details, refer to [15] or [21].

Three-dimensional Fano fibrations $\pi : X \longrightarrow T$ are classified into three kinds:

- $\dim(T) = 0$. In this case, X is called a Fano 3-fold.
- $\dim(T) = 1$. A Fano fibration X is called a del Pezzo fibration, in the sense that the generic fiber is a smooth del Pezzo surface (smooth Fano surface).
- $\dim(T) = 2$. A Fano fibration X is called a conic bundle, in the sense that the generic fiber is a smooth conic.

In the Minimal model program, a final outcome is always either a variety with numerically effective (in short, nef) canonical divisor or a Mori fiber space. This is the reason why Fano fibrations fascinate many algebraic geometers.

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Our study will be focused on del Pezzo fibrations, in particular, their birational maps. But we will consider local situations which mean del Pezzo fibrations over a discrete valuation ring.

We will not explain what the Minimal model program (in short, MMP) and the Log minimal model program (in short, LMMP) are, even though they are heavily required to follow this exposition. We may refer to several standard expositions such as [7], [15], [21], [20], [36] and [37]. But, any other requirements will be explained in the following sections in this chapter.

In Chapter 2, we will construct "nice" models of del Pezzo fibrations with relative Picard number 1 in a local situation. In the case of dimension 2, we have perfect models of Fano fibrations, so-called relatively minimal models. In fact, A. Corti and J. Kollár constructed models of del Pezzo fibrations which have mild singularities and reduced and irreducible special fibers ([10] and [18]). The main reason that we try to find another nice models is that we need normal special fibers for the inductive study of Fano fibrations. A. Corti and J. Kollár's models do not have normal special fibers in general. In our models, so-called Shokurov models, reduced special fibers are normal if the base field is algebraically closed. Compared with A. Corti and J. Kollár's models, our construction will ignore the reducedness of special fibers to save the normality. But we expect that it can be compensated with the minimal multiplicity of special fiber.

In Chapter 3, we study Fano varieties, in particular smooth ones. Generally, relevant global geometric information in lower dimensions gives us much information on higher dimensional local geometric objects. In this sense, studying "global" Fano varieties is a very appropriate step to "local" Fano fibrations. The thing that we study in Fano varieties is their anticanonical linear systems which reflect geometry on them. Roughly speaking, we will find an element in each anticanonical linear system, which has the "worst" singularities. Besides the study of anticanonical linear systems, certain special points on smooth hypersurfaces of degree $n \geq 3$ in $\mathbb{P}^n$, which are generalizations of Eckardt points on smooth cubic surfaces, will be investigated.

Our ultimate results will appear in Chapter 4. We will prove that if two del Pezzo fibrations of degree $d \leq 4$ over a discrete valuation ring with residue field of characteristic zero have smooth special fibers, then there is no non-trivial birational map between them, which is biregular on generic fibers (Theorem 4.3.4). For instance, let us consider cubic surfaces defined over a local ring of a smooth complex curve at a point. They are del Pezzo fibrations of degree 3 over a germ of a smooth complex curve. We assume that generic fibers are smooth. Our result says that any isomorphism between these cubic surfaces defined over the quotient field of the discrete valuation ring is also an isomorphism on the reduction by the maximal ideal, if cubic surfaces have smooth reductions by the maximal ideal. The strategy of the proof shows that our philosophy, “the global to the local”, works. The proof is based on the results of Chapter 3. Moreover we will prove several further results (Theorem 4.3.6, Propositions 4.5.1, 4.5.2 and 4.5.3).

1.2. Definitions and preliminary results.

In the present section, we will review definitions and preliminary facts which are mainly used in this dissertation. We assume that our base field is of characteristic zero. For convenience, all varieties and schemes are assumed to be $\mathbb{Q}$-factorial.

Let X be a normal variety. A *subboundary* $D = \sum d_i D_i$ on X is a $\mathbb{Q}$-divisor on X such that $d_i \leq 1$ for all i, where each D_i is a prime divisor. An effective subboundary is called a *boundary*. We use the notations $\lfloor D \rfloor$ (round down), $\lceil D \rceil$ (round up) and $\{D\}$ for $\sum \lfloor d_i \rfloor D_i$, $\sum \lceil d_i \rceil D_i$ and $\sum (d_i - \lfloor d_i \rfloor) D_i$, respectively.

A *log pair* (X, B) is a pair consisting of a normal variety X and a $\mathbb{Q}$-divisor B on X such that $K_X + B$ is a $\mathbb{Q}$-Cartier divisor. For any log pair (X, B), there exists resolution of singularities $f : Y \longrightarrow X$ such that the divisor consisting of the strict transformation of B via f and the exceptional divisors is simple normal crossing. Such a resolution is called a *log resolution* of (X, B). For any log resolution $f : Y \longrightarrow X$, we have the following formula:

$$K_Y + f_*^{-1}(B) = f^*(K_X + B) + \sum_E a(E; X, B)E,$$

where E's are f-exceptional divisors. The coefficient $a(E; X, B)$ is called *discrepancy* of E with respect to (X, B). It is easy to show that a discrepancy is independent of the choice of log resolution.

The following concepts indicate how singular a log pair is. We should note that these concepts have been developed from M. Reid ([33] and [34]), and strengthened through the Minimal model program and the Log minimal model program.

Definition 1.2.1. *A log pair (X, B) (or $K_X + B$) is called*

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{purely log terminal} \\ \text{Kawamata log terminal} \\ \text{log canonical} \end{cases} \quad \text{if } a(E; X, B) \begin{cases} > 0 \text{ and } B = 0 \\ \geq 0 \text{ and } B = 0 \\ > -1 \\ > -1 \text{ and } \lfloor B \rfloor \leq 0 \\ \geq -1 \end{cases}$$

for any exceptional divisor E over X.

There is one more concept related to singularities, which is frequently used in birational geometry.

Definition 1.2.2. *A log pair (X, B) (or $K_X + B$) is log terminal if there is a log resolution f of X such that discrepancies of f-exceptional divisors with respect to (X, B) are greater than -1.*

Remark 1.2.3. Since varieties are assumed to be $\mathbb{Q}$-factorial, log terminal pair coincides with strictly log terminal pair as used in [36].

Pure log terminal pairs are introduced to aid an inductive approach to varieties. It is "Inversion of adjunction" (see the next Theorem) that makes this induction possible.

Let X be a normal variety and let S be a prime divisor. For simplicity we assume that $K_X + S$ is log canonical in codimension 2. Then there is an effective $\mathbb{Q}$-divisor $\mathrm{Diff}_S(0)$, called the *different*, such that

$$(K_X + S)|_S = K_S + \mathrm{Diff}_S(0).$$

If B is a $\mathbb{Q}$-divisor which is $\mathbb{Q}$-Cartier in codimension 2, then the different $\mathrm{Diff}_S(B)$ for $K_X + S + B$ is defined by the formula

$$(K_X + S + B)|_S = K_S + \mathrm{Diff}_S(B).$$

Theorem 1.2.4. (Inversion of adjunction) *Let $S + B$ be an effective $\mathbb{Q}$-divisor on X such that S and B have no common components, S is reduced, and $\lfloor B \rfloor = 0$. Suppose that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then $K_X + S + B$ is purely log terminal near S if and only if S is normal and $K_S + \mathrm{Diff}_S(B)$ is Kawamata log terminal.*

Proof. See [36] and [20]. □

Corollary 1.2.5. *Let S be a prime divisor and let B, B' be effective $\mathbb{Q}$-divisors on X such that S and $B+B'$ have no common components and $\lfloor B \rfloor = 0$. Suppose that B' is $\mathbb{Q}$-Cartier and $K_X + S + B$ is purely log terminal. Then $K_X + S + B + B'$ is log canonical near S if and only if $K_S + \mathrm{Diff}_S(B + B')$ is log canonical.*

Proof. See [20]. □

Since we will use many tools which are related to log canonicity, it will be helpful for us to review more about log canonical singularities on log pairs.

One of the important notions is that of log canonical thresholds introduced by V. Shokurov in [36]. Roughly speaking, log canonical threshold is a sort of measure which shows how far from log canonicity a log pair is.

Definition 1.2.6. *Let (X, B) be a log canonical pair, $Z \subset X$ a closed subvariety and D an effective $\mathbb{Q}$-Cartier divisor on X. The log canonical threshold of D along Z with respect to $K_X + B$ is the number*

$$lct_Z(X, B; D) = \sup\{c | K_X + B + cD \text{ is log canonical near } Z\}.$$

It is easy to check $0 \leq lct_Z(X, B; D) \leq 1$. If $B = 0$, then we use $lct_Z(X; D)$ instead of $lct_Z(X, 0; D)$. For the case $Z = X$ we use the notation $lct(X, B; D)$ instead of $lct_X(X, B; D)$.

One of our tasks is to find elements in anticanonical linear systems of Fano varieties, which have the worst singularities. In general, it is not quite easy to find such elements. Instead of trying to find such elements, we introduce the following number measuring how bad elements of the anticanonical linear system of a variety can be.

Definition 1.2.7. *Let (X, B) be a log canonical pair with nonempty $| - (K_X + B)|$. The total log canonical threshold of (X, B) is the real number*

$$totallct(X, B) = \sup\{r \in \mathbb{R} | K_X + B + rD \text{ is log canonical for any } D \in | - (K_X + B)|\}.$$

Note that $0 \leq totallct(X, B) \leq 1$. In the case of $B = 0$, the total log canonical threshold of (X, B) will be denoted by $totallct(X)$ instead of $totallct(X, B)$.

Example 1.2.8. *Let $X = \mathbb{P}^1_{\mathbb{C}}$ and $B = \sum a_i P_i$ such that $0 \leq a_i \leq 1$ and $2 - \sum a_i > 0$. Then (X, B) is a log Fano variety, i.e., a log pair with ample $-(K_X + B)$. We can easily see that*

$$totallct(X, B) = \frac{1}{2 - \sum a_i + max\{a_i\}}.$$

On the other hand, we also need to find "good" elements in the anticanonical linear systems. Complements, which are also introduced by V. Shokurov in [36] and [38], will play such a role. Let X be a normal variety and let $D = S + B$ be a subboundary on X such that S and B have no common components, S is a reduced divisor, and $\lfloor B \rfloor \leq 0$.

Definition 1.2.9. *A divisor $K_X + D$, not necessarily log canonical, is n-complementary if there is a divisor D^+ such that*

1. *nD^+ is integral and $n(K_X + D^+)$ is linearly trivial,*
2. *$K_X + D^+$ is log canonical,*
3. *$nD^+ \geq nS + \lfloor (n+1)B \rfloor$.*

The divisor $K_X + D^+$ is called an n-complement of $K_X + D$.

Later, the following lemma will give us an important clue for connections between a "bad" and a "good" elements in anticanonical linear systems.

Lemma 1.2.10. *Let (X, S) be a pure log terminal pair with reduced $S \neq 0$. Let $f : X \longrightarrow Z$ be a contraction such that $-(K_X + S)$ is nef and big over Z. If $K_S + \mathrm{Diff}_S(0)$ is n-complementary, then an n-complement $K_S + \Theta$ of $K_S + \mathrm{Diff}_S(0)$ in a neighborhood of any fiber of f meeting S can be extended to an n-complement of $K_X + S$.*

Proof. See [29]. □

Let (X, B) be a log pair. As before, for a log resolution $f : Y \longrightarrow X$ we have

$$K_Y + f_*^{-1}(B) = f^*(K_X + B) + \sum a_i E_i,$$

where each E_i is an f-exceptional divisor. Define

$$\mathcal{I}(X, B) = f_* \mathcal{O}_Y(\lceil \sum a_i E_i - f_*^{-1}(B) \rceil).$$

Then $\mathcal{I}(X, B)$ is an ideal sheaf on X, independent of the choice of log resolution. The associated subscheme of X, denoted by $\mathcal{LCS}(X, B)$, is called the *log canonical singularity subscheme associated to* (X, B). We also define the *locus of log canonical singularities* to be $LCS(X, B) = \operatorname{Supp} \mathcal{LCS}(X, B)$. Note that $LCS(X, B) = \mathcal{LCS}(X, B)$ if (X, B) is log canonical. Any irreducible component E of $\sum a_i E_i - f_*^{-1} B$ with coefficient at most -1 is called a *log canonical place* and $Center_X(E) = f(E)$ is called a *center of log canonicity.* It is clear that

$$LCS(X, B) = \bigcup_{E\,:\,\text{log canonical place}} Center_X(E).$$

Now, we have reached the stage at which we can review vanishing theorems.

Theorem 1.2.11. (Kawamata-Viehweg vanishing) *Let X be a smooth variety and let $\pi : X \longrightarrow S$ be a proper morphism onto S. Suppose that a π-nef and π-big $\mathbb{Q}$-divisor D on X has fractional part $\{D\}$ the support of which is only normal crossing (not necessarily simple). Then $R^i \pi_* \mathcal{O}_X(K_X + \lceil D \rceil) = 0$ for all $i \geq 1$.*

Proof. See [15, Theorem 1-2-3]. □

The following useful vanishing theorem is proven by V. Shokurov.

Theorem 1.2.12. *Let (X, B) be a log pair with a boundary B. Let $\pi : X \longrightarrow S$ be a proper morphism and let D be a Cartier divisor on X such that*

$$D \equiv K_X + B + H,$$

where H a π-nef and π-big $\mathbb{R}$-divisor. Then $R^i \pi_*(X, \mathcal{I}(X, B) \otimes \mathcal{O}_X(D)) = 0$ for all $i \geq 1$.

Proof. See [1] or [2]. □

1.3. Notations.

From now on, we will use the abbreviations plt, klt, lt and lc for purely log canonical, Kawamata log terminal, log terminal and log canonical, respectively.

We fix a discrete valuation ring with a local parameter t which is always denoted by $\mathcal{O}$. Our favorite discrete valuation ring is a local ring of a complex smooth curve at a point. The quotient field and residue field of $\mathcal{O}$ are denoted by K and k. Unless otherwise stated, the field k is always assume to be of characteristic zero. We denote $T = \operatorname{Spec} \mathcal{O}$. The generic point and the closed point of T are denoted by η and o, respectively. For a scheme Z defined over T, we will use the notation Z_η and S_Z for its generic fiber and reduced special fiber respectively. When we consider a birational map between schemes over T, it is always assume to be biregular over generic fibers unless otherwise mentioned.

2. Models.

2.1. Introduction.

The aim of the present chapter is to construct a "nice" model for a given variety X_K defined over $\operatorname{Spec} K$. To construct models we will use the Log minimal model program (LMMP), which works up to dimension 3.

Definition 2.1.1. *Let X_K be a variety defined over* $\operatorname{Spec} K$*. A model of X_K is a scheme X defined over* $\operatorname{Spec} \mathcal{O}$ *the generic fiber of which is isomorphic to X_K.*

We are interested primarily in models of Fano varieties. For smooth del Pezzo surfaces, there are some nice models constructed by A. Corti and J. Kollár, respectively. A. Corti has constructed so-called standard models ([10]) of smooth del Pezzo surfaces. To show the existence of standard models, he uses Maruyama's elementary transformations ([25]), which are generalization of elementary transformations on ruled surfaces, and then examines the number of crepant divisors and the number of components of special fiber during Maruyama's elementary transformations in the ambient space. On the other hand, J. Kollár has constructed good models of smooth hypersurfaces which are called semistable models ([18]). In his construction the main tool is the invariant theory. The advantage of semistable models is that these models can be constructed over almost arbitrary discrete valuation ring. But we can define standard models only over a discrete valuation ring whose residue field is an algebraically closed field of characteristic zero.

For smooth del Pezzo surface X_K of degree 3, a semistable model of X_K is always a standard model if we assume that the residue field is an algebraically closed field of characteristic zero. But a standard model is not semistable in general. So we can say that semistable model is better than standard model when X_K is a smooth del Pezzo surface of degree 3. In some sense, semistable models make it possible for us to approach to standard models of del Pezzo surfaces with an arithmetic point of view.

The reason that we need another models of Fano varieties over $\mathcal{O}$ is that special fibers are required to be normal for the inductive study. Standard and semistable models do not take care of the normality of special fibers. Instead of ignoring reduced special fibers, we will insist on normal special fibers. But, we expect that nonreduced fibers can be compensated with the "minimal multiplicities" of special fibers.

2.2. Models of a Fano variety defined over K.

First of all, we fix a variety X_K defined over the field K. We assume that X_K is at worst $\mathbb{Q}$-factorial klt and that the Picard number is one. We define two models of X_K as follows.

Definition 2.2.1. *A scheme X defined over T is a lt model (resp. plt model) of X_K if*

(1) the generic fiber X_η is isomorphic to X_K, and

(2) (X, S_X) is lt (resp. plt).

We assume that the LMMP holds in the category of lt pairs of dimension $\dim X_K + 1$. According to [37], the LMMP in the category of log terminal pairs holds up to dimension 3. In higher dimension, it is still a conjecture.

With the LMMP, we define the following model of X_K.

Definition 2.2.2. *A scheme X defined over T is a final lt (flt) model of X_K if it is an outcome of the LMMP starting with a lt model of X_K and its reduced special fiber as boundary. In the case, exactly one of the followings occurs;*

(1) $K_X + S_X$ is nef over T.

(2) $-(K_X + S_X)$ is ample over T and the relative Picard number $\rho(X/T)$ is one.

The generic fibers cannot be contracted during the LMMP since the Picard number of X_K is one. Therefore, the outcome will be a model of X_K.

Lemma 2.2.3. *If X_K is a Fano variety, a flt model X over T of X_K corresponds to the case (2) in definition 2.2.2.*

Proof. Since the anticanonical divisor $-K_{X_\eta}$ is ample on the generic fiber X_η, $K_X + S_X$ cannot be nef over T. □

Lemma 2.2.4. *Let X_K is a Fano variety. A flt model of X_K is a plt model if the residue field k is algebraically closed.*

Proof. Let $(X/T, S_X)$ be a flt model of X_K. Since $-(K_X + S_X)$ is ample, the relative Picard number is one. Thus, the reduced special fiber of X/T is irreducible and hence X/T

is a plt model of X_K. □

If the residue field is not algebraically closed, then lemma 2.2.4 is not true in general because the reduced special fiber of a flt model may not be geometrically irreducible (see example 2.4.2). But a plt model is always a flt model even over algebraically nonclosed residue field.

Proposition 2.2.5. *Let X_K is a Fano variety. Then, a plt model of X_K is a flt model.*

Proof. Let X/T be a plt model of X_K. Since $(X/T, S_X)$ is a lt, we can apply the LMMP to $(X/T, S_X)$. Then there exists a birational map $f : X/T \dashrightarrow Y/T$, where Y/T is a flt model of X_K. Since S_X has only one component, the birational map f is a composition of finite number of log flips. Consider the following sequence of log flips.

$$X = X^{(0)} \dashrightarrow X^{(1)} \dashrightarrow X^{(2)} \cdots \dashrightarrow X^{(n-1)} \dashrightarrow X^{(n)} = Y$$

And we consider the last log flips. We have a log flipping contraction $g_n : X^{(n-1)} \longrightarrow Z^{(n-1)}$ and a contraction $g_n^+ : X^{(n)} \longrightarrow Z^{(n-1)}$. There exists a curve $C \subset X^{(n)}$ such that $g_n^+(C)$ is a point and $(K_{X^{(n)}} + S_{X^{(n)}}).C > 0$ by a property of log flips. But this is impossible since $-(K_{X^{(n)}} + S_{X^{(n)}})$ is ample. This implies X/T is a flt. □

Now, we can explain how to construct a plt model of X_K when X_K is a Fano variety. First of all, we have the following trivial proposition.

Proposition 2.2.6. *A lt model of X_K exists.*

Proof. Let X/T be a flat closure of X_K. Then $(X/T, S_X)$ is a model of X_K. By the log terminal resolution of singularities, we have a log terminal resolution which has no change in the generic fiber. Then, this will be a lt model of X_K. □

So we can always get a lt model of X_K. Even though we cannot say that we always have a plt model of any variety X_K, we can construct a plt model of a Fano variety X_K whenever the residue field is algebraically closed.

Theorem 2.2.7. *A plt model of a Fano variety X_K exists if the LMMP holds for dimension $d = \dim X_K + 1$ and the residue field k is algebraically closed.*

Proof. It follows from proposition 2.2.6 and lemma 2.2.4. □

If X_K is not a Fano variety, then lemma 2.2.4 dose not hold in general. Let us take examples.

Example 2.2.8. Consider elliptic fibration X/T of Kodaira type I_3. Then the reduced special fiber S_X is a wheel of three lines. Thus X/T is already a lt model. Its log canonical divisor $K_X + S_X$ is nef, actually, numerically trivial, and hence X/T is a flt model. But X/T is not a plt.

Meanwhile, even though X_K is not a Fano variety, we can get a plt model in some case, using the LMMP.

Example 2.2.9. This example comes originally from [34]. We start with elliptic fibration Y/T of Kodaira type IV. Its reduced special fiber S_Y consists of three concurrent lines passing through a point p. Let Y'/T be the blown-up centered at the point p (we assume that the point p is k-rational). Then the reduced special fiber $S_{Y'}$ consists of one -1-curve and three -3-curves which cross normally. So Y'/T is a lt model. These -3-curves are $-(K_{Y'} + S_{Y'})$-negative and -1-curve is $-(K_{Y'} + S_{Y'})$-positive. Thus we can contract these -3-curve to a model Z/T, using the LMMP. Then, Z/T is a plt model.

Now, we define one more model of X_K. Before this, we need a biregular invariance of a flt model.

Definition 2.2.10. *Let $\pi : X \longrightarrow T$ be a flt model of X_K with $\rho(X/T) = 1$. The Shokurov index of X/T is a natural number*

$$\iota(X/T) := \mathrm{mult}_{S_X}(\pi^*(o)).$$

Note that S_X is a Cartier divisor if $\iota(X/T) = 1$. In general, Shokurov indices of flt models are not bounded. For any natural number n, there is a plt model of a conic curve which has Shokurov index n.

Among flt models of X_K, we can consider a flt model of X_K with the smallest Shokurov index.

Definition 2.2.11. *A flt model X/T of X_K is called a Shokurov model (Sh-model) if $\iota(X/T)$ is the smallest among those of flt models of X_K. A Shokurov model of Shokurov index 1 is called a simple Sh-model.*

Example 2.2.12. We assume that the discrete valuation ring $\mathcal{O}$ is the local ring at a point $p \in C$ of a smooth complex curve C. Let X/T be a Sh-model of a smooth conic curve X_K. If $\iota(X/T) = 1$, then S_X is a smooth Cartier divisor and hence X/T is a $\mathbb{P}^1$-bundle over T.

Example 2.2.13. Let X be a subscheme of $\mathbb{P}^2_{\mathcal{O}}$ defined by a equation $x^2 + ty^2 + tz^2 = 0$. We assume that the discrete valuation ring $\mathcal{O}$ is the local ring at a point $p \in C$ of a smooth complex curve C. Then we can easily see that $K_X + S_X$ is plt and $\iota(X/T) = 2$. Furthermore, this surface has exactly two singular points on S_X which are of type A_1. But this is not a Sh-model. Actually, we can construct a birational map from X to a Sh-model. We consider the minimal resolution X' of X. Then the special fiber of X' consists of two -2-curves and one double -1-curve. So we can contract this double -1-curve to get a smooth surface Y. The special fiber of Y consists of two -1-curves. After contracting one of them, we get a Sh-model of X_η.

Example 2.2.14. In example 2.2.13, if the discrete valuation ring $\mathcal{O}$ is assumed to be the local ring at a point $p \in C$ of a smooth curve C defined over $\mathbb{Q}$, then we can check that X/T is an non-simple Sh-model of X_η.

Example 2.2.15. Let X be a subscheme of $\mathbb{P}^3_{\mathcal{O}}$ defined by a equation $x_0^3 + x_1^3 + x_2^3 + (1 + t)x_3^3 = 0$. We assume that the discrete valuation ring $\mathcal{O}$ is the local ring at a point $p \in C$ of a smooth complex curve C. Then, we see that the cubic surface in $\mathbb{P}^3_K$ defined by this equation has the Picard number one because no orbit of Galois group $\mathrm{Gal}(\bar{K}/K)$ action on 27 lines of X_K over $\bar{K}$ consists of disjoint lines. And X/T is a smooth 3-fold with smooth special fiber, and hence a plt model. Furthermore, it is a Sh-model of X_K since $\iota(X/T) = 1$.

2.3. Properties of flt models.

In this section, we will study some properties of flt models. Recall that a plt model is always flt.

First, we start with the following proposition which gives us some information about special fibers.

Proposition 2.3.1. *Let (X, B_X) be a log variety (not necessarily defined over T). If $K_X + B_X$ is lc, then $LCS(X, B_X)$ is a seminormal variety. Furthermore, if $K_X + B_X$ is lt, then it consists of normal components intersecting normally.*

Proof. See [1, Corollary 2.12] and [36, Corollary 3.8]. □

Corollary 2.3.2. *Let X/T be a flt model of a Fano variety X_K.*

(1) If k is algebraically closed, then S_X is a normal variety.

(2) If k is not algebraically closed, then S_X consists of normal components intersecting normally.

Proof. It is easy consequence of lemma 2.2.4 and proposition 2.3.1. □

Compared with Corti's standard models, flt models do not insist on reducedness of special fibers. One of purposes that we introduced Sh-models is to compensate the lost of the reducedness. By taking minimal Shokurov index, we can restrict singularities on models. For instance, if a plt Sh-model of conic curve has Shokurov index one, then it is a smooth variety. We can generalize this as follows.

Proposition 2.3.3. *Suppose that X_K has at worst terminal singularities. Let X/T be a plt model of X_K. Then,*

$$a(X) + 1 > \frac{1}{\iota(X/T)},$$

where $a(X)$ is the minimal discrepancy of X.

Proof. Let $f : Z \longrightarrow X$ be a log resolution of (X, S_X). Then we have

$$K_Z = f^*(K_X) + \sum_{Center_X E_i \subset S_X} a_i E_i + \sum_{Center_X F_j \not\subset S_X} b_j F_j,$$

$$f^*(S_X) = f_*^{-1}(S_X) + \sum \frac{c_i}{\iota(X/T)} E_i.$$

Since divisor $\iota(X/T)S_X$ is Cartier, each c_i is integer. By our assumption, $a_i - \frac{c_i}{\iota(X/T)} > -1$ for each i and $b_j > 0$ for each j. Consequently, for each i and each j, we get

$$a(E_i; X) + 1 > \frac{c_i}{\iota(X/T)} \geq \frac{1}{\iota(X/T)} \qquad \text{and} \qquad a(F_j : X) + 1 > 1 \geq \frac{1}{\iota(X/T)}.$$

□

Corollary 2.3.4. *Suppose that X_K has at worst terminal singularities. If a plt Sh-model of X_K has Shokurov index one, then it has at worst terminal singularities.*

Proof. It immediately follows from proposition 2.3.3. ☐

Proposition 2.3.5. *Suppose that X_K has at worst terminal singularities. If a plt Sh-model of X_K with Shokurov index one is Gorenstein, then the special fiber has at worst canonical singularities.*

Proof. Let $f : Z \longrightarrow X$ be a log resolution of (X, S_X). Then we have

$$K_Z + f_*^{-1}(S_X) = f^*(K_X + S_X) + \sum d_i E_i.$$

Since K_X and S_X are Cartier, $d_i \geq 0$ for each i. From adjunction, we have

$$K_{f_*^{-1}(S_X)} = f^*(K_{S_X}) + \sum d_i E_i|_{f_*^{-1}(S_X)}.$$

Note that S_X is normal. This shows our statement. ☐

2.4. Flt models of smooth conic curves.

The goal of this section is to classify singularities on flt models of smooth conic curves. To this end, let us start with following important examples.

Example 2.4.1. We have a very easy example, $\mathbb{P}^1_{\mathcal{O}}/T$ which is a plt model of $\mathbb{P}^1_K$. This plt model has Shokurov index one.

Example 2.4.2. This example comes from Châtelet surface ([8]). Let $a \in k \backslash k^2$ (here, we assume that our residue field k is algebraically nonclosed). We consider a subscheme Y of $\mathbb{P}^2_{\mathcal{O}}$ defined by $x^2 - ay^2 + tz^2 = 0$. Then Y/T is not a plt model. Its geometric special fiber consists of two lines which are conjugate over k. Thus one line cannot be contracted without contracting the other. This means that Y/T is a flt model which has Shokurov index one.

Example 2.4.3. Suppose that k contains a primitive n-th of unity ϵ. Let $\pi : \mathbb{P}^1_{\mathcal{O}} \longrightarrow T$ be a Sh-model of $\mathbb{P}^1_K$. Consider the following $\mathbb{Z}_n$-action on $\mathbb{P}^1_{\mathcal{O}}$

$$[x, y; t] \longrightarrow [x, \epsilon^q y; \epsilon t],$$

where $(n,q)=1$. Then the morphism $\tilde{\pi}:\tilde{X}=\mathbb{P}^1_{\mathcal{O}}/\mathbb{Z}_n \longrightarrow \tilde{T}=T/\mathbb{Z}_n$ is a plt model of $\tilde{X}_\eta$. Here, $\tilde{T}$ is a spectrum of new discrete valuation ring $\tilde{\mathcal{O}}$ which is the $\mathbb{Z}_n$-invariant subring of $\mathcal{O}$. Clearly, $\tilde{X}/\tilde{T}$ has Shokurov index n. And $\tilde{X}$ has exactly two singular points on $S_{\tilde{X}}$ which are types $\frac{1}{m}(1,q)$ and $\frac{1}{m}(1,-q)$.

Theorem 2.4.4. ([28]) *Suppose that the residue field is algebraically closed. Let $f: X \longrightarrow T$ be a plt model of smooth conic X_K.*

1. *If $\iota(X/T)=1$, then X is isomorphic to $\mathbb{P}^1_{\mathcal{O}}$.*
2. *If $\iota(X/T)=m>1$, then X has exactly two singular points on S_X which are of types $\frac{1}{m}(1,q)$ and $\frac{1}{m}(1,-q)$.*

Proof. If $\iota(X/T)=1$, then S_X is a smooth Cartier divisor and hence X is smooth. Since $\deg K_{S_X}=(K_X+S_X)\cdot S_X=-2$, we have $S_X\cong\mathbb{P}^1_k$. Consequently, X is isomorphic to $\mathbb{P}^1_{\mathcal{O}}$ by Nöther-Enriques Theorem.

Now, suppose that $\iota(X/T)=m>1$. The reduced special fiber S_X is of index m, that is, mS_X is a Cartier divisor. Thus we have the following m-fold cyclic covering.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\phi} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{T} & \xrightarrow{\pi} & T \end{array}$$

where $\tilde{X}\longrightarrow\tilde{T}\longrightarrow T$ is the Stein factorization. We can see that $\tilde{T}$ is the affine scheme of discrete valuation ring $\tilde{\mathcal{O}}=\Gamma(\tilde{X},\mathcal{O}_{\tilde{X}})$. We get

$$\phi^*(K_X+S_X)=K_{\tilde{X}}+S_{\tilde{X}}.$$

Since K_X+S_X is plt, $K_{\tilde{X}}+S_{\tilde{X}}$ is also plt. By our construction, $\iota(\tilde{X}/\tilde{T})=1$ and $S_{\tilde{X}}$ is a Cartier divisor. The first statement shows that $\tilde{X}$ is isomorphic to $\mathbb{P}^1_{\tilde{\mathcal{O}}}$. Example 2.4.3 proves the second. □

Proposition 2.4.5. *Suppose that the residue field k is algebraically closed. Then Sh-model of smooth conic curve X_K is $\mathbb{P}^1_{\mathcal{O}}$.*

Proof. By Tsen's theorem, smooth conic curve X_K has a K-rational point. Riemann-Roch theorem implies that smooth conic curve X_K is isomorphic to $\mathbb{P}^1_K$. Thus, projective space $\mathbb{P}^1_{\mathcal{O}}$ is a Sh-model of X_K. □

This proposition shows that there is no non-simple Sh-model of smooth conic curve if the residue field is algebraically closed. On the other hand, if the residue field is not algebraically closed, then there exists a smooth conic curve over K which has non-simple Sh-model.

Proposition 2.4.6. *Suppose that the residue field k is not algebraically closed. Then any flt model of smooth conic curve X_K with Shokurov index one is one of the following.*

1. *Conic bundle (in classical sense, i.e., special fiber is reduced and geometrically irreducible conic).*

2. *Flt model which has the special fiber consisting of two conjugate lines.*

Proof. Let X/T be such a flt model. If our log pair surface (X, S_X) is lt but not plt, then it immediately follows from the classification of log surfaces that X is smooth (see [21]). If X/T is a plt model, then X is also smooth by proposition 2.3.3. So, we can show that X can be embedded in $\mathbb{P}^2_{\mathcal{O}}$ by very ample linear system $|-K_X|$. Since $X \in \mathbb{P}^2_{\mathcal{O}}$ is a conic, the statement immediately follows. □

Proposition 2.4.7. *Suppose that the residue field k is not algebraically closed. If a flt of smooth conic curve X_K is not a plt model, then its Shokurov index is one.*

Proof. Let X/T be such a flt model. Since log pair surface (X, S_X) is lt but not plt, X is smooth. Thus, X can be embedded in $\mathbb{P}^2_{\mathcal{O}}$ by very ample linear system $|-K_X|$. Since $X \in \mathbb{P}^2_{\mathcal{O}}$ is a conic, its Shokurov index cannot be bigger than one. □

Corollary 2.4.8. *A plt model of smooth conic curve X_K always exists.*

Proof. Suppose that flt model X/T of X_K is not a plt model. Then, X is smooth and its special fiber consists of two conjugate -1-curves. And they cross at a point p normally. Take the blowup centered at p. Note that the point p is k-rational point because it is invariant under the Galois action. Then we have one -1-curve and two -2-curves. We can contract these two -2-curves to get a plt model of X_K with Shokurov index 2. $\square$

Remark 2.4.9. In example 2.4.2, the subscheme Z of $\mathbb{P}^2_{\mathcal{O}}$ defined by $t(x^2 - ay^2) + z^2 = 0$ is a plt model of Y_K with Shokurov index 2.

2.5. Plt models of smooth curves of genus 1.

In this section, we will study flt models of smooth curves of genus 1. We assume that residue field k is algebraically closed throughout this section.

As mentioned before, we cannot always get a plt model for given elliptic curve. But we have an invariant of elliptic curve E_K defined over K, which can tell us whether E_K has a plt model or not.

Definition 2.5.1. *Let E_K be an elliptic curve. Then, we have a Weierstrass equation which defines E_K.*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for E_K. Let

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_4 = a_3^2 + 4a_6, \quad b_8 = \frac{(b_2b_6 - b_4^2)}{4},$$

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_8.$$

We define j-invariant of E_K by

$$j(E_K) = \frac{(b_2^2 - 24b_4)^3}{\Delta} \in K.$$

We have normalized valuation v on K, where we assign ∞ to $v(0)$.

Theorem 2.5.2. ([16],[17],[27]) *Let E_K be an elliptic curve defined over K, and let X/T be a minimal proper regular model for E_K. Then the special fiber S_X has one of the following forms.*

I_0 *S_X is a smooth curve of genus 1.*

I_1 *S_X is a rational curve with a node.*

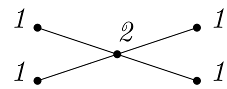
$\mathrm{I_n}$ *S_X consists of n smooth rational curves arranged in the shape of an n-gon, where n* ≥ 2.

II *S_X is a rational curve with cusp.*

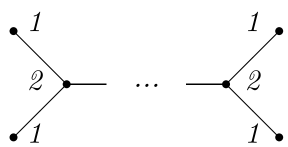
III *S_X consists of two smooth rational curves which tangentially at a single point.*

IV *S_X consists of three smooth rational curves intersecting at a single point.*

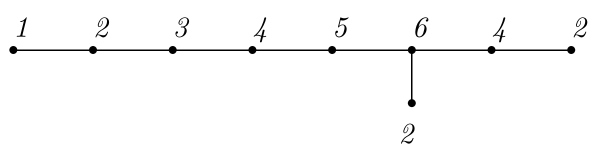
$\mathrm{I_0^*}$ *S_X consists of five smooth rational curves which have the following configuration.*



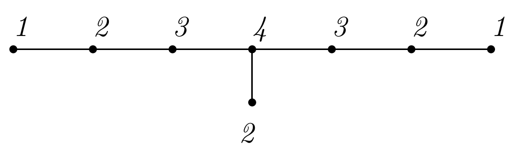
$\mathrm{I_n^*}$ *S_X consists of* $\mathrm{n}+5$ *smooth rational curves which have the following configuration.*



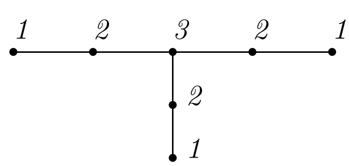
II^* *S_X consists of nine smooth rational curves which have the following configuration.*



III^* *S_X consists of eight smooth rational curves which have the following configuration.*



IV^* *S_X consists of seven smooth rational curves which have the following configuration.*



Here, • denotes smooth rational curve and the numbers beside them denote multiplicities. Any smooth rational curve in this classification is a -2*-curve.*

Proof. See [16],[17], or [27]. □

Remark 2.5.3. *Because we assume that E_K has a K-rational point, a model X/T of E_K has a section. Therefore, multiple fibers cannot appear in this classification.*

Lemma 2.5.4. *Let E_K be a elliptic curve defined over K. A minimal model of E_K has the special fiber of type $\mathrm{I_n}$ or $\mathrm{I_n^*}$, $\mathrm{n} \geq 1$ if and only if $v(j(E_K)) < 0$.*

Proof. See [40]. □

Proposition 2.5.5. *Let E_K be an elliptic curve defined over K. It has a plt model if and only if $v(j(E_K)) \geq 0$.*

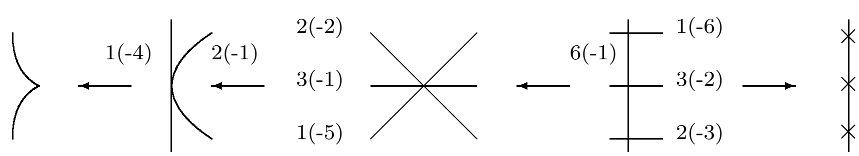
Proof. We see that there exists a minimal proper regular model X/T for E_K. By lemma 2.5.4, $v(j(E_K)) \geq 0$ if and only if X/T has the special fiber S_X of type $\mathrm{I_0}$, II, III, IV, $\mathrm{I_0^*}$, $\mathrm{II^*}$, $\mathrm{III^*}$, or $\mathrm{IV^*}$. We can construct a plt model in each case by simple observation. The following show each plt model Y/T corresponding to each case.

Case $\mathrm{I_0}$).

Model $X/T = Y/T$ is already a plt model. The reduced special fiber S_Y is an elliptic curve. The Shokurov index $\iota(Y/T) = 1$.

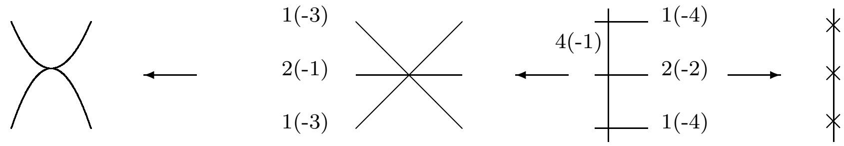
Case II).

We have a plt model Y/T which has three singular points of types $\mathrm{A_1}$, $\frac{1}{6}(1,1)$, and $\frac{1}{3}(1,1)$. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 6$.



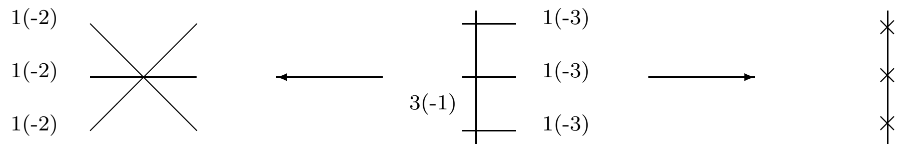
Case III).

There exists a plt model Y/T which has two singular points of type $\frac{1}{4}(1,1)$ and one singular point of type $\mathrm{A_1}$. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 4$.



Case IV).

There is a plt model Y/T which has three singular points of type $\frac{1}{3}(1,1)$. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 3$.



Case I_0^*).

We have a plt model Y/T which has four singular points of type A_1. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 2$.

Case II^*).

There is a plt model Y/T which has three singular points of types A_1, A_2, and A_5. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 6$.

Case III^*).

We have a plt model Y/T which has two singular points of type A_3 and one singular point of type A_1. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 4$.

Case IV^*).

There exists a plt model Y/T which has three singular points of type A_2. The reduced special fiber S_Y is a smooth rational curve. The Shokurov index $\iota(Y/T) = 3$.

In the case of $\mathrm{I_n}$, any sequence of blowups of X/T has the special fiber which has a wheel in its configuration. This implies that there is no plt model for this case.

In the case of $\mathrm{I_n^*}$, any sequence of blowups of X/T has the special fiber which has two disjoint forks in its configuration. This implies that there is no plt model for this case. □

From now on, we assume that smooth curve E_K of genus 1 has no K-rational point. This implies that any model of E_K has no section. Let X/T be a minimal proper regular model of E_K. Then its special fiber is multiple. And we know that the special fiber is the multiple of I_0 or I_n^*, where $\mathrm{n} \geq 1$. So we have the following assertion.

Proposition 2.5.6. *Let E_K be a smooth curve of genus 1 over K with no K-rational point. It has a plt model if and only if its minimal proper regular model has the special fiber consisting of multiple of smooth elliptic curve.*

Proof. This is clear from the proof of proposition 2.5.5. □

2.6. Further research problems.

In the study of Sh-model of del Pezzo surfaces, Shokurov index will be one of fascinating problems. Throughout the investigation of Shokurov index, we may approach to "relative minimal" type models. Actually, we expect that minimal Shokurov indices are bounded in the case of del Pezzo surfaces. Also, it is expected that Sh-models with minimal Shokurov index belong to a bounded family. The expectation follows from the fact that Shokurov indices control singularities. As a matter of fact, we showed that minimal discrepancy can be bounded below by some number depending only on Shokurov index.

3. Log canonical thresholds on Fano varieties.

3.1. Introduction.

As mentioned before, lc thresholds measure how far from log canonicity log pairs are. We may find lc thresholds in several other branches of mathematics, which are disguised as different appearances. For example, we consider the log canonical thresholds $lct_0(\mathbb{C}^n, 0; D)$ of $D = (f = 0)$ at the origin with respect to $(\mathbb{C}^n, 0)$, where f is a nonconstant holomorphic function near the origin. Then, we can see ([19]) that this number is exactly the same as the following number:

$$\sup\{c | |f|^{-c} \text{ is locally } L^2 \text{ near the origin}\}.$$

A Bernstein-Sato polynomial is another example. Bernstein-Sato polynomials appear in differential operator theory, in particular, $\mathcal{D}$-modules. Let us briefly explain what Bernstein-Sato polynomials are. It is known that for any convergent power series $f \in \mathbb{C}\{z_1, \cdots z_n\}$, there are a nonzero polynomial $b(s) \in \mathbb{C}[s]$ and a linear differential operator $Q = \sum_{I,j} f_{I,j} s^j \frac{\partial^I}{\partial z_I}$ such that

$$b(s)f^s = Qf^{s+1},$$

where each $f_{I,j}$ is a convergent power series. For fixed f, such $b(s)$'s form an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is called the Bernstein-Sato polynomial of f. We can see that $lct_0(D; \mathbb{C}^n, 0)$ is the absolute value of the largest root of the Bernstein-Sato polynomial of f ([19]).

In this chapter, we study lc thresholds on Fano varieties, which will significantly contribute to our main study. More precisely speaking, our aim is to compute total lc thresholds of several kinds of Fano varieties such as del Pezzo surfaces and hypersurfaces of degree n in $\mathbb{P}^n$. In addition, as a byproduct, we get results on some special points on smooth hypersurfaces of degree 4 in $\mathbb{P}^4$, which are a generalization of Eckardt points on smooth cubic surfaces. We conjecture the same results on smooth hypersurfaces of degree $n \geq 5$ in $\mathbb{P}^n$ (Conjecture 3.4.2). And we prove them under the assumption of the Log minimal model program.

Before we start, it should be mentioned that many ideas for these results are suggested by I. Cheltsov.

3.2. Log canonical thresholds on del Pezzo surfaces.

Nonsingular del Pezzo surfaces are objects with which we are very familiar. Furthermore, we understand singular del Pezzo surfaces very well. For example, [4], [12], [13], and [35] give us rich information. In this section, we will study some classical result on anticanonical linear systems on del Pezzo surfaces from a modern point of view. Strictly speaking, we investigate all possible singular effective anticanonical divisors on smooth del Pezzo surfaces. From this investigation, we can compute total lc thresholds of smooth del Pezzo surfaces.

Lemma 3.2.1. *Let S be a smooth del Pezzo surface of degree $d \leq 4$. Then, $K_S + C$ is lc in codimension 1 for any $C \in |-K_S|$.*

Proof. Let $C = \sum_{i=1}^n m_i C_i \in |-K_S|$, where C_i's are distinct integral curves on S and each $m_i \geq 1$.

First, we claim that if C is not irreducible, then each C_i is isomorphic to $\mathbb{P}^1$. Suppose that C_i is not isomorphic to $\mathbb{P}^1$. Then, the self-intersection number of C_i is greater than 0. Because $-K_S$ is ample, C is connected. So, we have

$$2p_a(C_i) - 2 = (C_i + K_S) \cdot C_i = (1 - m_i)C_i^2 - \sum_{i \neq j} m_j C_j \cdot C_i < 0,$$

which is contradiction. Thus, each component is a smooth rational curve.

Since $d = C \cdot (-K_S) = \sum_{i=1}^n m_i C_i \cdot (-K_S)$ and $-K_S$ is ample, we have $\sum_{i=1}^n m_i \leq d$.

If $d = 1$, then $n = 1$ and $m_1 = 1$.

If $d = 2$, then we have three possibilities C_1, $C_1 + C_2$, and $2C_1$. But the last case is absurd because the Fano index of S is one.

Suppose $d = 3$. Then possibilities are C_1, $C_1 + C_2$, $C_1 + C_2 + C_3$, $C_1 + 2C_2$, $2C_1$, and $3C_1$. With the Fano index one, we can get rid of the last two cases. For the case of $C = C_1 + 2C_2$, we consider the equation $3 = K_S^2 = (C_1 + 2C_2)^2 = C_1^2 + 4C_1 \cdot C_2 + 4C_2^2$. Since $(C_1 + 2C_2) \cdot (-K_S) = 3$, we have $C_1 \cdot (-K_S) = C_2 \cdot (-K_S) = 1$, and hence $C_1^2 = C_2^2 = -1$. Thus, $C_1 \cdot C_2 = 2$. But, this implies contradiction $-2 = 2p_a(C_1) - 2 = C_1 \cdot (-2C_2) = -4$.

Finally, we suppose that $d = 4$. We have eleven candidates, C_1, $C_1 + C_2$, $C_1 + C_2 + C_3$, $C_1 + C_2 + C_3 + C_4$, $C_1 + 2C_2$, $C_1 + 3C_2$, $C_1 + C_2 + 2C_3$, $2C_1 + 2C_2$, $2C_1$, $3C_1$, and $4C_1$. Again, we can exclude the last four candidates by Fano index. For the case of $C = C_1 + 3C_2$, we consider the equation $4 = K_S^2 = (C_1 + 3C_2)^2 = C_1^2 + 6C_1 \cdot C_2 + 9C_2^2$. As before, we can see $C_1^2 = C_2^2 = -1$. So, we have contradiction $3C_1 \cdot C_2 = 7$. Let's consider the case of $C = C_1 + 2C_2$. Since $(C_1 + 2C_2) \cdot (-K_S) = 4$, $C_1^2 = 0$ and $C_2^2 = -1$. Then, we have $4 = (C_1 + 2C_2)^2 = -4 + 4C_1 \cdot C_2$. But, $-2 = p_a(C_1) - 2 = -2C_1 \cdot C_2$. Finally, we consider $C = C_1 + C_2 + 2C_3$. Then, each C_i is -1-curve. Since $4 = (C_1 + C_2 + 2C_3)^2 = C_1^2 + C_2^2 + 4C_3^2 + 2(C_1 \cdot C_2 + 2C_1 \cdot C_3 + 2C_2 \cdot C_3)$, we have $5 = C_1 \cdot C_2 + 2C_1 \cdot C_3 + 2C_2 \cdot C_3$. But, $-2 = 2p_a(C_1) - 2 = -(C_2 + 2C_3) \cdot C_1$, and hence $3 = 2C_2 \cdot C_3$. But this is impossible. □

Let S be a smooth del Pezzo surface with Fano index r. Then, there is an ample integral divisor H, called the fundamental class of S, such that $-K_S = rH$. A curve C on S is called a line (resp. conic and cubic) if $C \cdot H = 1$ (resp. 2 and 3).

Proposition 3.2.2. *Let S be a smooth del Pezzo surface of degree $d \leq 4$, and let $C \in |-K_S|$. Suppose that $K_S + C$ is worse than lc.*

1. *If $d = 1$, then C is a cuspidal rational curve.*
2. *If $d = 2$, then C is one of the following:*
   - *$C = C_1 + C_2$, where C_1 and C_2 are lines intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*
   - *C is a cuspidal rational curve.*
3. *If $d = 3$, then C is one of the following:*
   - *$C = C_1 + C_2 + C_3$, where C_1, C_2, and C_3 are lines intersecting at one point with $C_1 \cdot C_2 = C_1 \cdot C_3 = C_2 \cdot C_3 = 1$.*
   - *$C = C_1 + C_2$, where C_1 and C_2 are a line and a conic intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*
   - *C is a cuspidal rational curve.*
4. *If $d = 4$, then C is one of the following:*
   - *$C = C_1 + C_2 + C_3$, where C_1 and C_2 are lines, and C_3 is a conic intersecting at one point with $C_1 \cdot C_2 = C_1 \cdot C_3 = C_2 \cdot C_3 = 1$.*
   - *$C = C_1 + C_2$, where C_1 and C_2 are a line and a cubic intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*

- $C = C_1 + C_2$, *where* C_1 *and* C_2 *are conics intersecting tangentially at one point with* $C_1 \cdot C_2 = 2$.
- C *is a cuspidal rational curve.*

Proof. Note that if C is irreducible, then arithmetic genus $p_a(C)$ of C is one. If C is not irreducible, then each component is isomorphic to $\mathbb{P}^1$. And we can see the intersection numbers of two different components of C are less than or equal to 2.

We can easily check the cases of degree 1 and 2.

Now, we suppose that $d = 3$. And we suppose that $C = C_1 + C_2 + C_3$. Since $3 = (C_1 + C_2 + C_3) \cdot (-K_S)$, each C_i is a line. From $2 = 2 - 2p_a(C_1) = C_1 \cdot (C_2 + C_3)$ and $3 = C_1^2 + C_2^2 + C_3^2 + 2C_1 \cdot (C_2 + C_3) + 2C_2 \cdot C_3$, we get $C_2 \cdot C_3 = 1$. Similarly, we can get $C_1 \cdot C_2 = C_1 \cdot C_3 = 1$. Since $K_S + C$ is not lc, these three lines intersect each other at one point.

If C has less than 4 components, then we can show our statement with the same method as above.

The only remaining that we have to show is that $K_S + C$ is lc if $d = 4$ and $C = C_1 + C_2 + C_3 + C_4$. Since each C_i is a line, we get

$$4 = C^2 = -4 + 2(C_1 \cdot C_2 + C_1 \cdot C_3 + C_1 \cdot C_4 + C_2 \cdot C_3 + C_2 \cdot C_4 + C_3 \cdot C_4).$$

And, we have $C_1 \cdot (C_2 + C_3 + C_4) = 2 - 2p_a(C_1) = 2$, $C_2 \cdot (C_1 + C_3 + C_4) = 2$, $C_3 \cdot (C_1 + C_2 + C_4) = 2$, and $C_4 \cdot (C_1 + C_2 + C_3) = 2$. With these 5 equations and connectedness of C, we can see that C is a normal crossing divisor. Thus, $K_S + C$ is lc. □

Corollary 3.2.3. *Let* S *be a smooth del Pezzo surface of degree* $d \leq 4$.

- *If* $d = 1$, *then* $totallct(S) \geq \frac{5}{6}$.
- *If* $d = 2$, *then* $totallct(S) \geq \frac{3}{4}$.
- *If* $d = 3$ *or* 4, *then* $totallct(S) \geq \frac{2}{3}$.

Proof. If C is three smooth curves intersecting each other at a single point transversally, then $lct(X; C) = \frac{2}{3}$. If $C = C_1 + C_2$ where C_i's are smooth curves intersecting tangentially with $C_1 \cdot C_2 = 2$, then we have $lct(X; C) = \frac{3}{4}$. For the case of a cuspidal rational curve,

$lct(X; C) = \frac{5}{6}$. Thus, our statement immediately follows from Proposition 3.2.2. □

Remark 3.2.4. Let S be a smooth del Pezzo surface of degree d. Then we can check the following table:

d	*Remarks*	$totallct(S)$
9	*Fano index 3*	$\frac{1}{3}$
8	*Fano index 1*	$\frac{1}{3}$
8	*Fano index 2*	$\frac{1}{2}$
7		$\frac{1}{2}$
6		$\frac{1}{2}$
5		$\frac{1}{2}$
4		$\frac{2}{3}$
3	*S has an Eckardt point.*	$\frac{2}{3}$
3	*Generic.*	$\frac{3}{4}$
2	*S has an effective anticanonical divisor with a tacnode*	$\frac{3}{4}$
2	*Generic.*	$\frac{5}{6}$
1		$\frac{5}{6}$

Note that having an Eckardt points in del Pezzo surfaces of degree 3 is a codimension 1 condition. Similarly, having an effective anticanonical divisor with a tacnode is a codimension 1 condition.

Remark 3.2.5. If S be a smooth del Pezzo surface of degree 1, then $|-K_S|$ has exactly one base point. We can easily check that any element in $|-K_S|$ is smooth at this point.

3.3. Lower bounds of total log canonical thresholds.

Let W be a smooth hypersurface of degree m in $\mathbb{P}^n$ and H be a hyperplane section of W, where $n \geq 4$. It follows from Lefschetz theorem that the Picard group of W is a free abelian group generated by a hyperplane section H, i.e., $\mathrm{Pic}(W) = \mathbb{Z}H$. Therefore, a hyperplane section H of W is irreducible and reduced.

Lemma 3.3.1. *For any curve C on H,* $\mathrm{mult}_C H = 1$.

Proof. Let p be a general point in $\mathbb{P}^n \backslash W$. We consider a cone P_p with the vertex p and the base C. Then we have

$$P_p \cap W = C \cup R_p,$$

where R_p is the residual curve of degree $(m-1)\deg(C)$. Curves C and R_p intersect at $(m-1)\deg(C)$ different points (see [31]). Since H is a hyperplane section of $W \subset \mathbb{P}^n$, we have

$$H \cdot R_p = \deg(R_p) = (m-1)\deg(C).$$

On the other hand,

$$H \cdot R_p \geq \deg(R_p)\mathrm{mult}_C H = (m-1)\deg(C)\mathrm{mult}_C H.$$

These imply the statement. □

Corollary 3.3.2. *A hyperplane section H has only isolated singularities. In particular, it is normal.*

Proof. The first statement immediately follows from Lemma 3.3.1. Since H is a smooth in codimension 1 hypersurface of a smooth variety, it is normal. □

Theorem 3.3.3. *The lc threshold of H in W is greater than or equal to $\lambda = \min\{\frac{n-1}{m}, 1\}$.*

Proof. Let $0 < \alpha < \lambda$. We may consider log pair $(\mathbb{P}^{n-1}, \alpha H)$ instead of $(W, \alpha H)$. Suppose that $K_{\mathbb{P}^{n-1}} + \alpha H$ is not klt. Then, $\mathcal{L} = \mathcal{LCS}(\mathbb{P}^{n-1}, \alpha H)$ is a zero-dimensional subscheme. Now, we consider a Cartier divisor $D = K_{\mathbb{P}^{n-1}} + \alpha H + (\lambda - \alpha)H'$, where H' is a generic element in $|H|$. Note that

$$\mathcal{O}_{\mathbb{P}^{n-1}}(D) = \begin{cases} \mathcal{O}_{\mathbb{P}^{n-1}}(-1) & \text{if } m-n \geq -1 \\ \mathcal{O}_{\mathbb{P}^{n-1}}(m-n) & \text{otherwise.} \end{cases}$$

By Theorem 1.2.12, we have an exact sequence;

$$H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(D)) \longrightarrow H^0(\mathcal{L}, \mathcal{O}_{\mathcal{L}}(D)) \longrightarrow 0.$$

But the first term is zero even though the second term is not. This is contradiction. □

Corollary 3.3.4. *Suppose that $n = m$. Then, the total lc threshold of W is at least $\frac{n-1}{n}$.*

Proof. This immediately follows from Theorem 3.3.3. □

Proposition 3.3.5. *If the lc threshold α of H in W is not 1, then the locus of lc singularities $LCS(W, \alpha H)$ of $(W, \alpha H)$ consists of a single point.*

Proof. The proof is similar as that of Theorem 3.3.3. □

Example 3.3.6. Suppose that hypersurface W in $\mathbb{P}^n$ is given by equation

$$x_0^m - x_1^m + \sum_{i=2}^{n} x_i^m = 0$$

and hyperplane section H is given by $x_0 - x_1 = 0$. Then, the log canonical threshold of H is λ. Thus, our λ is the sharp lower bound for lc thresholds of hyperplane sections of smooth hypersurfaces of degree m in $\mathbb{P}^n$.

3.4. Numerical characterization of Eckardt points.

An Eckardt point is a point on a smooth cubic surface at which three lines on this surface intersect each other. In other words, it is a point p on the surface such that there is an effective anticanonical divisor which is a cone with vertex p and base consisting of three different points. From this point of view, we can generalize Eckardt points on smooth hypersurface of degree n of $\mathbb{P}^n$.

Definition 3.4.1. *Let X be a smooth hypersurface of degree n in $\mathbb{P}^n$, where $n \geq 3$. A point p on X is called an (generalized) Eckardt point if there is an element S in $|-K_X|$ which is a cone in $\mathbb{P}^{n-1}$ over a smooth hypersurface of degree n of $\mathbb{P}^{n-2}$ with vertex p.*

Clearly, this generalized Eckardt point coincides with classical one when $n = 3$.

Now, we want to characterize these Eckardt points from birational geometric point of view. Our main motivation for this problem is a Eckardt point on a smooth cubic surface. We can easily see that a smooth cubic surface has an Eckardt point if and only if its total log canonical threshold is $\frac{2}{3}$. So, it is very natural that we expect the following conjecture.

Conjecture 3.4.2. *Let X be a smooth hypersurface of degree n in $\mathbb{P}^n$, where $n \geq 3$. A point p on X is an Eckardt point and an element $S \in |-K_X|$ is a cone as used in Definition 3.4.1 if and only if a log pair $(X, \frac{n-1}{n}S)$ is not klt and the locus of lc singularities $LCS(X, \frac{n-1}{n}S)$ of $(X, \frac{n-1}{n}S)$ is one point $\{p\}$.*

It is clear that the "only if" part always holds. The problem is the other part. Of course, the conjecture holds for $n = 3$. Let X be a smooth hypersurface of degree n in $\mathbb{P}^n$, where $n \geq 4$. Let S be a hyperplane section of X. The goal of this section is to prove Conjecture 3.4.2 for $n = 4$. Under the assumption of the Log Minimal model Program in dimension $\leq n-1$, we will show that log pair $(X, \frac{n-1}{n}S)$ is not klt if and only if divisor S in $\mathbb{P}^{n-1}$ is a cone over smooth hypersurface of degree n in $\mathbb{P}^{n-2}$.

From now on, we suppose that $(X, \frac{n-1}{n}S)$ is not klt. Then, $LCS(X, \frac{n-1}{n}S)$ is not empty and consists of only finite number of points (in fact, only one point by Proposition 3.3.5). So, we may forget about hypersurface X and deal only with the log pair $(\mathbb{P}^{n-1}, \frac{n-1}{n}S)$.

Throughout this section we assume that the Log Minimal Model Program (LMMP) holds for dimension $\leq n-1$.

Lemma 3.4.3. *There is a birational morphism $f : V \longrightarrow \mathbb{P}^{n-1}$ satisfying the following:*

- *f is an isomorphism outside of $LCS(\mathbb{P}^{n-1}, \frac{n-1}{n}S) = \{p\}$,*
- *V has $\mathbb{Q}$-factorial terminal singularities, and*
- *there is an effective f-exceptional $\mathbb{Q}$-divisor E on V such that the support of E coincides with that of the f-exceptional locus, $\lfloor E \rfloor \neq 0$, and $K_V + \frac{n-1}{n}f_*^{-1}S + E = f^*(K_{\mathbb{P}^{n-1}} + \frac{n-1}{n}S)$.*

Proof. It is an application of the LMMP. It is not hard to show. □

We fix such a birational morphism $f : V \longrightarrow \mathbb{P}^{n-1}$. Let $\tilde{S} = f_*^{-1}(S)$. Since a log pair $(\mathbb{P}^{n-1}, \frac{n-1}{n}S)$ is lc, a log pair $(V, \frac{n-1}{n}\tilde{S} + E)$ is also lc. We see that $K_V + \frac{n-1}{n}\tilde{S} + E$ is not nef and that $-(K_V + \frac{n-1}{n}\tilde{S} + E)$ is not ample. Therefore, there is an extremal contraction $g : V \longrightarrow W$ such that $-(K_V + \frac{n-1}{n}\tilde{S} + E)$ is g-ample and W is not a point. Because $-(K_V + \frac{n-1}{n}\tilde{S} + E)$ is g-ample and f-numerically trivial, any curve contracted by g is not contained in the fibers of f.

Lemma 3.4.4. *Suppose that extremal contraction g contracts a subvariety F of V to a subvariety Z of W. Then,* $\dim F - \dim Z = 1$.

Proof. Suppose that $\dim F - \dim Z > 1$. Since $F \cap E \neq \emptyset$ and V is $\mathbb{Q}$-factorial, there is a curve on $F \cap E$, which is contracted by both f and g. But this is impossible. $\square$

Proposition 3.4.5. *If extremal contraction $g : V \longrightarrow W$ is a Mori fiber space, then g is a conic bundle.*

Proof. It immediately follows from Lemma 3.4.4. $\square$

The following lemma is due to V. V. Shokurov's paper which is in preparation ([39]). It generalizes X. Benveniste and S. Mori's results ([3] and [26]) under the assumption of existence of flips.

Lemma 3.4.6. *Suppose that Y has at worst $\mathbb{Q}$-factorial terminal singularities. Let $h : Y \longrightarrow Z$ be a birational contraction. If a curve C is a irreducible component of exceptional locus of h, then $K_Y \cdot C > -1$.*

Proof. It is enough to consider the statement over an analytic neighborhood of $h(C) = q \in Z$. Suppose that $K_Y \cdot C \leq -1$. We choose a divisor H on Y with $H \cdot C = 1$. Then $(K_Y + H) \cdot C \leq 0$. We consider K_Y-flip of h;

$$\begin{array}{ccc} Y & \stackrel{\phi}{\dashrightarrow} & Y^+ \\ & {}_{h}\searrow \quad \swarrow_{h^+} & \\ & Z & \end{array}$$

Let H^+ be the birational transform of H to Y^+ via ϕ. Since we have $\dim Ex(h) + \dim Ex(h^+) \geq \dim Y - 1$ (see [15]), we get $\dim Ex(h^+) = \dim Y - 2$. Since $H^+ \cdot C' < 0$ for any curve C' on $Ex(h^+)$, we have $Ex(h^+) \subset H^+$. Let E be the exceptional divisor of blow-up centered at a component of $Ex(h^+)$. We may assume that the center of E on Y is not contained in H. We have the following inequality:

$$a(E; Y, H) \leq a(E; Y^+, H^+) \leq 0.$$

This is contradiction because Y is terminal. $\square$

Lemma 3.4.7. *Extremal contraction g is not a small contraction whose exceptional locus has a curve as an irreducible component.*

Proof. Suppose that g is small contraction whose exceptional locus has a curve C as an irreducible component. We have $K_V \cdot C > -1$ by Lemma 3.4.6. On the other hand,

$$(K_V + \frac{n-1}{n}\tilde{S}) \cdot C = (f^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1) - E) \cdot C \leq -1.$$

Thus, $\tilde{S} \cdot C < 0$ and $C \subset \tilde{S}$, and hence

$$(K_V + \tilde{S}) \cdot C < (K_V + \frac{n-1}{n}\tilde{S}) \cdot C.$$

Let $\nu : \hat{S} \longrightarrow \tilde{S}$ be a normalization of $\tilde{S}$. By adjunction, we have

$$K_{\hat{S}} + \mathrm{Diff}_{\tilde{S}}(0) = \nu^*((K_V + \tilde{S})|_{\tilde{S}}).$$

Since $\tilde{S}$ is smooth in a generic point of curve C, curve C cannot be contained in $\mathrm{Diff}_{\tilde{S}}(0)$, and hence $K_{\hat{S}} \cdot \nu_*^{-1}C < -1$. On the other hand, $K_{\hat{S}} \cdot \nu_*^{-1}C \geq -1$ since the curve $\nu_*^{-1}C$ is contractible. □

So far, we proved that extremal contraction g is a contraction of a subvariety F of V to a subvariety Z of W with $\dim F - \dim Z = 1$ and $\dim Z \geq 1$. Let C be a general fiber of morphism g over Z. Then, we have

$$(K_V + \frac{n-1}{n}\tilde{S}) \cdot C = (f^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1) - E) \cdot C = -\deg(f_*C) - E \cdot C < -\deg(f_*C),$$

because a curve C should meet the exceptional locus of f.

Lemma 3.4.8. *Suppose that extremal contraction g is a contraction of a subvariety F of V to a subvariety Z of W with $\dim F - \dim Z = 1$ and $\dim Z \geq 1$. Then, the subvariety F is a prime divisor of V if g is not a conic bundle.*

Proof. We suppose that extremal contraction g is not a conic bundle. Let C be a general enough fiber of morphism g. Inequality

$$-1 \leq K_V \cdot C = (K_V + \frac{n-1}{n}\tilde{S}) \cdot C - \frac{n-1}{n}\tilde{S} \cdot C < -\deg(f_*C) - \frac{n-1}{n}\tilde{S} \cdot C$$

implies that $F \subset \tilde{S}$. If the codimension of subvariety F of V is greater than 1, then we get

$$-1 \geq -\deg(f_*C) > (K_V + \frac{n-1}{n}\tilde{S}) \cdot C = (K_V + \tilde{S}) \cdot C - \frac{1}{n}\tilde{S} \cdot C > -1 - \frac{1}{n}\tilde{S} \cdot C > -1,$$

where the second last inequality is implied by Lemma 3.4.6. This is absurd. Therefore, subvariety F has codimension 1. □

Now, we know that extremal contraction g is either a conic bundle or a contraction of divisor F of V to a subvariety Z of W with $\dim Z = n-3$.

Theorem 3.4.9. *If the LMMP holds for dimension $\leq n-1$, then Conjecture 3.4.2 holds for n.*

Proof. Suppose that g is a conic bundle. Then $\lfloor E \rfloor \cap C \neq \emptyset$ since no component of divisor E lies in the fibers of g. Therefore,

$$\deg(f_*C) = -(K_V + \frac{n-1}{n}\tilde{S} + E)\cdot C = 2 - \frac{n-1}{n}\tilde{S}\cdot C - E\cdot C < 2.$$

Consequently, f_*C is a line on S. This implies that S is a cone in $\mathbb{P}^{n-1}$.

Now, we suppose that a morphism g is not a conic bundle. Then it is a contraction of divisor F of V to a subvariety Z of W with $\dim Z = n-3$. In this case, the proof of Lemma 3.4.8 shows that $F = \tilde{S}$. Therefore, we have

$$-\deg(f_*C) > (K_V + \frac{n-1}{n}\tilde{S})\cdot C = (K_V + \tilde{S})\cdot C - \frac{1}{n}\tilde{S}\cdot C = -2 + \frac{1}{n}\tilde{S}\cdot C > -2.$$

Thus, f_*C is a line on S. Consequently, S is a cone in $\mathbb{P}^{n-1}$. □

Corollary 3.4.10. *Conjecture 3.4.2 holds for $n = 4$.*

Proof. Since the LMMP for dimension 3 is proven, Theorem 3.4.9 implies the statement. □

Corollary 3.4.11. *The total lc threshold of smooth quartic X in $\mathbb{P}^4$ is totallct$(X) = \frac{3}{4}$ if and only if the quartic X has an Eckardt point.*

Proof. It immediately follows from Corollary 3.4.10. □

3.5. Further research problems.

An immediate question is how to prove Conjecture 3.4.2 without the assumption of the LMMP. Besides Conjecture 3.4.2 we have several interesting questions in this area.

In Section 3.3 it is proven that total lc thresholds of smooth hypersurfaces of degree $n \geq 3$ in $\mathbb{P}^n$ are at least $\frac{n-1}{n}$. It is natural that we should ask what the other lc thresholds of hyperplane sections on them are.

It follows from [24] that we have finitely many lc thresholds of hyperplane sections on Fano hypersurfaces of degree n in $\mathbb{P}^n$. The paper [24] shows that there are only finitely many Bernstein-Sato polynomials for polynomials of degree n in m variables. Since the lc thresholds of hyperplane sections can be derived from roots of Bernstein-Sato polynomials of polynomials defining hyperplane sections, the result of [24] implies the our assertion.

In [23], an algorithm finding all Bernstein-Sato polynomials of all polynomials in m variables of degree at most n is introduced. Unfortunately, it is not quite practical for the case $n \geq 3$ or $m \geq 3$. So, in this direction, the first challenging problem is to find all lc thresholds of hyperplane sections on smooth quartic 3-folds. This problem is much related to singularities on normal quartic surfaces. In terms of Arnol'd, these singularities are completely classified ([11]). But it is not easy to get lc thresholds from this information. In fact, author has almost done with this computation. The results will be announced soon. Since the method is primitive, it is hopeless to apply for higher dimensional cases. More powerful method needs to invent for the computation.

An interesting question has arisen from the observation of cubic surfaces. As mentioned in Remark 3.2.4, a generic cubic surface has total lc threshold $\frac{3}{4}$. At this moment, our curiosity asks the following question:

What is the total log canonical threshold of generic quartic 3-fold?

In the case of cubic surfaces, we can easily get it through simple combinatorical problem. To be strict, any smooth cubic surface is a blowup of $\mathbb{P}^2$ at 6 points in general position. Taking this into consideration, we can obtain generic total lc threshold. In the case of quartics, the main obstacle in computation is the fact that we do not have such a nice geometric description But, once we solve this problem, we may propose a general conjecture on the total log canonical threshold of generic hypersurface of degree n in $\mathbb{P}^n$.

4. Birational maps of del Pezzo fibrations.

4.1. Introduction.

It is a classical result that any $\mathbb{P}^1$-bundle over a smooth complex curve C can be birationally transformed to a $\mathbb{P}^1$-bundle over C by elementary transformation. Here, we may ask if it is also possible in the 3-fold case. In other words, is it true that any smooth del Pezzo fibration over a smooth curve can be birationally transformed to another smooth del Pezzo fibration? This is the main motivation question for the present chapter.

We are interested primarily in local cases. Strictly speaking, we consider birational maps between Fano fibrations over T. Of course, we can birationally transform any $\mathbb{P}^1$-bundle over a germ of a smooth complex curve (C, o) into another $\mathbb{P}^1$-bundle over (C, o). But, something different happens in higher dimensional cases. We will show that any del Pezzo fibration of degree at most 4 with smooth special fiber cannot be birationally transformed to another del Pezzo fibration with smooth special fiber (Theorem 4.3.4). In addition, we will answer to the same questions when Fano fibrations have smooth hypersurfaces of degree n in $\mathbb{P}^n$ over K as generic fibers. These results are based on the study of total lc thresholds of Fano varieties which are done in Chapter 3. This method can also explain why it is possible to transform $\mathbb{P}^1$-bundle over (C, o).

Even though it is not easy to find examples, there are del Pezzo fibrations of degree $d \leq 4$ over T with smooth special fibers which can be birationally transformed into other del Pezzo fibration with normal reduced and irreducible special fiber. But, as in the Minimal model program over 3-folds, we have to allow some mild singularities, such as terminal ones. The last section will be dedicated to such examples.

4.2. Properties of certain birational maps.

Let X/T be a $\mathbb{Q}$-factorial Gorenstein model of a smooth variety defined over K which satisfies the following three conditions.

- (Special fiber condition)

  The special fiber S_X is a reduced and irreducible variety with nonempty anticanonical linear system. Moreover, log pair (X, S_X) is plt.

- (1-complement condition)

For any $C \in |-K_{S_X}|$, there exists 1-complement $K_{S_X} + C_X$ of K_{S_X} such that C_X does not contain any center of log canonicity of $K_{S_X} + C$.

- (Surjectivity condition)

  Any 1-complement of K_{S_X} can be extended to a 1-complement of $K_X + S_X$.

It immediately follows from the Special fiber condition and Corollary 2.3.4 that X has at worst terminal singularities. Moreover, the special fiber S_X is a variety over k with Gorenstein canonical singularities.

Before we start the main argument, let us state an important lemma which is known as Negativity lemma.

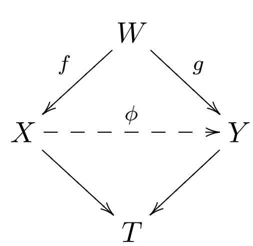
Lemma 4.2.1. *Let $\psi : Z \longrightarrow Z'$ be a birational morphism, not necessarily defined over T, with exceptional divisors E_i. Suppose that*

$$\sum a_i E_i \equiv_\psi H + D,$$

where H is a ψ-nef $\mathbb{Q}$-Cartier divisor, D is a effective $\mathbb{Q}$-Cartier divisor and no E_i appears in D. Then, all $a_i \leq 0$. Moreover, let $p \in Z'$ be a point such that D or H is not numerically trivial on $\psi^{-1}(p)$, and E_i a divisor such that $\psi(E_i) = p$. Then $a_i < 0$.

Proof. See [9]. □

Let $\phi : X \dashrightarrow Y$ be a birational map over T, where X and Y are $\mathbb{Q}$-factorial Gorenstein models of a smooth variety defined over K which satisfy the above three conditions. Suppose that $\phi : X \dashrightarrow Y$ is not an isomorphism in codimension 1. We fix a resolution of indeterminacy of $\phi : X \dashrightarrow Y$ as follows.



Let $\widetilde{S_X}$ and $\widetilde{S_Y}$ be proper transformations of S_X and S_Y by f and g, respectively. Since the birational map ϕ is not an isomorphism in codimension 1, $\widetilde{S_X}$ is a g-exceptional divisor and $\widetilde{S_Y}$ is f-exceptional.

Lemma 4.2.2. *Let $K_X+S_X+D_X$ be a 1-complement of K_X+S_X. Let $D_Y=\phi_*D_X$. For any prime divisor E over X,*

$$a(E;X,qS_X+D_X)=a(E;Y,\alpha_qS_Y+D_Y),$$

where q is any given number and $\alpha_q=-a(\widetilde{S_Y};X,qS_X+D_X)$. Moreover, log canonical divisor $K_Y+S_Y+D_Y$ is linearly trivial.

Proof. Suppose that E is a divisor on W. Note that $f_*^{-1}D_X=g_*^{-1}D_Y=D_W$. Then we have

$$K_W+q\widetilde{S_X}+D_W=f^*(K_X+qS_X+D_X)-\alpha_q\widetilde{S_Y}+\sum a_iE_i,$$

and

$$K_W+\alpha_q\widetilde{S_Y}+D_W=g^*(K_Y+\alpha_qS_Y+D_Y)+b\widetilde{S_X}+\sum b_iE_i,$$

where each E_i is f-exceptional and g-exceptional. From these, we get

$$f^*(K_X+qS_X+D_X)-g^*(K_Y+\alpha_qS_Y+D_Y)=(q+b)\widetilde{S_X}+\sum(b_i-a_i)E_i.$$

Since $K_X+qS_X+D_X$ is numerically trivial, we have

$$(q+b)\widetilde{S_X}+\sum(b_i-a_i)E_i\equiv_g 0.$$

By the Negativity lemma, $b=-q$ and $b_i=a_i$. This proves the first statement.

Since ϕ is biregular on generic fiber, it is clear that D_Y is linearly equivalent to $-K_Y$. Thus, the second statement follows from the fact that S_Y is linearly trivial. □

Lemma 4.2.3. *There exists a 1-complement $K_{S_X}+C_X$ (resp. $K_{S_Y}+C_Y$) of K_{S_X} (resp. K_{S_Y}) that does not contain the center of S_Y (resp. S_X) on X (resp. Y).*

Proof. Let $K_Y+S_Y+L_Y$ be a 1-complement of K_Y+S_Y. By Lemma 4.2.2, $a(\widetilde{S_Y};X,S_X+L_X)\leq -1$, where $L_X=\phi_*^{-1}L_Y$. Clearly, the center of $\widetilde{S_Y}$ on X is contained in $C=L_X|_{S_X}$. By inversion of adjunction, the center of $\widetilde{S_Y}$ is a center of log canonicity singularities of $K_{S_X}+C$. Furthermore, $K_{S_X}+C$ is linearly trivial by Lemma 4.2.2. Therefore, the 1-complement condition implies the statement. □

Lemma 4.2.4. *There is a 1-complement $K_X+S_X+D_X$ (resp. $K_Y+S_Y+H_Y$) of K_X+S_X (resp. K_Y+S_Y) such that D_X (resp. H_Y) does not contain the center of S_Y (resp. S_X).*

Proof. It immediately follows from Lemma 4.2.3 and the Surjectivity condition. □

From now on, we fix 1-complements $K_X+S_X+D_X$ and $K_Y+S_Y+H_Y$ of K_X+S_X and K_Y+S_Y, respectively, which satisfy the condition in Lemma 4.2.4. We will use the notation D_Y, D_W, H_X and H_W for ϕ_*D_X, $f_*^{-1}D_X$, $\phi_*^{-1}H_Y$ and $g_*^{-1}H_Y$, respectively. Note that $g_*^{-1}D_Y = f_*^{-1}D_X$ and $g_*^{-1}H_Y = f_*^{-1}H_X$.

Now, we define the following condition:

- (Total lc threshold condition)

  The inequality $\tau_X+\tau_Y>1$ holds, where $\tau_X = totallct(S_X)$ and $\tau_Y = totallct(S_Y)$.

Theorem 4.2.5. *Under the Total lc threshold condition, the birational map ϕ is an isomorphism in codimension 1.*

Proof. Suppose that ϕ is not an isomorphism in codimension 1. We pay attention to the following eight equations;

$$K_W = f^*(K_X)+a\widetilde{S_Y}+\sum a_iE_i, \qquad \widetilde{S_X} = f^*(S_X)-b\widetilde{S_Y}-\sum b_iE_i,$$

$$D_W = f^*(D_X)-\sum c_iE_i, \qquad H_W = f^*(H_X)-e\widetilde{S_Y}-\sum e_iE_i,$$

$$K_W = g^*(K_Y)+n\widetilde{S_X}+\sum n_iE_i, \qquad \widetilde{S_Y} = g^*(S_Y)-m\widetilde{S_X}-\sum m_iE_i,$$

$$D_W = g^*(D_Y)-l\widetilde{S_X}-\sum l_iE_i, \qquad H_W = g^*(H_Y)-\sum r_iE_i.$$

First of all, $b=m=1$, since S_X and S_Y are reduced and irreducible. Since D_X does not contain the center of $\widetilde{S_Y}$ on X, we have $\mathrm{mult}_{\widetilde{S_Y}}D_X=0$. For the same reason, we also have $\mathrm{mult}_{\widetilde{S_X}}H_Y=0$.

By Lemma 4.2.2, we get $n+a-l = a(\widetilde{S_X};Y,-aS_Y+D_Y) = a(\widetilde{S_X};X,D_X)=0$ and $a+n-e = a(\widetilde{S_Y};X,-nS_X+H_X) = a(\widetilde{S_Y};Y,H_Y)=0$, and hence $a+n=l=e$. Since X and Y have at worst terminal singularities, $a+n=l>0$.

Since $K_Y + S_Y + D_Y$ is linearly trivial by Lemma 4.2.2, $(K_Y + S_Y + D_Y)|_{S_Y} = K_{S_Y} + D_Y|_{S_Y}$ is linearly trivial. Thus, $D_Y|_{S_Y} \in |-K_{S_Y}|$. Consequently, it follows from inversion of adjunction that $K_Y + S_Y + \tau_Y D_Y$ is lc. By the same reason, $K_X + S_X + \tau_X H_X$ is lc.

Now, we have $a(\widetilde{S_Y}; X, S_X + \tau_X H_X) = a - 1 - \tau_X e \geq -1$ and $a(\widetilde{S_X}; Y, S_Y + \tau_Y D_Y) = n - 1 - \tau_Y l \geq -1$. But, $l = a + n \geq \tau_X e + \tau_Y l = (\tau_X + \tau_Y)l > l$ by the Total lc threshold condition. Since $l > 0$, this is impossible. □

4.3. Applications.

As an easy application of Theorem 4.2.5, we get the following well-known example.

Example 4.3.1. Let Z be a $\mathbb{P}^1$-bundle over T. Suppose that the special fiber S_Z has no k-rational point. In particular, the residue field k is not algebraically closed. Then, there is no birational transform of Z into another $\mathbb{P}^1$-bundle over T, because the special fiber S_Z satisfies the Total lc condition. If S_Z has a k-rational point, then the Total lc condition fails. Moreover, it can be birationally transformed into another $\mathbb{P}^1$-bundle over T by elementary transformations.

It is interesting to compare Example 4.3.1 with W.-L. Chow and S. Lang's result ([6]).

Theorem 4.3.2. *Let C_1 and C_2 be subschemes on $\mathbb{P}^n_{\mathcal{O}}$ such that generic fibers are smooth curves of genus > 0 in $\mathbb{P}^n_K$. We suppose that special fibers S_{C_1} and S_{C_2} are smooth. Then any birational map between C_1 and C_2 is biregular. Here, fields K and k are not necessarily of characteristic zero.*

Proof. Note that we assume that birational maps over T are biregular over generic fibers. For the proof, see [6]. □

For the higher dimensional applications, we start with the following lemma.

Lemma 4.3.3. *Let X and Y be varieties, not necessarily defined over T, with $\mathbb{Q}$-factorial singularities. Let H be an ample divisor on X. If $\phi : X - \to Y$ is a birational map, isomorphim in codimension one, such that $\phi_* H$ is nef on Y, then ϕ^{-1} is a morphism. In addition, if $\phi_* H$ is ample, then ϕ is an isomorphism.*

Proof. See [9]. □

Theorem 4.3.4. *Let X and Y be del Pezzo fibrations of degree $d \leq 4$ over T. Suppose that each scheme-theoretic special fiber is smooth. Then any birational map between X and Y over T which is biregular over generic fiber is a biregular morphism.*

Proof. Since $-K_X$ and $-K_Y$ are ample over T, the Surjectivity condition follows from Lemma 1.2.10. By the same reason, it follows from Lemma 4.3.3 that the birational map ϕ cannot be an isomorphism in codimension 1 unless it is biregular.

It is enough to check the 1-complement condition and the Total lc threshold condition by Theorem 4.2.5. The Total lc threshold condition immediately follows from Corollary 3.2.3. If $2 \leq d \leq 4$, then the 1-complement condition follows from base-point-freeness of anti-canonical linear systems. In the case of degree 1, it can be derived from Remark 3.2.5.

□

Corollary 4.3.5. *Let X be a del Pezzo fibration over T of degree ≤ 4 with smooth special fiber. Then, the birational automorphism group of X/T is the same as the biregular automorphism group of X/T.*

Proof. Note that we always assume that birational map is biregular on generic fiber. The statement immediately follows from Theorem 4.3.4. □

Theorem 4.3.6. *Let X and Y be smooth Fano fibrations over T satisfying the following;*

- *Their generic fibers are isomorphic to a smooth hypersurface of degree n in $\mathbb{P}^n_K$, where $n \geq 3$.*
- *Special fibers S_X and S_Y are smooth.*

Then, any birational map of X into Y over T which is biregular on generic fiber is biregular.

Proof. The proof of Theorem 4.3.4 works. In particular, the Total lc threshold condition is satisfied by Corollary 3.3.4. □

Remark 4.3.7. Birational rigidity was proven for any smooth hypersurface of degree n in $\mathbb{P}^n$ and generic hypersurface of degree m of $\mathbb{P}^m$, where $4 \leq n \leq 8$ and $m \geq 9$. For these, refer to [5], [14], [30], and [32]. With birational rigidity, we can easily prove Theorem 4.3.6 in the case of $n \leq 8$. Also, we can get weaker statement than Theorem 4.3.6 for $n \geq 9$.

4.4. Examples.

If we allow some mild singularities on del Pezzo fibrations, then we can find birational maps of del Pezzo fibrations over T with reduced and irreducible special fiber. In each example, note that one of two del Pezzo fibrations has terminal singularities. Before considering these examples, we will state an easy lemma which helps us to understand our examples.

Lemma 4.4.1. *Let* $f(x_1, \cdots, x_m, y_1, \cdots, y_n) = g(x_1, \cdots, x_m) + h(y_1, \cdots, y_n)$ *be a holomorphic function near* $0 \in \mathbb{C}^{m+n}$ *and let* $D_f = (f = 0)$ *on* $\mathbb{C}^{m+n}$, $D_g = (g = 0)$ *on* $\mathbb{C}^m$, *and* $D_h = (h = 0)$ *on* $\mathbb{C}^n$. *Then*

$$lct(\mathbb{C}^{m+n}, D_f) = min\{1, lct(\mathbb{C}^m, D_g) + lct(\mathbb{C}^n, D_h)\}.$$

Proof. See [22]. □

Example 4.4.2. This example comes from [10] and [18]. Let X and Y be subschemes of $\mathbb{P}^3_{\mathcal{O}}$ defined by equations $x^3 + y^3 + z^2w + w^3 = 0$ and $x^3 + y^3 + z^2w + t^{6n}w^3 = 0$, respectively, where n is a positive integer. Note that X is smooth and Y has single singular point of type cD_4 at $p = [0, 0, 0, 1]$. Then, we have a birational map ρ_n of X into Y defined by $\rho_n([x, y, z, w]) = [t^{2n}x, t^{2n}y, t^{3n}z, w]$. Now, we consider a divisor $D \in |-K_X|$ defined by $z = w$. This divisor D is a sort of good divisor because $K_X + S_X + D$ is lc and $D|_{S_X}$ is an elliptic curve on S_X. But, the birational transform $\rho_{n*}(D)$ of D by ρ_n is worse than before. First, $\rho_{n*}(D)|_{S_Y}$ is three lines intersecting each other at a single point (Eckardt point) transversally on S_Y. Furthermore, we can see that $\rho_{n*}(D)$ on Y is defined by $z = t^{3n}w$. The log canonical threshold of $\rho_{n*}(D)$ is $\frac{4n+1}{6n}$ by Lemma 4.4.1, and hence $K_Y + \rho_{n*}(D)$ cannot be lc.

Example 4.4.3. Let Z and W be subschemes of $\mathbb{P}^3_{\mathcal{O}}$ defined by equations $x^3 + y^2z + z^2w + t^{12m}w^3 = 0$ and $x^3 + y^2z + z^2w + w^3 = 0$, respectively, where m is a positive integer. Here,

Z has a singular point of type cE_6 at $[0, 0, 0, 1]$ and W is smooth. We have a birational map ψ_m of Z into W defined by $\psi_m([x, y, z, w]) = [t^{2m}x, t^{3m}y, z, t^{6m}w]$. Again, we consider a divisor $H \in |-K_Z|$ defined by $z = w$. For the same reason as above, H is a good divisor. But, the log canonical threshold of the birational transform $\psi_{m*}(H)$ of H by ψ_m is $\frac{5m+1}{6m}$. Therefore, if $m > 1$, then $K_W + \psi_{m*}(H)$ cannot be lc. Note that $\psi_{m*}(H)|_{S_W}$ is a cuspidal rational curve on S_W.

Example 4.4.4. We consider birational map $\varphi_m = \psi_m^{-1}$ from W to Z, where W, Z and ψ_m are the same as in Example 4.4.3. And, we pay attention to smooth divisor $L \in |-K_W|$ on W defined by $x = 0$. Then, we can see that $\varphi_{m*}(L)|_{S_Z}$ consists of a line and a conic intersecting each other tangentially. And, the log canonical threshold of $\varphi_{m*}(L)$ is $\frac{9m+1}{12m}$, and hence $K_Z + \varphi_{m*}(L)$ is not lc.

The following two examples were constructed by M. Grinenko. One is a del Pezzo fibration of degree 2, and the other is of degree 1.

Example 4.4.5. Let X and Y be subschemes of $\mathbb{P}^3_{\mathcal{O}}(1, 1, 1, 2)$ defined by equations $w^2 + x^3y + x^2yz + z^4 + t^4xy^3 = 0$ and $w^2 + x^3y + xy^3 + x^2yz + t^2z^4 = 0$, respectively, where w is of weight 2. The map $\phi : X \dashrightarrow Y$ defined by $\phi(x, y, z, w) = (x, t^2y, z, tw)$ is birational. Subscheme X has a singular point of type cD_5 at $[0, 1, 0, 0]$. Subscheme Y has two singular points of types cD_6 and cA_1 at $[0, 0, 1, 0]$ and $[1, 0, -1, 0]$, respectively.

Example 4.4.6. Let Z be a subscheme of $\mathbb{P}^3_{\mathcal{O}}(1, 1, 2, 3)$ defined by equation $w^2 + z^3 + xy^5 + t^4x^5y = 0$, where z and w are of weight 2 and 3, respectively. Then, we have a birational automorphism α of Z defined by $\alpha(x, y, z, w) = (y, t^2x, t^2z, t^3w)$. Note that Z has a singular point of type cE_8 at $[1, 0, 0, 0]$.

Example 4.4.7. Let Z and Z' be subschemes of $\mathbb{P}^3_{\mathcal{O}}(1, 1, 2, 3)$ defined by equations $w^2 + z^3 + xy^5 + x^5y = 0$ and $w^2 + z^3 + xy^5 + t^{24}x^5y = 0$, respectively, where z and w are of weight 2 and 3, respectively. Note that the special fiber of Z' is smooth. There is a birational map $\beta : Z' \longrightarrow Z$ defined by $\beta(x, y, z, w) = (x, t^6y, t^2z, t^3w)$.

4.5. Further research problems.

As shown in several examples, we observe honest birational maps between simple Sh-models of smooth del Pezzo surface defined over K. Our ultimate goal is to factorize such birational maps into composition of “elementary” birational maps. In the case of $\mathbb{P}^1$-bundles over a curve, elementary transformations explain these factorizations. As a matter of fact, there are “elementary” birational maps, so-called elementary links, which can factorize every birational maps between 3-fold Mori fiber spaces ([9]). But, it seems not to be concrete as the case of $\mathbb{P}^1$-bundle over a curve. We expect very sensitive relations between total lc thresholds of special fibers and “elementary” birational maps. And this point of view will bring us concrete factorizations.

For this purpose, the very next step will be to apply the method of total lc thresholds to the case of mild singularities. Our previous argument shows that Theorem 4.3.4 holds even for singular special fibers as long as the Total lc threshold condition is satisfied. Author has studied in this direction and observed the following phenomena. From now on, we assume that the residue field k is algebraically closed.

Proposition 4.5.1. *Let X/T and Y/T be simple Sh-models of a smooth del Pezzo surface of degree 1 defined over K. Suppose that we have a birational map $f : X \dashrightarrow Y$ over T. In addition, we assume that X has a smooth special fiber S_X. Then, exactly one of the following occurs:*

1. *Birational map f is biregular.*
2. *The special fiber S_Y of Y must have only one singular point on it. Moreover, it is a Du Val singularity of type E_8.*

Proof. We see that a smooth del Pezzo surface has total lc threshold $\frac{5}{6}$. It implies that the total lc threshold of S_Y is at most $\frac{1}{6}$. On the other hand, total lc thresholds of Gorenstein del Pezzo surfaces of degree 1 only with canonical singularities are at least $\frac{1}{6}$. Furthermore, considering blow-up construction of del Pezzo surfaces, we can show that a Gorenstein del Pezzo surface of degree 1 only with canonical singularities has total lc threshold $\frac{1}{6}$ if and only if it has only one singular point and this singular point is of type E_8. □

This is what exactly happens in Example 4.4.7. On the other hand, we can observe in Examples 4.4.2, 4.4.3, and 4.4.4 that smooth fibers have an Eckardt point or an effective anticanonical divisor with tacnode.

Proposition 4.5.2. *Let X/T and Y/T be simple Sh-models of a smooth del Pezzo surface of degree 2 defined over K. Suppose that we have a birational map $g : X \dashrightarrow Y$ over T. In* *addition, we assume that X has a smooth special fiber S_X. If the special fiber S_X has no effective anticanonical divisor with tacnode, then exactly one of the following occurs:*

1. *Birational map g is biregular.*

2. *The special fiber S_Y of Y must have only one singular point on it. Moreover, it is a Du Val singularity of type E_7.*

Proof. Note that the total lc threshold of S_X is $\frac{5}{6}$ by the assumption (Remark 3.2.4). Similar argument as in Proposition 4.5.1 shows statement. □

Proposition 4.5.3. *Let X/T and Y/T be simple Sh-models of a smooth del Pezzo surface of degree 3 defined over K. Suppose that we have a birational map $h : X \dashrightarrow Y$ over T. In addition, we assume that X has a smooth special fiber S_X. If the special fiber S_X has no Eckardt point, then exactly one of the following occurs:*

1. *Birational map h is biregular.*

2. *The special fiber S_Y of Y must have only one singular point on it. Moreover, it is a Du Val singularity of type E_7 or A_5.*

Proof. Note that the total lc threshold of S_X is $\frac{3}{4}$ by the assumption (Remark 3.2.4). Similar argument as in Proposition 4.5.1 shows statement. □

These statements are clear evidences that it is worth to study total lc thresholds of Gorenstein del Pezzo surfaces. It leads us to more information on birational maps between del Pezzo fibrations.

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