Sobolev resolvent estimates for the Laplace-Beltrami operator on compact manifolds

by

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Abstract

In this dissertation we systematically summarize the studies conducted in [2] and [9] about the possible region $\mathcal{R}_g \subset \mathbb{C}$ on which we have uniform $L^p \to L^q$ estimates for resolvent operators $(\Delta_g + \zeta)^{-1}, \zeta \in \mathcal{R}_g$ on a compact Riemannian manifold (M, g) with dimension $n \geq 2$. These studies answer a question regarding the largest possible region $\mathcal{R}_g$ on a general compact manifold, raised by Kenig, Dos Santos Ferreira and Salo in [3], and extend this result on the torus $\mathbb{T}^n$ and compact manifolds with non-positive curvature, using the half-wave operator $e^{it\sqrt{-\Delta_g}}$ and techniques from the wave equation.

Primary Reader: Christopher D. Sogge

Secondary Reader: Hans Lindblad

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Dedication

I dedicate this dissertation to my parents Yinghe Shao, Qiuxiang Xie and my fiancé Jianling Guo in China. Thank you all for your constant love and support.

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Chapter 1

Introduction

1.1 Historical results

The Laplacian operator is defined on a Riemannian manifold (M, g) as follows:

$$\Delta_g u = |g|^{-1/2} \sum_{i,j=1}^{n} \partial_i \left(|g|^{1/2} g^{ij} \right) \partial_j u, \quad \forall u \in C^\infty(M).$$

Here

$$g = \sum_{i,j}^{n} g_{ij} dx_i dx_j$$

is the Riemannian metric tensor in local coordinate charts $(x_1, x_2, \ldots, x_n)$. It is well known this operator is self-adjoint and negative as

$$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle, \quad \forall f, g \in C^\infty(M)$$

and

$$\langle \Delta f, f \rangle \leq 0, \quad \forall f \in C^\infty(M).$$

Here we want to make a remark about the term self-adjoint. The Laplacian is in fact only symmetric and essentially self-adjoint if we want to emphasize that it is a densely defined operator on Hilbert space $L^2(M)$ and we often abuse language a little by using the same symbol Δ_g to denote its Friedrichs extension on the Sobolev space in $L^2(M)$. However we have no intention to go through these functional analysis details as it may unnecessarily distract the readers' attention.

Now let us consider the Euclidean space $M = \mathbb{R}^n$. It is a standard result in spectral theory that $(-\infty, 0]$ is the (continuous) spectrum of Laplacian $\Delta_{\mathbb{R}^n}$. So we know if we choose a complex number $\zeta \notin [0, \infty)$ then the resolvent operator $(\Delta_{\mathbb{R}^n} + \zeta)^{-1}$ is a well-defined -2 order pseudodifferential operator (traditionally in analysis we use the "positive" Laplace $-\Delta_g$ since it is more convenient to deal with a positive operator). See for example [5], [13]. Therefore we know

$$||(\Delta + \zeta)^{-1} f||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C_\zeta ||f||_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \forall f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n), \zeta \notin [0, \infty). \tag{1.1}$$

Here we explain a little about the exponents $p = 2n/(n+2)$ and $q = 2n/(n-2)$. At first these two exponents are on the line of duality $1/p + 1/q = 1$, and they are also on the "Sobolev" line $1/p - 1/q = 2/n$. The first line is important because in the field of analysis where $L^p \to L^q$ estimates are concerned, usually the best and the most important results are achieved on the duality line, due to the fact that L^p and $L^{p'}$ are dual space of each other. The second line is named after the classical Sobolev estimate:

$$||f||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C||\Delta_{\mathbb{R}^n} f||_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \quad \forall f \in W^{2,2n/(2n+2)}(\mathbb{R}^n). \tag{1.2}$$

When $n \geq 3$, if we let $K(x,y)$ denote the integral kernel of an arbitrary -2 order pseudodifferential operator P, then it is bounded in the following sense:

$$|K(x,y)| \leq C|x-y|^{2-n}, \quad x \neq y.$$

Here C is a constant depending on the operator P. The proof of this result is classical and can be found in [14]. So by the Hardy-Littlewood-Sobolev inequality, which is a generalization of (1.2), we may also consider the generalization of (1.1) when $1/p - 1/q = 2/n$, p, q are within a certain range. In this thesis, we usually use the term "off-duality" estimates to refer to such generalized estimates.

It is noteworthy that in (1.1) it is not at all obvious whether the constants C_ζ can be taken independently from ζ. However in 1987, Kenig, Ruiz and Sogge surprisingly proved this to be true in [7] by establishing

$$||f||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C||(\Delta+\zeta)f||_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \quad \forall f \in W^{2,2n/(n+2)}(\mathbb{R}^n), \quad \forall \zeta \in \mathbb{C}. \tag{1.3}$$

Notice that these estimates are even true when $\zeta \geq 0$, compared with the trivial estimates (1.1).

The fact that ζ can be chosen to be any complex number partially attributes to the fact that the operator $-\Delta_{\mathbb{R}^n}$ has *no* eigenvalues, so there is no danger for the right side in (1.3) to be vanishing unless trivially $f = 0$. On the other hand, if we consider a compact Riemannian manifold (M, g) then the condition is vastly different. It is well known that the spectrum of the Laplacian $-\Delta_g$ on such a manifold consists only of isolated eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \ldots$ (see for example [6], [13]). So $\mathcal{R}_g = \mathbb{C}$ is obviously impossible for a compact manifold. However it is still interesting to ask what the largest possible region $\mathcal{R}_g \subset \mathbb{C}$ for a compact manifold can be, on which we have uniform resolvent estimates

$$||f||_{L^{\frac{2n}{n-2}}(M)} \leq C_{\mathcal{R}_g}||(\Delta_g + \zeta)f||_{L^{\frac{2n}{n+2}}(M)}, \forall \zeta \in \mathcal{R}_g, f \in W^{2,2n/(n+2)}(M), \tag{1.4}$$

or equivalently,

$$||(\Delta_g + \zeta)^{-1}f||_{L^{\frac{2n}{n-2}}(M)} \leq C_{\mathcal{R}_g}||f||_{L^{\frac{2n}{n+2}}(M)}, \forall \zeta \in \mathcal{R}_g, f \in C^{\infty}(M). \tag{1.5}$$

The most ideal case, compared with (1.3), would certainly be able to take

$$\mathcal{R}_g = \mathbb{C} \backslash \{\lambda_0, \lambda_1, \ldots\}.$$

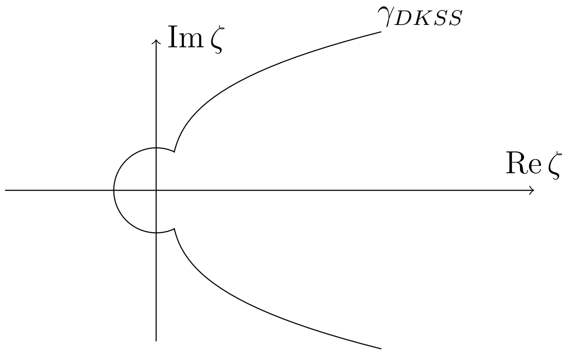


Figure 1.1: $\mathcal{R}_g$ for torus $\mathbb{T}^n$

However, as we are about to see in Theorem 1.2.1 this hope is far too optimistic and cannot be true.

The first result on this subject is due to Shen. In [10], by using the Poisson formula he proved that on torus $\mathbb{T}^n, n \geq 3$, we can essentially take $\mathcal{R}_g$ to be the region on the left side of a small disk centered at the origin, and a parabola opening to the right, as illustrated in figure 1.1.

Here we want to make a remark that the disk centered at the origin is merely for avoiding the first eigenvalue $\lambda_0 = 0$ and its shape is of no special purpose. So one can replace it with any other region as long as it contains the origin. The essence of this region $\mathcal{R}_g$ is the parabola, or equivalently the comparison between the real and imaginary parts of its boundary, which in this case would be

$$|\operatorname{Re}\zeta| \approx |\operatorname{Im}\zeta|^2, \quad \zeta \in \text{ boundary of } \mathcal{R}.$$

Later in [3], Dos Santos Ferreira, Kenig and Salo greatly generalized Shen's work to any compact Riemannian manifold of dimension $n \geq 3$ with the same region. They used the classical Hadamard parametrix for operator $(\Delta_g + \zeta)^{-1}$ which is similar to the techniques used in [7]. The appearance of the parabolic boundary of $\mathcal{R}_g$ given by two different methods seems to be quite interesting. So analogous to the Euclidean case, they asked if it is possible to improve the uniform resolvent estimates to the region bounded by γ_{opt} as shown in the following figure 1.2:

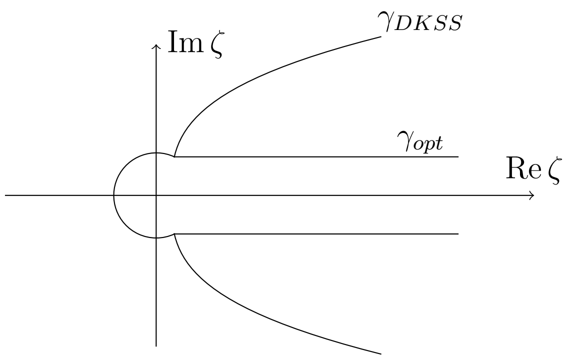


Figure 1.2: DKS question about the best possible $\mathcal{R}_g$

Answering this question is the starting point of this thesis.

1.2 Our results

In [2] we first showed that the region bounded by γ_{DKSS} in figure 1.1 is essentially optimal on the round spheres therefore cannot be improved:

Theorem 1.2.1. *Let S^n denote the standard sphere with dimension $n \geq 3$. Then if*

$\lambda \gg 1$ *we have*

$$||(\Delta_{S^n} + (\lambda + i\mu)^2)^{-1}||_{L^{\frac{2n}{n+2}}(S^n) \to L^{\frac{2n}{n-2}}(S^n)} \approx (\mathrm{dist}(\lambda + i\mu, \mathrm{Spec}(\sqrt{-\Delta_{\mathrm{S^n}}})))^{-1}.$$

Here we write $\zeta = (\lambda + i\mu)^2, \lambda > 0$ for technical convenience. So the parabola defined by $|\mathrm{Re}\,\zeta| \approx |\mathrm{Im}\,\zeta|^{1/2}$ now can be written, with respect to λ, μ, as $\lambda > 0, |\mu| \approx 1$.

Theorem 1.2.1 claims that one cannot essentially improve γ_{DKSS} at least on the round spheres, therefore answers the DKS problem negatively. The main idea of its proof is the observation that the resolvent estimates is majorized when the spectrum of $\sqrt{-\Delta_{S^n}}$ is close to λ. This inspired us to connect resolvent estimates to spectral analysis, which turned out to be successful.

In [2] we also tried to determine if the region bounded by γ_{opt} in figure 1.1 is possible on certain manifolds. Though we could not give a decisive answer to this question, we did derive a necessary condition which made the existence of such manifolds highly unlikely:

Theorem 1.2.2. *Suppose that* $0 < \varepsilon(\lambda) \leq 1$ *as* $\lambda \to \infty$ *and suppose further that* $\varepsilon(\lambda) \searrow 0$ *and when* $\lambda \gg 1$

$$||(\Delta_g + \lambda^2 + i\lambda\varepsilon(\lambda))^{-1}||_{L^{\frac{2n}{n+2}}(M) \to L^{\frac{2n}{n-2}}(M)} < C.$$

Then we have

$$N(\lambda + \varepsilon(\lambda)) - N(\lambda - \varepsilon(\lambda)) \in O(\varepsilon(\lambda)\lambda^{n-1}).$$

Here $N(\lambda)$ is the eigenvalue counting function which counts the number of eigenvalues not exceeding λ of $\sqrt{-\Delta_g}$. If we recall the classical Weyl formula about the asymptoti expansion of this function we have the following:

$$N(\lambda) = c_g\lambda^n + R_g, \quad R_g \in O(\lambda^{n-1}). \tag{1.6}$$

Here c_g is a constant depending on the manifold and its geometry. Notice that γ_{opt} corresponds to the region defined by $\mu \approx \lambda^{-1}$, so Theorem 1.2.2 leads to the conclusion: if on certain manifolds M the γ_{opt} curve can be achieved with regard to uniform resolvent estimates, then we must have

$$N(\lambda + 1/\lambda) - N(\lambda - 1/\lambda) \in O(\lambda^{n-2}).$$

Currently we have no idea how to show this is rigorously impossible. However, due to the complexity in calculating the precise expression of the remainder term R_g in the Weyl formula on even the simplest manifolds like S^n, the only likely approach to prove the existence of such manifolds is to prove $R_g \in O(\lambda^{n-2})$. We generally believe this is impossible. In fact, improving the reminder term R_g in the Weyl formula has been the central topic of spectral analysis for almost a century and so far the most famous results we have achieved are the following two theorems:

Theorem 1.2.3 ([1])**.** *If a manifold (M, g) has non-positive curvature, then we have*

$$R_g \in O(\lambda^{n-1}/\log\lambda).$$

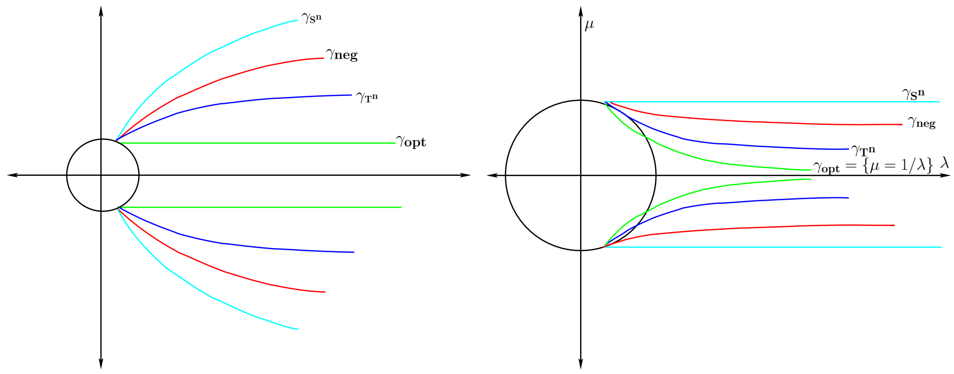
And,

Theorem 1.2.4 ([4])**.** *On the torus $\mathbb{T}^n$ we have*

$$R_g \in O(\lambda^{n-1-\frac{n-1}{n+1}}).$$

There is a folklore conjecture that on the torus $\mathbb{T}^n$ of dimension n we may achieve $R_g \in O(\lambda^{n-2+\varepsilon})$ in which $\varepsilon > 0$ is an arbitrarily small number. However, even this courageous conjecture is still far weaker compared with the conclusion $R_g \in O(\lambda^{n-2})$ implied by γ_{opt}. So we may safely expect that γ_{opt} to be impossible and the difference between γ_{DKSS} and γ_{opt} is essentially the difference between the Euclidean space and compact manifolds. It remains, of course, a very interesting and perhaps deep question to prove or disprove the existence of such manifolds in a rigorous way.

Having considered two extreme possibilities of the region $\mathcal{R}_g$, it is then natural to ask whether there exists some Riemannian manifolds (M, g) on which improvements on the region $\mathcal{R}_g$ can be made. In [2] we conducted investigation on this matter and obtained the following two theorems:



(a) $(\Delta_g + \zeta)^{-1}$ (b) $(\Delta_g + (\lambda + i\mu)^2)^{-1}$

Theorem 1.2.5. *Assume we have a Riemannian manifold (M, g) of dimension $n \geq 3$ with non-positive sectional curvatures. Then we have*

$$||f||_{L^{\frac{2n}{n-2}}(M)} \leq C||(\Delta_g + (\lambda + i\mu)^2)f||_{L^{\frac{2n}{n+2}}(M)}, \quad \lambda \gg 1, |\mu| \geq (\log \lambda)^{-1}.$$

And,

Theorem 1.2.6. *Let $\mathbb{T}^n$ denote the n-dimensional torus with $n \geq 3$, then we have*

$$||f||_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \leq C||(\Delta_{\mathbb{T}^n} + (\lambda + i\mu)^2)f||_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}, \quad \lambda \gg 1, |\mu| \geq \lambda^{-\frac{1}{n+1}}.$$

We make a summary about the regions $\mathcal{R}_g$ for various manifolds on which we can have uniform resolvent estimates in the following figures:

The key step in establishing Theorem 1.2.5 and 1.2.6 is the following result in [2] which relates the possible improved resolvent estimates with the shrinking spectral projection estimates:

Theorem 1.2.7. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Suppose that $0 < \varepsilon(\lambda) \leq 1$ decreases monotonically to zero as $\lambda \to +\infty$ and that $\varepsilon(2\lambda) \geq \frac{1}{2}\varepsilon(\lambda)$, $\lambda \geq 1$. Then one has the uniform shrinking spectral projection estimates*

$$\Big\| \sum_{|\lambda-\lambda_j|\leq\varepsilon(\lambda)} E_j f \Big\|_{L^{\frac{2n}{n-2}}(M)} \leq C\varepsilon(\lambda)\lambda\|f\|_{L^{\frac{2n}{n+2}}(M)}, \quad \lambda \geq 1, \tag{1.7}$$

if and only if one has the uniform improved resolvent estimates

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C\big\|(\Delta_g + (\lambda + i\mu)^2)u\big\|_{L^{\frac{2n}{n+2}}(M)},$$
$$\lambda, \mu \in \mathbb{R},\ \lambda \geq 1,\ |\mu| \geq \varepsilon(\lambda),\ u \in C^\infty(M). \tag{1.8}$$

By more refined analysis, the above results can be easily generalized to the off-duality case. In [2] we proved the following theorem concerning off-duality uniform resolvent estimates:

Theorem 1.2.8. *Let M be a compact Riemannian manifold of dimension $n \geq 3$. Then, if $1/p - 1/q = 2/n$, we have the following uniform resolvent estimates*

$$||f||_{L^q(M)} \leq C||(\Delta_g + (\lambda + i\mu)^2)f||_{L^p(M)} \tag{1.9}$$

if $p \leq 2(n+1)/(n+3)$ and $q \geq 2(n+1)/(n-1)$, and $\lambda, |\mu| \geq 1$. In particular, the constant C does not depend on λ, μ.

And similarly the improved resolvent estimates were also generalized to

Theorem 1.2.9. *If M is a boundaryless Riemannian compact manifold with dimension ≥ 3 and of non-positive sectional curvature, then for $1/p - 1/q = 2/n, p < 2(n+1)/(n+3), q > 2(n+1)/(n-1)$ we have the following uniform resolvent estimates*

$$||f||_{L^q(M)} \leq C||(\Delta_g + (\lambda + i\mu)^2)f||_{L^p(M)} \tag{1.10}$$

if $\lambda \gg 1$ and $|\mu| > (\log(\lambda))^{-1}$.

Theorem 1.2.10. *Let $\mathbb{T}^n$ denote the flat torus with $n \geq 3$. Then for a (p, q) pair satisfying $1/p - 1/q = 2/n$ and $p \leq 2(n+1)/(n+3), q \geq 2(n+1)/(n-1)$ there exists a function in p, which we denote by $\varepsilon_n(p)$, such that when $1/p$ is ranging from $(n+3)/2(n+1)$ to $(n+2)/2n$ ($\overline{AF}$ in figure 1.3) it increases from from 0 to $1/(n+1)$, and symmetrically decreases from $1/(n+1)$ to 0 when $(n+2)/2n \leq 1/p \leq (n^2+3n+4)/2n(n+1)$ ($\overline{FA'}$ in figure 1.3), and we have the following improved resolvent estimates*

$$||f||_{L^q(\mathbb{T}^n)} \leq C||(\Delta_{\mathbb{T}^n} + (\lambda + i\mu)^2)f||_{L^p(\mathbb{T}^n)}, \quad \lambda > 1, |\mu| \geq \lambda^{-\varepsilon(p)}. \tag{1.11}$$

The exact form for $\varepsilon_n(p)$ is given as:

$$\varepsilon_n(p) = \frac{\frac{n(n-1)t+n-1}{2(n+1)} - \frac{n-1}{n+1}(1-t) - \frac{n-1}{2}t}{\frac{n+1}{2}t + 1 - t}, \; t = \frac{2(n+1)}{n-1}\frac{1}{p} - \frac{n+3}{n-1}. \tag{1.12}$$

when $(1/p, 1/q)$ is below the line of duality.

Here is the $(1/p, 1/q)$ exponent diagram for the off-duality generalization results:

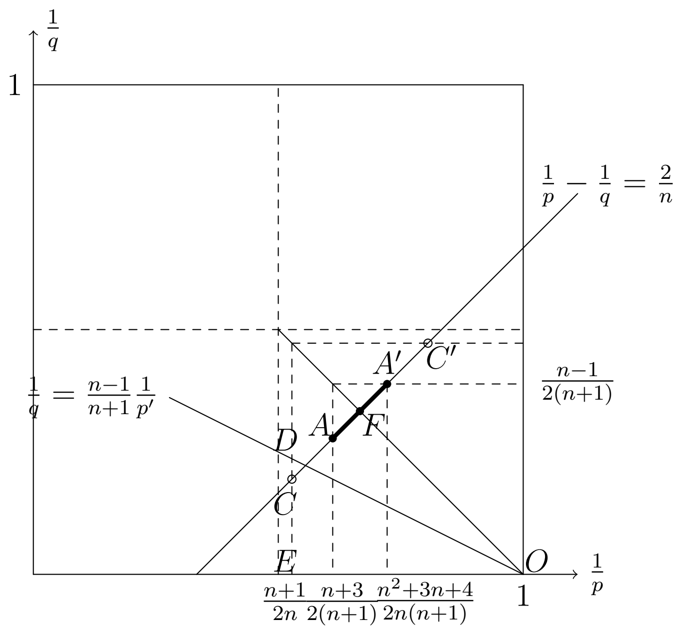


Figure 1.3: Off-duality pairs

$\overline{AA'}$ will be from the non-local operator (defined later), which happens to be the global range mentioned in Theorem 1.2.8. The Carleson-Sjölin argument used for the local operator will give us the segment $\overline{DO}$ and Young's inequality can give us the segment $\overline{EO}$. Therefore we can interpolate to obtain the segment $\overline{CC'}$ with both end points removed as the range for the local part. Since in general $\overline{AA'} \subset \overline{CC'}$, the resolvent estimate range for a compact manifold is therefore, as far as we can prove, constrained in $\overline{AA'}$. Notice that point $F = (\frac{n+2}{2n}, \frac{n-2}{2n})$ is the $(1/p, 1/q)$ pair considered in [3] and [7].

We are also able to investigate the uniform resolvent estimates for surfaces $(n = 2)$ as follows:

Theorem 1.2.11. *Let (M, g) be a Riemannian surface of dimension 2, then we have for $\lambda \gg 1, |\mu| \geq 1$ the following uniform resolvent estimates:*

$$||f||_{L^\infty(M)} \leq C||(\Delta_g + (\lambda + i\mu)^2)f||_{H^1(M)}.$$

Here $H^1(M)$ is the Hardy space on compact manifolds defined in [15].

This thesis is organized as follows: in Chapter 2 we will use the half-wave operator technique to prove Theorem 1.2.8. Our technique differs from the original one used in [3]: we take the Fourier transform with respect to the time variable t, instead of the space variable as in [3], of the half-wave operator $e^{it\sqrt{-\Delta_g}}$ and split the resolvent into local and non-local parts. The advantage of our technique is we can more easily relate the non-local part to the geometry using global harmonic analysis. Chapter 3 consists of several important applications of the argument developed in Chapter 2: showing the sharpness of γ_{DKSS} on round spheres; showing Theorem 1.2.2, then finally establishing the key theorem in this thesis, Theorem 1.2.7. Finally in Chapter 4 we then move on to apply Theorem 1.2.7 to improving the uniform resolvent estimates, say Theorem 1.2.9 and Theorem 1.2.10.

Throughout this paper $\delta(p) = n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ for $1 \leq p \leq \infty$ and unless specified otherwise we generally assume $1 \leq p \leq 2 \leq q \leq \infty$.

Chapter 2

Uniform resolvent estimates on general Riemannian manifolds

2.1 Splitting of resolvent operator

Our key interpretation of the resolvent operator $(\Delta_g+\zeta)^{-1}$ and its Schwartz kernel $\mathfrak{G}(x,y)$ is as follows. In the sense of spectral theory we have

$$\mathfrak{G}(x,y) = \sum_j \frac{e_j(x)e_j(y)}{\zeta - \lambda_j^2}, \quad x, y \in M, \;\; \zeta \in \mathbb{C}\backslash\overline{\mathbb{R}_+}.$$

Since the formula involves λ_j^2, and since the map $\mathbb{C} \ni z \to z^2 \in \mathbb{C}$ is onto, it is natural to write ζ as a square

$$\zeta = (\lambda + i\mu)^2, \quad \lambda > 0, \mu \in \mathbb{R}, \ \mu \neq 0,$$

as we mentioned in the Introduction, so that we can use the following

Lemma 2.1.1. *For all $\mu \in \mathbb{R}$, $\mu \neq 0$*

$$\int_{-\infty}^{\infty} \frac{e^{-i\tau t}}{\mu - i\tau} \, d\tau = 2\pi \operatorname{sgn}\mu H(\mu t) e^{-\mu t}, \tag{2.1}$$

where $H(t)$ is the Heaviside function which equals one for $t \geq 0$ and 0 for $t < 0$. Also, for all $\lambda, \mu \in \mathbb{R}$, $\mu \neq 0$,

$$\int_{-\infty}^{\infty} \frac{e^{-i\tau t}}{\tau^2 - (\lambda + i\mu)^2} \, d\tau = \frac{i\pi \operatorname{sgn}\mu}{\lambda + i\mu} e^{i(\operatorname{sgn}\mu)\lambda|t|} e^{-|\mu t|}. \tag{2.2}$$

Proof. To prove (2.1), we may assume that $\mu > 0$ since the other case follows from reflection. Then if $0 < \varepsilon < \mu$, Cauchy's integral formula yields

$$\int_{-\infty}^{\infty} \frac{e^{-i\tau t}}{\mu - i\tau} \, d\tau = e^{-(\mu - \varepsilon)t} \int_{-\infty}^{\infty} \frac{e^{-it\tau}}{\varepsilon - i\tau} \, d\tau.$$

Letting $\varepsilon \searrow 0$, the right side tends to $e^{-\mu t}$ times the Fourier transform of $i(\tau + i0)^{-1}$, and since the latter is $2\pi H(t)$, we get (2.1).

Since

$$\frac{1}{\tau^2-(\lambda+i\mu)^2}=\frac{1}{2(\lambda+i\mu)}\Big(\frac{1}{\tau-\lambda-i\mu}-\frac{1}{\tau+\lambda+i\mu}\Big),$$

(2.1) implies (2.2). Alternatively, since for $\mu>0$ the right side of (2.2) is

$$\frac{\pi i}{\lambda+i\mu}e^{i(\lambda+i\mu)|t|},$$

we conclude that for $\lambda=0$ and $\mu>0$, (2.2) is just the formula for the Fourier transform of the Poisson kernel:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-i\tau t}\frac{\varepsilon}{\tau^2+\varepsilon^2}\,d\tau=e^{-\varepsilon|t|},\quad \varepsilon>0.$$

Since both sides of (2.2) are analytic in the half-plane $\{z=\lambda+i\mu,\ \mu>0\}$, we conclude that (2.2) must be valid when $\mu>0$. The formula for $\mu<0$ follows from this and the fact that the left side of (2.2) is an even function of $z=\lambda+i\mu$. $\square$

Using the formula (2.2) we get from Fourier's inversion formula

$$\begin{aligned}\big(\lambda_j^2-(\lambda+i\mu)^2\big)^{-1}&=\frac{\pi i\,\mathrm{sgn}\mu}{2\pi(\lambda+i\mu)}\int_{-\infty}^{\infty}e^{i(\mathrm{sgn}\,\mu)\lambda|t|}e^{-|\mu t|}\,e^{i\lambda_j t}\,dt\\&=\frac{i\,\mathrm{sgn}\mu}{\lambda+i\mu}\int_0^{\infty}e^{i(\mathrm{sgn}\,\mu)\lambda t}e^{-|\mu|t}\cos t\lambda_j\,dt.\end{aligned}$$

Consequently, if , as usual, we set, $P = \sqrt{-\Delta_g}$, then we have for $f \in C^\infty$ the formula

$$\left(\Delta_g + (\lambda + i\mu)^2\right)^{-1} f = \frac{\text{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn}\,\mu)\lambda t} e^{-|\mu|t} \left(\cos tP\right) f \, dt. \tag{2.3}$$

This expression indicates that the largest singularity of the resolvent operator happens when $t = 0$, so it is natural to split the operator according to the time variable t as a short-time local part and a long-time non-local part. To handle the local operator we use a Littlewood-Paley argument to obtain precise information about the singularity, and break the latter's spectrum up into unit-length pieces and sum their L^p bounds up together. Fortunately due to the smallness of the singularity of the non-local operator we will not lose too much information.

Now consider a function $\rho(t) \in C_0^\infty(\mathbb{R})$ with support restricted in $|t| \leq 1$. Then we define the local and non-local operators as

$$\mathfrak{S}_{loc}(P) = \frac{\text{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t) e^{i\lambda t} e^{-|\mu|t} \cos tP dt \tag{2.4}$$

and

$$r_{\lambda,\mu}(P) = \frac{\text{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty (1 - \rho(t)) e^{i\lambda t} e^{-|\mu|t} \cos tP dt. \tag{2.5}$$

In the following two sections we are going to investigate these two operators using local and global techniques respectively.

2.2 Local operator

Our main theorem in this section is

Theorem 2.2.1. *The local operator $\mathfrak{S}_{loc}(P)$ is uniformly bounded for $\lambda > 1$ and $\mu \neq 0$ from $L^p(M)$ to $L^q(M)$ if $1/p - 1/q = 2/n$ and $p < 2n/(n+1), q > 2n/(n-1)$, or more straightforwardly when $(1/p, 1/q)$ is on the segment $\overline{CC'}$ in figure 1.3 with both end points removed.*

If we are trying to improve the uniform resolvent estimates, say proving the uniform boundedness of $(\sqrt{-\Delta_g} + (\lambda + i\mu)^2)^{-1}$ with $|\mu| \ll 1$, then this theorem tells us instead of considering the entire resolvent operator, we can simply focus on its non-local part, which has a strong connection to the underlying geometry as we are about to see.

Now let us prove Theorem 2.2.1. For simplicity we only prove the case when $\mu > 0$ and the other case is purely symmetric. We first dyadically split the local resolvent into

$$\mathfrak{S}_{loc}(x,y) = \sum_{j=0}^{\infty} S_j(x,y),$$

in which

$$S_j f = \frac{1}{i(\lambda + i\mu)} \int_0^{\infty} \beta(\lambda 2^{-j} t)\rho(t) e^{i\lambda t - \mu t} \cos tP dt, \quad j \geq 1 \tag{2.6}$$

and

$$S_0 f = \frac{1}{i(\lambda + i\mu)} \int_0^{\infty} (1 - \sum_{j=0}^{\infty} \beta(\lambda 2^{-j} t))\rho(t) e^{i\lambda t - \mu t} \cos tP dt. \tag{2.7}$$

Here the function $\beta \in C_0^\infty(\mathbb{R}^1)$ satisfies the following properties

$$\beta(t) = 0, t \notin [1/2, 2], |\beta(t)| \leq 1, \text{ and } \sum_{-\infty}^{\infty} \beta(2^{-j}t) = 1 \tag{2.8}$$

Roughly speaking, S_0 is the worse part of the local operator as its time support is close to the singular point $t = 0$. So we take advantage of its $O(\lambda^{-1})$ smallness of the support to directly estimate its kernel. For simplicity we denote $\tilde{\rho}(\lambda t) = 1 - \sum_0^\infty \rho(\lambda 2^{-j} t)$. We want to prove:

Lemma 2.2.2. *The multiplier $S_0(\tau)$ defined as*

$$S_0(\tau) = \frac{1}{i(\lambda + i\mu)} \int_0^\infty \tilde{\rho}(\lambda t)\rho(t) e^{i\lambda t} e^{-\mu t} \cos t\tau dt \tag{2.9}$$

is a -2 order symbol function with symbol norm independent from λ or μ.

Proof. Due to the small t support of the integrand we know that when $|\tau| < 1$ the integral along with its τ derivatives in (2.9) are uniformly bounded. Therefore we need only to prove that

$$|\frac{d^j}{d\tau^j} S_0(\tau)| \leq C_j \tau^{-2-j}, |\tau| \geq 1. \tag{2.10}$$

Let us first prove the case $j = 0$, which may help the readers understand how to handle the general case. Due to the fact that

$$\cos t\tau = \frac{1}{\tau}\frac{d}{dt}\sin t\tau, \quad \sin t\tau = -\frac{1}{\tau}\frac{d}{dt}\cos t\tau, \tag{2.11}$$

we can do integration by parts twice and end up with some integral boundary terms. Combining the fact that the integrand has a small t support $t \leq 4\lambda^{-1}$, $e^{-\mu t}\mu$ is integrable uniformly in μ and $|\lambda + i\mu| \geq \lambda$ or μ we immediately see that both the boundary terms and the integrals are uniformly bounded. This proves (2.10) when $j = 0$.

Now after taking j times τ derivatives we have

$$\frac{d^j}{d\tau^j}S_0 = \frac{\pm 1}{i(\lambda + i\mu)}\int \tilde{\rho}(\lambda t)\rho(t)e^{i\lambda t}e^{-\mu t}t^j \cos t\tau dt$$

when j is even, and with $\cos t\tau$ being replaced by $\sin t\tau$ when j is odd. Then similar to the $j = 0$ case we take integration by parts in t for $j + 2$ times. Thanks to the presence of t^j no matter j is even or odd the boundary terms would be non-vanishing only at the final step, which can be estimated similarly as in the case $j = 0$. So for simplicity we assume j is even and ignore the boundary terms.

Now by Leibniz's formula we have, for $\alpha, \beta, \gamma \geq 0$,

$$\begin{aligned}\frac{d^j}{d\tau^j} S_0 &= \sum_{\alpha+\beta+\gamma=j+2} \frac{C_{\alpha\beta\gamma}}{\tau^{j+2}(\lambda+i\mu)} \int_0^\infty \frac{d^\alpha}{dt^\alpha}(e^{-\mu t}) \frac{d^\beta}{dt^\beta}(t^j) \frac{d^\gamma}{dt^\gamma}(\tilde{\rho}(\lambda t)\rho(t)e^{i\lambda t}) \cos t\tau dt \\ &= \frac{1}{\tau^{j+2}(\lambda+i\mu)}\Big(\sum_{\alpha+\beta=j+2} + \sum_{\alpha+\beta=j+1} + \sum_{\alpha+\beta\leq j} \Big) \\ &= \frac{1}{\tau^{j+2}(\lambda+i\mu)}(I+II+III).\end{aligned}$$

Notice that in I we have $\gamma = 0$, in II we have $\gamma = 1$ and in III we have $\gamma \geq 2$. A simple check will show that the terms in I are those containing $\mu^{k+2}t^k, k \geq 0$, therefore can be estimated using variable scaling $t \to \mu^{-1}t$ and the fact that $|\lambda + i\mu| > \mu$. Similarly the terms in II are those containing $\mu^{k+1}t^k, k \geq 0$ and can be handled using the same scaling and the fact $|\lambda + i\mu| > \lambda$. Finally, it is easy to check

$$III = \sum_{\alpha+\beta\leq j} \frac{C_{\alpha\beta}}{\tau^{j+2}(\lambda+i\mu)} \int_0^\infty e^{-\mu t}(\mu t)^\alpha t^{j-\beta-\alpha} \left(\frac{d}{dt}\right)^{j-\beta-\alpha+2} (\tilde{\rho}(\lambda t)\rho(t)e^{i\lambda t}) \cos t\tau dt.$$

Using the facts that the factors $e^{-\mu t}(\mu t)^\alpha$ are uniformly bounded, on the support of the integrands we have $t < \lambda^{-1}$ and $|\lambda + i\mu| > \lambda$ the proof is therefore complete. □

With the aid of this lemma, we know that $S_0(P)$, defined in the sense of spectral theory by $S_0(P)f = \sum_{j=0}^\infty S_0(\lambda_j)E_j f, f \in C^\infty(M)$ in which E_j is the j-th eigenspace projection associated with eigenvalue λ_j of P, is therefore a -2 order pseudodifferential operator (see for example [12], Theorem 4.3.1). And in particular its symbol norms are uniformly bounded from λ or μ. This then leads us to the following kernel estimate (see for example Proposition 1 on the page 241 of [14]) if we assume that $n \geq 3$:

$$|S_0(x,y)| \leq C d_g(x,y)^{2-n}.$$

By the Hardy-Littlewood-Sobolev inequality, S_0 is a $L^p \to L^q$ bounded operator on the Sobolev line $\{(1/p, 1/q) : 1/p - 1/q = 2/n\}$.

To deal with $S_j, j \geq 1$, we need the following proposition about oscillatory integrals:

Proposition 2.2.3. *Let $n \geq 2$ and assume that $a \in C^\infty(\mathbb{R}_+)$ satisfies the Mihlin-type (for more details about these singular functions, see for example [14]) condition that for each $j = 0, 1, \ldots, 2$*

$$|\frac{d^j}{ds^j} a(s)| \leq A_j s^{-j}, s > 0 \tag{2.12}$$

Then there are constants B, B_j, which depend only on the size of finitely many of the constants A_j so that for every $\omega \in \mathbb{C}$ such that for $\operatorname{Im}\omega \neq 0$ and $1/4 < |x| < 4$ we have

$$\int_{\mathbb{R}^n} \frac{a(|\xi|)e^{ix\cdot\xi}}{|\xi| - \omega} d\xi = |x|^{1-n} a_{1,\omega}(x) + \sum_{\pm} e^{\pm i(\operatorname{Re}\omega)|x|} |\operatorname{Re}\omega|^{\frac{n-1}{2}} |x|^{-\frac{n-1}{2}} a^{\pm}_{2,\omega}(|x|), \tag{2.13}$$

where $|a_{1,\omega}(x)| = B$ is a bounded smooth function, $|\frac{d^j}{ds^j} a^{\pm}_{2,\omega}(s)| \leq B_j s^{-j}$. Therefore $a^{\pm}_{2,\omega}(|x|)$ have bounded derivatives under our assumption on $|x|$.

Proof. If $\operatorname{Re}\omega = 0$, then we take $\xi \to \varepsilon\xi$ scaling to prove (2.13) since now the oscillatory integral is the Fourier transform of a -1 order Mihlin-type symbol function. We then assume from now on that $\mathrm{Re}\,\omega \neq 0$, which allows us to take the scaling $\xi \to \mathrm{Re}\,\omega\xi$. Notice that if $|\mathrm{Im}\,\omega/\mathrm{Re}\,\omega| \geq 1$ or $|\xi| \notin (1/4, 4)$ then we again have a -1 order Mihlin-type symbol function with symbol norms uniformly bounded, so we need only to consider the following integral

$$|\mathrm{Re}\,\omega|^{n-1} \int \frac{\beta(|\xi|)a(\mathrm{Re}\,\omega|\xi|)e^{i\mathrm{Re}\,\omega x\cdot\xi}}{|\xi| - 1 + i\varepsilon}, \quad 0 < |\varepsilon| < 1 \tag{2.14}$$

in which $\beta(r)$ is a smooth function supported in $(1/4, 4)$ and equal to 1 in $(1/2, 2)$. For simplicity we write $\beta(|\xi|)a(\mathrm{Re}\,\omega|\xi|)$ as $\alpha(|\xi|)$. Now when $|\mathrm{Re}\,\omega| \cdot |x| < 1$, we can use the property that the function $\alpha(|\xi|)e^{i\mathrm{Re}\,\omega x\cdot\xi}$ has bounded ξ derivatives to show that the integral in (2.14) will be uniformly bounded under such assumption. Therefore we need only to consider the case when $|\mathrm{Re}\,\omega||x| \geq 1$.

Recall the following standard formula about the Fourier transform of sphere S^{n-1}:

$$\int_{S^{n-1}} e^{x\cdot\omega} d\sigma(\omega) = \sum_{\pm} |x|^{-\frac{n-1}{2}} c_{\pm}(|x|)e^{\pm i|x|}, \tag{2.15}$$

in which we have

$$|\frac{d^j}{dr^j}c_{\pm}(r)| \geq r^{-j}, r \leq 1/4. \tag{2.16}$$

So using polar coordinates and this formula we obtain the following integrals:

$$\sum_{\pm}(|\frac{\operatorname{Re}\omega}{x}|)^{\frac{n-1}{2}}\int_{\frac{1}{4}}^{4}\frac{\alpha(r)c_{\pm}(r|\operatorname{Re}\omega||x|)}{r-1+i\varepsilon}r^{\frac{n-1}{2}}e^{\pm ir\operatorname{Re}\omega|x|}dr,\quad 0<|\varepsilon|<1. \tag{2.17}$$

Now by using Lemma 2.1.1 we are easily able to prove:

Lemma 2.2.4.

$$\int\frac{e^{-irt}}{r-1+i\varepsilon}dr=2\pi iH(\varepsilon t)e^{-it}e^{-|\varepsilon t|}, \tag{2.18}$$

in which $H(t)$ is the Heaviside function.

Now we can regard the integrals in (2.17) as convolutions between right side of equation in (2.18) and $b_{\pm}(t,|x|)$, in which $b_{\pm}(t,|x|)$ denotes the r Fourier transform of function $\alpha(r)c_{\pm}(r\operatorname{Re}\omega|x|)r^{\frac{n-1}{2}}$. Notice that in particular we have

$$|D_x^{\gamma}D_t^{\alpha}b_{\pm}(t,|x|)|\leq C_{N,\gamma,\alpha}(1+t)^{-N} \tag{2.19}$$

due to the support of $|x|$ and the assumption that $|\operatorname{Re}\omega|\cdot|x|>1$. Without loss of generality we assume $\operatorname{Re}\omega>0$ and $\varepsilon>0$. The other cases can be handled similarly. Under such assumption the convolutions can be written as

$$2\pi ie^{\mp i\operatorname{Re}\omega|x|}(e^{-\operatorname{Re}\omega\varepsilon|x|}\int_{-\infty}^{\operatorname{Re}\omega|x|}e^{it}e^{\varepsilon t}b_{\pm}(t,|x|)dt). \tag{2.20}$$

Therefore the proof to the proposition will be complete as long as we can show that the functions in the parentheses have uniformly bounded x derivatives.

In fact, we have

$$\begin{aligned}
&\partial_x^\gamma (e^{-\mathrm{Re}\,\omega\varepsilon|x|}\int_{-\infty}^{\mathrm{Re}\,\omega|x|} e^{it}e^{\varepsilon t}b_\pm(t,|x|)dt) \\
=&\sum_{|j|+|k|=|\gamma|-1} C_{j,k}\partial_x^j(\mathrm{Re}\,\omega\varepsilon|x|e^{-\mathrm{Re}\,\omega\varepsilon|x|})\partial_x^k(e^{i\mathrm{Re}\,\omega|x|}b_\pm(\mathrm{Re}\,\omega|x|,|x|)) \\
&+\sum_{j+k=\gamma} C_{j,k}\partial_x^j(e^{i\mathrm{Re}\,\omega\varepsilon|x|})\int_{-\infty}^{\mathrm{Re}\,\omega|x|} e^{it}e^{-\varepsilon t}\partial_x^k(b_\pm(t,|x|))dt \\
=&I+II.
\end{aligned}$$

By the fact that $\varepsilon < 1, |x| \approx 1$ and (2.2) we immediately see that I is uniformly bounded. Now for II, after a $t \to -t + \mathrm{Re}\,\omega$ variable substitution the terms in it are majorized by the following integral

$$(\mathrm{Re}\,\omega\varepsilon)^j\int_0^\infty e^{-\varepsilon t}|b_\pm^{(k)}(-t+\mathrm{Re}\,\omega,|x|)|dt \le C_{j,k,N}\int_0^\infty (\varepsilon\mathrm{Re}\,\omega)^j(1+\varepsilon t)^{-j}(1+|t-\mathrm{Re}\,\omega|)^{-N}dt. \tag{2.21}$$

By arguing the ratio between t and $\mathrm{Re}\,\omega$ we can immediately show that II is also uniformly bounded. So the proof is complete. □

Now, let $\varepsilon = \frac{2^j}{\lambda}$, so the kernel of S_j can be written as

$$S_j(x,y) = \frac{1}{\lambda+i\mu}\int_0^\infty \beta(t/\varepsilon)\rho(t)e^{\lambda t}e^{-\mu t}\cos tP(x,y)dt. \tag{2.22}$$

We should notice that due to the finite propagation speed of the wave operator $\cos tP$

the kernels $S_j(x,y)$ are actually supported in the region where $|d_g(x,y)/\varepsilon| < 2$, so we have

$$S_j(x,y) = \rho(x,y,\varepsilon)S_j(x,y) \tag{2.23}$$

in which we recall that ρ is a smooth bump function supported when $d_g(x,y) \leq 4\varepsilon$. By such consideration we can therefore restrict the support of all the following operators to such a small region.

By Euler's formula, in geodesic coordinates we have

$$\cos tP(x,y) = \sum_{\pm} \int_{\mathbb{R}^n} e^{i\kappa(x,y)\cdot\xi} e^{i\pm t|\xi|} \alpha_{\pm}(t,x,y,|\xi|)d\xi + Q(t,x,y) \tag{2.24}$$

in which $Q(t,x,y)$ is a smooth function with compact support in t,x,y, $\kappa(x,y)$ are the geodesic coordinates of x about y and $\alpha_{\pm}(t,x,y,|\xi|)$ are 0-order symbol functions in ξ, say they satisfy:

$$|\lambda_{\lambda}^{\gamma_1}\lambda_{t,x,y}^{\gamma_2}\lambda_{\pm}(t,x,y,|\xi|) \leq C_{\gamma_1,\gamma_2}(1+|\xi|)^{-|\gamma_1|}. \tag{2.25}$$

Notice that here the amplitude functions $\alpha_{\pm}$ were chosen to be radial in ξ, which is critical in our later argument. This can be easily done through the classical Hadamard parametrix construction, see for example [13] Chapter 1 and 2 for details. So if we replace the operator $\cos tP$ in (2.22) with $Q(t,x,y)$ we have a operator with $L^p \to L^q$ norm belonging to $O(\lambda^{-1}\varepsilon)$. Also due to the t support in (2.22) we would have that there are only $O(\log_2 \lambda)$ many S_j not vanishing. So summing over so many j will give us a bounded operator over the Sobolev line immediately.

So by abusing language a little bit, we can replace the wave operator $\cos tP$ with the Fourier integral representation in (2.24) and consider this new operator only. Now take a $(t, \xi) \to (\varepsilon t, \xi/\varepsilon)$ scaling in (2.22) we then obtain

$$S_j^{\pm}(x,y) = \frac{\varepsilon^{1-n}}{\lambda + i\mu} \int_0^\infty \int_{\mathbb{R}^n} \beta(t)\rho(\varepsilon t)e^{i\lambda\varepsilon t}e^{-\mu\varepsilon t}\alpha_{\pm}(\varepsilon t, x, y, |\xi|/\varepsilon)e^{i\frac{\kappa(x,y)}{\varepsilon}\cdot\xi \pm t|\xi|}d\xi dt \tag{2.26}$$

for $d_g(x,y)/\varepsilon < 4$. However, when $d_g(x,y)/\varepsilon \le 1/4$, an integration by parts argument with respect to ξ would show that

$$|S_j^{\pm}(x,y)| \le \varepsilon^{1-n}\lambda^{-1} \le d_g(x,y)^{2-n}2^{-j},$$

which is a (p,q) bounded operator over the Sobolev line after summation over j. So we are reduced to considering the operator $K_j^{\pm} = \beta(\frac{d_g(x,y)}{2\varepsilon})S_j^{\pm}(x,y)$ in which $\beta(r)$ is supported when $r \in (1/2, 2)$.

Let $a_\varepsilon^{\pm}(\tau, x, y, |\xi|)$ denote the inverse Fourier transform of

$$t \to \beta(t)\rho(\varepsilon t)\alpha_{\pm}(\varepsilon t, x, y, |\xi|/\varepsilon),$$

in which

$$|D_\xi^\gamma a_\varepsilon^{\pm}(\tau, x, y, |\xi|)| \le C_{N,\gamma}(1+\tau)^{-N}|\xi|^{-|\tau|}. \tag{2.27}$$

Then after the Fourier transform in (2.26) we would have

$$K_j^{\pm}(x,y) = \varepsilon^{2-n} \int \int_{\mathbb{R}^n} e^{i\frac{\kappa(x,y)}{\varepsilon}\cdot\xi} \frac{a_\varepsilon^{\pm}(\tau,x,y,|\xi|)}{(\pm|\xi| - \tau - \varepsilon\lambda - i\varepsilon\mu)(\pm|\xi| - \tau + \varepsilon\lambda + i\varepsilon\mu)} d\xi d\tau. \tag{2.28}$$

Now if we split the fraction in the integrand as follows:

$$\begin{aligned}&\frac{1}{(\pm|\xi| - \tau - \varepsilon\lambda - i\varepsilon\mu)(\pm|\xi| - \tau + \varepsilon\lambda + i\varepsilon\mu)}\\ =&\frac{1}{2(\varepsilon\lambda + i\varepsilon\mu)}\left(\frac{1}{\pm|\xi| - \tau - \varepsilon\lambda - i\varepsilon\mu} - \frac{1}{\pm|\xi| - \tau + \varepsilon\lambda + i\varepsilon\mu}\right)\end{aligned}$$

then we would have, after applying Proposition 2.2.3,

$$K_j^{\pm}(x,y) = 2^{-j}\varepsilon^{2-n}a_{1,\omega}(x,y) + 2^{-j}\varepsilon^{2-n}\sum_{\pm}\int e^{\pm i(\tau\pm\varepsilon\lambda)\frac{d_g(x,y)}{\varepsilon}} \cdot |\tau \pm \varepsilon\lambda|^{\frac{n-1}{2}} a_\varepsilon(\tau,x,y) d\tau, \tag{2.29}$$

in which $a_{1,\omega}(x,y)$ is a uniformly bounded smooth function with support when $d_g(x,y)/\varepsilon \in (1/4,4)$, and $a_\varepsilon(\tau,x,y)$ is virtually the function $a_{2,\omega}$ in (2.13). In particular, the latter satisfies the following properties

$$|\partial_{x,y}^{\gamma} a_\varepsilon(\tau,x,y)| \leq C_{\gamma,N}(1+\tau)^{-N}\varepsilon^{-|\gamma|} \tag{2.30}$$

also due to the fact that $d_g(x,y)/\varepsilon \approx 1$.

Now for the $2^{-j}\varepsilon^{2-n}a_{1,\omega}(x,y)$ term, a simple calculation shows that its $L^1 \to L^\infty$ norm is bounded by $2^{-j}\varepsilon^{2-n}$ and the $L^p \to L^p$ norm bounded by $2^{-j}\varepsilon^2, 1 \leq p \leq \infty$.

So interpolation shows that these operators sum up to a (p,q) bounded operator on the Sobolev line. Therefore we are reduced to analyzing the second term in (2.29). After carrying out the τ integral, we immediately see that we are further reduced, without loss of generality, to estimating the (p,q) bounds over the Sobolev line for the following operators

$$T_j(x,y)=2^{-j}\varepsilon^{2-n}2^{j\frac{n-1}{2}}e^{i\lambda d_g(x,y)}a_\varepsilon(x,y)=\lambda^{\frac{n-3}{2}}\varepsilon^{-\frac{n-1}{2}}e^{i\lambda d_g(x,y)}a_\varepsilon(x,y) \tag{2.31}$$

in which the smooth function $a_\varepsilon(x,y)$ is supported when $d_g(x,y)/\varepsilon\in(1/4,4)$, and

$$|\partial^\gamma_{x,y}a_\varepsilon(x,y)|\le C_\gamma\varepsilon^{-|\gamma|}. \tag{2.32}$$

Now the rest part will be a standard procedure as in [12] since we know the function $d_g(x,y)$ satisfies the Carleson-Sjölin condition, and this was also done similarly in [3]. See [7] also for the Euclidean case. For the reader's convenience we state it briefly as follows. First, (2.30) reminds us that if we scale x,y back to unit-length by $(x,y)\to(\varepsilon x,\varepsilon y)$ we have

$$|\partial^\gamma_{x,y}a_\varepsilon(\varepsilon x,\varepsilon y)|\le C_\gamma. \tag{2.33}$$

Also the phase function now is $\lambda d_g(\varepsilon x,\varepsilon y)=2^j d_g(\varepsilon x,\varepsilon y)/\varepsilon$ and $d_g(\varepsilon x,\varepsilon y)/\varepsilon$ will satisfy Carleson-Sjölin condition, i.e. the hypersurface $\nabla_x d_g(\varepsilon x,\varepsilon y_0)/\varepsilon$ has curvature bounded away from zero. So, if $f(y)$ is a test function, then we have, if

$$\frac{1}{q} = \frac{n-1}{n+1}(1-\frac{1}{p}), \quad 1 \le p \le 2, \tag{2.34}$$

that

$$\begin{aligned}
&||T_j(x,y)f(y)||_q \\
=&\lambda^{\frac{n-3}{2}}\varepsilon^{-\frac{n-1}{2}}(\int|\int e^{-i\lambda d_g(x,y)}a_\varepsilon(x,y)f(y)dy|^p dx)^{\frac{1}{q}} \\
=&\lambda^{\frac{n-3}{2}}\varepsilon^{-\frac{n-1}{2}}\varepsilon^{n+\frac{n}{q}}(\int|\int e^{-i2^j d_g(\varepsilon x,\varepsilon y)/\varepsilon}a_\varepsilon(\varepsilon x,\varepsilon y)f(\varepsilon y)dy|^p dx)^{\frac{1}{q}} \\
\lesssim&\lambda^{\frac{n-3}{2}}\varepsilon^{-\frac{n-1}{2}}\varepsilon^{n+\frac{n}{q}}2^{\frac{-jn}{q}}\varepsilon^{-\frac{n}{p}}||f||_p \\
=&\lambda^{-2+n(\frac{1}{p}-\frac{1}{q})}2^{j(\frac{n+1}{2}-\frac{n}{p})}||f||_p.
\end{aligned} \tag{2.35}$$

Here the inequality is due to the standard $n \times n$ Carleson-Sjölin estimate in [12] (Theorem 2.2.1). On the other hand, if we apply Young's inequality to the kernel $T_j(x,y)$ as in (2.31), we also see that the kernel on the line $(1/p, 0)$ will be $L^p \to L^q$ bounded by the same norm as in (2.35) with corresponding exponent. Now a simple interpolation shows that our local operator is $L^p \to L^q$ bounded when $(1/p, 1/q)$ are on $\overline{CC'}$ in figure 1.3, with end points removed.

This completes the proof of Theorem 2.2.1 and therefore reduces the resolvent estimate problem to the non-local part.

2.3 Non-local operator

Recall the non-local operator is defined as:

$$r_{\lambda,\mu}(P) = \frac{1}{i(\lambda + i\mu)} \int_0^\infty (1 - \rho(t))e^{i\lambda t}e^{-\mu t} \cos tP dt \tag{2.36}$$

in which we assume that $\mu \geq 1$. This operator is easier than its local counterpart to handle with and we can prove its boundedness with a slightly larger (p, q) range:

Theorem 2.3.1. *The non-local operator $r_{\lambda,\mu}(P)$ defined in (2.36) is a uniformly bounded operator from $L^p(M)$ to $L^q(M)$ if $1/p - 1/q = 2/n$ and $p \leq 2(n+1)/n+3, q \geq 2(n+1)/(n-1)$. Notice that this is the segment $\overline{AA'}$ given in figure 1.3.*

The proof is based on the following lemma:

Lemma 2.3.2. *Given a fixed compact Riemannian manifold of dimension $n \geq 3$ there is a constant C so that whenever $\alpha \in C(\mathbb{R}_+)$ and let*

$$\alpha_k(P)f = \sum_{\lambda_j \in [k-1,k)} \alpha(\lambda_j)E_j f, k = 1, 2, \ldots$$

then we have

$$||\alpha_k(P)f||_{L^q(M)} \leq Ck(\sup_{\tau \in [k-1,k)} |\alpha(\tau)|)||f||_{L^p(M)} \tag{2.37}$$

if $1/p - 1/q = 2/n$ and $p \leq 2(n+1)/(n+3), q \geq 2(n+1)/(n-1)$.

Before proving this lemma, let us recall the following theorem in [12] and [11]:

Theorem 2.3.3. *If χ_λ denotes the spectral cluster projection of the operator $P = \sqrt{-\Delta_g}$, namely $\chi_\lambda f = \sum_{\lambda_j \in [\lambda-1,\lambda)} E_j f$, then we have the following estimates*

$$||\chi_\lambda f||_{L^2(M)} \leq C(1+\lambda)^{\delta(p)}||f||_{L^p(M)}, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}, \tag{2.38}$$

and

$$||\chi_\lambda||_{L^q(M)} \leq C(1+\lambda)^{\delta(q)}||f||_{L^2(M)}, \quad \frac{2(n+1)}{n-1} \leq q \leq \infty \tag{2.39}$$

in which $\delta(p) = n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.

Now since $\alpha_k(P) = \chi_k \alpha_k(P) \chi_k$, we have

$$\begin{aligned} ||\alpha_k(P)f||_{L^q(M)} &\lesssim k^{\delta(q)}||\alpha_k(P)\chi_k f||_{L^2(M)} \\ &\lesssim k^{\delta(q)}(\sup_{\tau\in[k-1,k)}|\alpha(\tau)|)||\chi_k f||_{L^2(M)} \\ &\lesssim k^{\delta(q)+\delta(p)}(\sup_{\tau\in[k-1,k)}|\alpha(\tau)|)||f||_{L^p(M)}. \end{aligned}$$

On the Sobolev line we happen to have $\delta(p) + \delta(q) = 1$. This completes the proof to the lemma. Now to prove the Theorem 2.3.1, we just need to notice that under our assumption $\mu \geq 1$ a simple integration by parts argument and Euler's formula show that we have the following estimates:

$$|r_{\lambda,\mu}(\tau)| \leq C_N \lambda^{-1}((1+|\lambda-\tau|)^{-N} + (1+|\lambda+\tau|)^{-N}), \quad \lambda, \mu \geq 1, N = 1, 2, \ldots. \tag{2.40}$$

So now the Lemma (2.3.2) comes into application immediately if we take $N = 3$ as follows:

$$||r_{\lambda,\mu}(P)||_{L^p \to L^q} \le C \sum_{k=1}^{\infty} k\lambda^{-1}(1 + |\lambda - k|)^{-3} \le C' \tag{2.41}$$

This completes the proof of Theorem 2.3.1. A simple calculation shows the line segment $\overline{CC'}$ always contains $\overline{AA'}$ as shown in figure 1.3. So if we combine Theorem 2.2.1 and 2.3.1 together, based on the assumption $|\mu| \ge 1$, the proof of Theorem 1.2.8 is automatically complete.

2.4 $n = 2$ case

In this section we are going to prove Theorem 1.2.11. Let us recall that on the Euclidean space $\mathbb{R}^n$ the Riesz transforms $R_j f$ of functions f are defined via Fourier transform by

$$\widehat{R_j f}(\xi) = \frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

And the Hardy space $H^1(\mathbb{R}^n)$ is defined to be those functions $f \in L^1(\mathbb{R}^n)$ such that

$$||f||_{H^1} = ||f||_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} ||R_j f||_{L^1(\mathbb{R}^n)} < +\infty. \tag{2.42}$$

Similarly, as in [15], on a compact manifold M we fix a smooth measure dx on M equivalent to Lebesgue measure in every coordinate patch. Let $\{\varphi_i\}$ be a smooth partition of unity subordinate to a system of coordinate neighborhoods and let $\{\psi_i\}$

be a smooth function supported in a coordinate neighborhood satisfying $\psi_i\varphi_=\varphi_i$. We then define $H^1(M)$ to be the subspace of functions f in $L^1(M)$ such that $\psi_i R_j(\varphi_j f)$ is integrable for all i, j. Roughly speaking $H^1(M)$ is just the space which locally looks like $H^1(\mathbb{R}^n)$ over each coordinate patch. Therefore by choosing a local coordinate system and Lebesgue measure dx, we can simply work as in the Euclidean space.

Recall that during the previous sections we split the resolvent on a compact manifold into three pieces

$$\mathfrak{S}_{loc}(P) + r_{\lambda,\mu}(P) = S_0(P) + \sum_{j\geq 1} S_j + r_{\lambda,\mu}(P).$$

A careful examination on the proof to Theorem 1.2.8 will reveal that when $n = 2$, in which case the "Sobolev" line degenerates into a single point $O = (1, 0)$ in figure 1.3, the only place where the proof breaks down is the S_0 part. The other two pieces are bounded $L^1 \to L^\infty$ operators and since the Hardy space H^1 can be embedded into L^1 we know that they are also $H^1 \to L^\infty$ bounded, which leaves us only the S_0 part to deal with. Recall that due to the fact $S_0(\tau)$ is a -2 order symbol, the pseudodifferential operator $S_0(P)$ will have kernel bounded by $|x - y|^{2-n}$ when $x \neq y$ and $n \geq 3$. Similar to the fundamental solution of the classical Laplacian on $\mathbb{R}^2$, when $n = 2$ the kernel of $S_0(P)$ is bounded by $\log |x - y|^{-1}$ when $x \neq y$. See [14] or [16]. So in order to prove Theorem 1.2.11 we need only to prove that when $f \in H^1$

we have

$$||\int \log|x-y|f(y)dy||_{L^\infty(M)} \leq C||f||_{H^1(M)}. \tag{2.43}$$

Again we need only to prove it for the Euclidean case, which is standard since

$$|\int \log|x-y|f(y)dy| \leq ||\ \log|x-\cdot|\ ||_{BMO}||f||_{H^1}.$$

Here the BMO norm is defined as

$$||f||_{BMO} = \sup_B \int_B |f(y)-f_B|dy, f \in L^1_{loc}(\mathbb{R}^n), \tag{2.44}$$

in which B are all the balls and f_B is the average of f on the ball B. It is well known that $\log|y|$ is within BMO space and a parallel transform or reflection on the y variable will not change its BMO norm. The proof to (2.43) is then complete immediately.

Remark: Recall that in [2], the requirement that $n \geq 3$ is due to the fact that when we were dealing with the local resolvent operator, we used a global version of Lemma 2.3.2 which involves a Littlewood-Paley theory argument. It is well-known that the Littlewood-Paley theory breaks down between L^1 and L^∞. However, in [8] Seeger and Sogge also proved that the theory is valid between H^1 and BMO spaces therefore can lead to the same result. Our approach is somehow simpler.

Chapter 3

Further analysis on resolvent estimates

3.1 Sharp resolvent estimates on spheres

In this section we are going to prove the sharpness of γ_{DKSS} when $M = S^n$, the round spheres. Before we prove Theorem 1.2.1, it is convenient to prove the following lemma at first:

Lemma 3.1.1.

$$||(I - \chi_{[\lambda-1,\lambda+1]}) \circ (\Delta_g + (\lambda + i\mu)^{-1})f||_{L^{\frac{2n}{n-2}}(M)} \leq C||f||_{L^{\frac{2n}{n+2}}(M)},$$

in which $\lambda > 1, \mu \neq 0$.

This lemma tells us that the major contribution for the resolvent operator with respect to L^p estimates essentially comes from its cut-off when the spectrum of $\sqrt{-\Delta_g}$ is close to λ.

Proof. Let us first prove Lemma 3.1.1 when $|\mu| \geq 1$, which is easy. Since we have proved when $|\mu| \geq 1$ the resolvent operator $(\Delta_g + (\lambda + i\mu)^2)^{-1}$ is $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ bounded, we need only to show

$$||\chi_{[\lambda-1,\lambda+1]} \circ (\Delta_g + (\lambda + i\mu)^2)^{-1}||_{L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}} \geq C.$$

This follows immediately from Lemma 2.3.2. After we proved the case when $|\mu| \geq 1$, it is obvious that now we need only to prove the following estimates

$$||(I - \chi_{[\lambda-1,\lambda+1]}) \circ \left((\Delta_g + (\lambda + i)^2)^{-1} - (\Delta_g + (\lambda + i\mu)^2)^{-1}\right)||_{L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}} \leq C,$$

when $\lambda \geq 1, \mu \in [-1, 1]\backslash\{0\}$. This can be done using the following fact:

$$\begin{aligned}
&\frac{1}{\tau^2 - (\lambda + i)^2} - \frac{1}{\tau^2 - (\lambda + i\mu)^2} \\
=&\frac{1}{2\tau}\left(\frac{1}{\tau - \lambda - i} + \frac{1}{\tau + \lambda + i}\right) - \frac{1}{2\tau}\left(\frac{1}{\tau - \lambda - i\mu} + \frac{1}{\tau + \lambda + i\mu}\right) \\
=&\frac{1}{2\tau}\frac{i(1-\mu)}{(\tau - \lambda - i)(\tau - \lambda - i\mu)} + \frac{1}{2\tau}\frac{i(\mu - 1)}{(\tau + \lambda + i)(\tau + \lambda + i\mu)},
\end{aligned}$$

when $|\tau - \lambda| \geq 1$ and $|\mu| \leq 1$. An application of Lemma 2.3.2 now completes the proof immediately.

Using this lemma, it is very easy to prove Theorem 1.2.1 as follows. It is a standard result (see for example [12]) that the eigenvalues of operator $\sqrt{-\Delta_{S^n}}$ are $\left\{\sqrt{k(k+n-1)}\right\}$, $k = 1, 2, \ldots,$ in particular the gaps between of these isolated eigenvalues are comparable to 1 when k is large. So by Lemma 3.1.1 we need only to find an integral k_0 such that $\lambda - \sqrt{k_0(k_0+n-1)} < 1/2$ and then prove

$$\left\|\chi_{[\lambda-1/2,\lambda+1/2]} \circ (\Delta_{S^n} + (\lambda+i\mu)^2)^{-1}\right\|_{L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}} \approx |\sqrt{k_0(k_0+n-1)} - (\lambda+i\mu)|^{-1}. \tag{3.1}$$

Due to the fact that $\sqrt{k_0(k_0+n-1)}$ is the only eigenvalue in $[\lambda-1/2, \lambda+1/2]$ the left side of (3.1) is actually equal to

$$(-k_0\sqrt{k_0+n-1} + \lambda + i\mu)^2)^{-1} H_{k_0}$$

in which H_{k_0} is the spectral projection. It was shown in [11] that

$$\|H_{k_0}\|_{L^{\frac{2n}{n+2}}(S^n) \to L^{\frac{2n}{n-2}}(S^n)} \approx k_0.$$

So the left side of (3.1) is comparable to

$$| - k_0(k_0+n-1) + (\lambda+i\mu)^2|^{-1} k_0 \approx |\sqrt{k_0(k_0+n-1)} - (\lambda+i\mu)|^{-1}.$$

So the proof is complete.

Remark: It is shown in [2] that by using a standard result about the spectral distribution of the Zoll manifolds due to Weinstein [17], one can adopt the proof above easily to show that γ_{DKSS} is essentially optimal for the Zoll manifolds as well.

$\square$

3.2 A necessary condition for γ_{opt}

In this section we are going to prove Theorem 1.2.2. Before that, it is convenient to prove the following lemma at first:

Lemma 3.2.1. *Suppose $\beta \in C_0^\infty(\mathbb{R})$ satisfies*

$$\beta(\tau) = 0, \quad \tau \notin [1/4, 4].$$

Then if $1 \le q \le r \le \infty$, there is a constant $C = C(r,q)$ so that

$$\|\beta(P/\lambda)f\|_{L^r(M)} \le C\lambda^{n(\frac{1}{q}-\frac{1}{r})}\|f\|_{L^q(M)}, \quad \lambda \ge 1. \tag{3.2}$$

We make a remark that by using the lemma we can verify our assertion that if

$$\left\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}f\right\|_{L^p(M)} \le C_p\,\varepsilon(\lambda)\lambda^{2\delta(p)}\|f\|_{L^{\frac{p}{p-1}}(M)}, \tag{3.3}$$

is valid for some finite exponent then it must be valid for $p = \infty$, in which $\varepsilon(\lambda)$ satisfies the properties stated in Theorem 1.2.7. We just choose a β as in the lemma satisfying $\beta(\tau) = 1$ for $\tau \in [1/2, 2]$. It then follows that for large λ we have $\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]} = \beta(P/\lambda) \circ \chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]} \circ \beta(P/\lambda)$ and so applying the lemma twice with exponents (q, r) being equal to (p, ∞) and $(1, p/(p-1))$, where p as is in (3.3) yields

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^1(M)\to L^\infty(M)} \leq C\lambda^{\frac{2n}{p}} \|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^{\frac{p}{p-1}}(M)\to L^p(M)}.$$

Recall the definition of $\delta(p)$, we have $2\delta(p) + 2n/p = n - 1 = 2\delta(\infty)$, we conclude that (3.3) implies that

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^1(M)\to L^\infty(M)} \leq C'\varepsilon(\lambda)\lambda^{n-1},$$

as we asserted before.

Proof of Lemma 3.2.1. Let $K_\lambda(x, y)$ denote the kernel of $\beta(P/\lambda)$. Then the proof of Theorem 4.3.1 in [12] shows that for every $N = 1, 2, 3, \dots$ there is a constant C_N which is independent of λ so that

$$|K_\lambda(x, y)| \leq C_N \lambda^n (1 + \lambda d_g(x, y))^{-N}, \quad \lambda \geq 1.$$

Consequently, (3.2) follows from an application of Young's inequality. $\square$

Now, to prove Theorem 1.2.2 we need to show that if there is a sequence $\tau_k \to \infty$

and $\varepsilon(\tau_k) \searrow 0$ with $\varepsilon(\tau_k) > 0$ and

$$(\varepsilon(\tau_k)\tau_k^{n-1})^{-1}\big[N(\tau_k+\varepsilon(\tau_k)-N(\tau_k-\varepsilon(\tau_k)-)\big] \longrightarrow \infty, \tag{3.4}$$

then

$$\big\|(\Delta_g+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}\big\|_{L^{\frac{2n}{n+2}}(M)\to L^{\frac{2n}{n-2}}(M)} \longrightarrow \infty. \tag{3.5}$$

We know from Lemma 3.1.1 that $(I-\chi_{[\tau_k-1,\tau_k+1]})\circ(\Delta_g+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}$ has $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ norm bounded by a uniform constant. Consequently, we would have (3.5) if we could show that

$$\big\|\chi_{[\tau_k-1,\tau_k+1]}\circ(\Delta_g+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}\big\|_{L^{\frac{2n}{n+2}}(M)\to L^{\frac{2n}{n-2}}(M)} \longrightarrow \infty.$$

Arguing as in the remark following Lemma 3.2.1 shows that the operator norm in the left is bounded from below by $c\tau_k^{-(n-2)}$ times the $L^1(M) \to L^\infty(M)$ operator norm for some positive constant c. Consequently, we would be done if we could show that

$$\tau_k^{-(n-2)}\big\||\chi_{[\lambda-1,\lambda+1]}\circ(\Delta_g+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}\big\|_{L^1(M)\to L^\infty(M)} \longrightarrow \infty. \tag{3.6}$$

The kernel of the operator is

$$\sum_{|\lambda_j-\tau_k|\le 1}(-\lambda_j^2+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}e_j(x)e_j(y),$$

and since the $L^1(M) \to L^\infty(M)$ norm is the supremum of the kernel, we deduce that the left side of (3.6) majorizes

$$\tau_k^{-(n-2)} \sup_{x\in M} \left| \sum_{|\lambda_j-\tau_k|\le 1} (-\lambda_j^2+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}|e_j(x)|^2 \right|.$$

Since the imaginary part of $(-\lambda_j^2+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}$ is $-\tau_k\varepsilon(\tau_k)/((\tau_k^2-\lambda_j^2)^2+\tau_k^2(\varepsilon(\tau_k))^2)$, we deduce that for large k we have

$$\begin{aligned}
\tau_k^{-(n-2)} \big\| |\chi_{[\lambda-1,\lambda+1]} \circ (\Delta_g+\tau_k^2+i\tau_k\varepsilon(\tau_k))^{-1}\big\|_{L^1(M)\to L^\infty(M)} \\
&\ge \tau_k^{-(n-2)} \sup_{x\in M} \sum_{|\lambda_j-\tau_k|\le 1} \frac{\tau_k\varepsilon(\tau_k)}{(\tau_k^2-\lambda_j^2)^2+\tau_k^2(\varepsilon(\tau_k))^2}|e_j(x)|^2 \\
&\ge \tau_k^{-(n-2)} \sup_{x\in M} \sum_{|\lambda_j-\tau_k|\le \varepsilon(\tau_k)} \frac{\tau_k\varepsilon(\tau_k)}{(\tau_k^2-\lambda_j^2)^2+\tau_k^2(\varepsilon(\tau_k))^2}|e_j(x)|^2 \\
&\ge \frac{1}{10}\tau_k^{-(n-1)}(\varepsilon(\tau_k))^{-1} \sup_{x\in M} \sum_{|\lambda_j-\tau_k|\le \varepsilon(\tau_k)} |e_j(x)|^2 \\
&\ge \frac{1}{10\mathrm{Vol}_g(M)}\,\tau_k^{-(n-1)}(\varepsilon(\tau_k))^{-1}\int_M \sum_{|\lambda_j-\tau_k|\le \varepsilon(\tau_k)} |e_j(x)|^2\,dV_g \\
&= \frac{1}{10\mathrm{Vol}_g(M)}\,\tau_k^{-(n-1)}(\varepsilon(\tau_k))^{-1}\big[N(\tau_k+\varepsilon(\tau_k))-N(\tau_k-\varepsilon(\tau_k)-)\big],
\end{aligned}$$

and since, by assumption, the last quantity tends to ∞ as $k\to\infty$, we get (3.6), which completes the proof.

3.3 Relation between resolvent estimates and spectral estimates

In this section we are going to prove Theorem 1.2.7. We shall first see how (1.7) implies (1.8). In view of Theorem 1.2.8, especially the case when $p = 2n/(n+2), q = 2n/(n-2)$, to prove (1.8), it suffices to just consider the case where

$$\varepsilon(\lambda) \le \mu \le 1. \tag{3.7}$$

Also, in view of Theorem 2.2.1, if $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(t) = 1,\ t \le 1/2, \quad \rho(t) = 0,\ t \ge 1/2, \tag{3.8}$$

then it suffices to verify that

$$\begin{aligned} \Big\| \int_0^\infty e^{i\lambda t} \big(\cos t\sqrt{-\Delta_g} f\big)\, (1-\rho(t)) e^{-\mu t}\, dt \Big\|_{L^{\frac{2n}{n-2}}(M)} & \\ \le C\lambda \|f\|_{L^{\frac{2n}{n+2}}(M)}, \quad \varepsilon(\lambda) \le \mu \le 1. & \end{aligned} \tag{3.9}$$

To use (1.7) to prove the resolvent estimates, we need the following simple lemma.

Lemma 3.3.1. *Suppose that $0 < \mu \le 1$ and that ρ is as in (3.8). Then for every*

$N = 1, 2, 3, \dots$ there is a constant C_N so that

$$\left| \int_0^\infty e^{i\lambda t \pm i\tau t}(1-\rho(t))e^{-\mu t}\, dt \right| \\ \leq C_N \left[(1+|\lambda \pm \tau|)^{-N} + \mu^{-1}(1+\mu^{-1}|\lambda \pm \tau|)^{-N} \right]. \quad (3.10)$$

Proof. If $|\lambda \pm \tau| \leq \mu$, the result is trivial. So we may assume $|\lambda \pm \tau| \geq \mu$. If we then integrate by parts and use Leibniz's rule, we find that the left side of (3.10) is majorized by

$$|\lambda \pm \tau|^{-N} \sum_{j+k=N} \int_0^\infty \mu^j e^{-\mu t} \left| \frac{d^k}{dt^k}(1-\rho(t)) \right| dt.$$

If $k \neq 0$, the summand is dominated by the first term in the right side of (3.10), in view of (3.8). For the remaining case where $j = N$ and $k = 0$, it is clearly dominated by the second term in the right side of (3.10). $\square$

Now let us try to prove Theorem 1.2.7 To show that (1.7) implies (1.8) we need to verify (3.9). By Lemma 3.3.1, the operator in the left side of this inequality is of the form

$$\sum_{j=0}^\infty m_{\lambda,\mu}(\lambda_j)E_j f,$$

where for every $N = 1, 2, 3, \dots$ there is a constant C_N so that for $0 < \mu \leq 1$

$$|m_{\lambda,\mu}(\lambda_j)| \leq C_N \Big[(1+|\lambda - \lambda_j|)^{-N} + \mu^{-1}(1+\mu^{-1}|\lambda - \lambda_j|)^{-N} \Big].$$

Since our assumption (1.7) implies that there is a uniform constant C so that

$$\Big\|\sum_{|\lambda-\lambda_j|\le\mu} E_j f\Big\|_{L^{\frac{2n}{n-2}}(M)} \le C\lambda\mu\|f\|_{L^{\frac{2n}{n+2}}(M)}, \quad \lambda\ge 1,\ \varepsilon(\lambda)\le\mu\le 1,$$

(3.9) follows from the proof of Lemma 2.3.2.

To prove the converse, we note that if (1.8) were valid, then we would have the uniform bounds

$$\varepsilon(\lambda)\lambda\Big\|\sum_{j=0}^{\infty}\big((\lambda_j^2-\lambda^2+\varepsilon(\lambda)^2)^2+(2\varepsilon(\lambda)\lambda)^2\big)^{-1}E_jf\Big\|_{L^{\frac{2n}{n-2}}(M)} \le C\|f\|_{L^{\frac{2n}{n+2}}(M)}, \quad (3.11)$$

due to the fact that

$$\frac{4i\varepsilon(\lambda)\lambda}{(\lambda_j^2-\lambda^2+\varepsilon(\lambda)^2)^2+(2\varepsilon(\lambda)\lambda)^2} = \frac{1}{\lambda_j^2-(\lambda+i\varepsilon(\lambda))^2} - \frac{1}{\lambda_j^2-(\lambda-i\varepsilon(\lambda))^2}.$$

This and a T^*T argument in turn implies that

$$\sqrt{\varepsilon(\lambda)\lambda}\,\Big\|\sum_{j=0}^{\infty}\big((\lambda_j^2-\lambda^2+\varepsilon(\lambda)^2)^2+(2\varepsilon(\lambda)\lambda)^2\big)^{-\frac12}E_jf\Big\|_{L^2(M)} \le C\|f\|_{L^{\frac{2n}{n+2}}(M)}.$$

As

$$\sqrt{\varepsilon(\lambda)\lambda}\,\big((\lambda_j^2-\lambda^2+\varepsilon(\lambda)^2)^2+(2\varepsilon(\lambda)\lambda)^2\big)^{-\frac12} \ge \frac{1}{10}\big(\varepsilon(\lambda)\lambda\big)^{-\frac12}, \quad \text{if } |\lambda-\lambda_j|\le\varepsilon(\lambda),$$

orthogonality and the preceding inequality imply that

$$\Big\| \sum_{|\lambda_j-\lambda|\le\varepsilon(\lambda)} E_j f\Big\|_{L^2(M)} \le C\sqrt{\varepsilon(\lambda)\lambda}\,\|f\|_{L^{\frac{2n}{n+2}}(M)}. \tag{3.12}$$

Since by another T^*T argument (3.12) is equivalent to (1.7), the proof is complete.

Now we need to point out that the equivalence in Theorem 1.2.7 relies heavily on the duality between $p = 2n/(n+2)$ and $q = 2n/(n-2)$ and consequently a standard TT^* argument. In the off-duality case such argument is not available anymore. However we can still manage to adopt the proof above by composing $L^p \to L^2$ and $L^2 \to L^q$ together just like the proof to Lemma 2.3.2. The result is the following theorem which is Theorem 1.3 in [9]:

Theorem 3.3.2. *Let M be a compact Riemannian manifold of dimension $n \ge 3$. Suppose that for $2n(n+1)/(n^2+3n+4) \le p \le 2(n+1)/(n+3)$ we have a function $0 < \varepsilon_p(\lambda) \le 1$ decreasing monotonically to 0 as $\lambda \to \infty$ and $\varepsilon_p(2\lambda) \ge \varepsilon_p(\lambda)/2$, for λ sufficiently large. Then if we have*

$$|| \sum_{|\lambda-\lambda_j|\le\varepsilon_p(\lambda)} E_j f||_{L^{p'}(M)} \le C\varepsilon_p(\lambda)\lambda^{2\delta(p)}||f||_{L^p(M)}, \lambda \gg 1, \tag{3.13}$$

we also have the following resolvent estimates for $1/p - 1/q = 2/n, p \le 2(n+1)/(n+3), q \ge 2(n+1)/(n-1)$:

$$||f||_{L^q(M)} \le C||(\Delta_g + (\lambda+i\mu)^2)f||_{L^p(M)}, |\mu| \ge \max\{\varepsilon_p(\lambda), \varepsilon_{q'}(\lambda)\}, \lambda \gg 1. \tag{3.14}$$

Currently we are not able to prove any equivalence just as in the duality case. This is a typical phenomenon in analysis where a off-duality L^p estimates is harder than the dual case. Nonetheless this theorem is still sufficient for us to proceed since our main concern is to use the shrinking spectral estimates on certain manifold with special geometry to prove the improved resolvent estimates, not vice versa.

Chapter 4

Improved resolvent estimates on manifolds with various geometry

4.1 Manifolds with non-positive curvature

In this section we are going to prove Theorem 1.2.9. Our technique is based on an L^p variant of the earlier supernorm bounds implicit of Bérard in [1], and the composition technique we used in the proof of Lemma 2.3.2.

Now by Theorem 3.3.2 we need only to prove the following result:

Theorem 4.1.1. *If* (M, g) *is a compact manifold of dimension* $n \geq 3$ *with non-*

positive sectional curvature then we have, when $p < \frac{2(n+1)}{n+3}$,

$$||\sum_{|\lambda_j-\lambda|<1/\log\lambda} E_j f||_{L^2(M)} \le C(\log\lambda)^{-\frac{1}{2}}\lambda^{\delta(p)}||f||_{L^p(M)}, \quad \lambda \gg 1. \tag{4.1}$$

Before we go through the details of the proof, we want to point out that if we can prove (4.1) with $\log\lambda$ replaced by $\varepsilon\log\lambda$ in which the fixed ε is smaller than 1 and only depends on M and p, then L^2 orthogonality would immediately show that (4.1) can be proved with a larger constant C possibly depending on p. But certainly this is harmless for us.

So we need to prove (4.1) with $\varepsilon\log\lambda$ replacing $\log\lambda$, say

$$||\sum_{|\lambda_j-\lambda|<1/(\varepsilon\log\lambda)} E_j f||_{L^2(M)} \le C(\varepsilon\log\lambda)^{-\frac{1}{2}}\lambda^{\delta(p)}||f||_{L^p(M)} \tag{4.2}$$

in which the specific value of ε is about to be determined later. First, we claim that if we choose an even nonnegative function $a \in \mathcal{S}(\mathbb{R})$ satisfying $a(r) = 1, |r| \le 1/2$ and having its Fourier transform supported in $(-1, 1)$, then in order to (4.2) it is sufficient to prove that for the multiplier

$$a(\varepsilon\log\lambda(\lambda - P)) = \frac{1}{2\pi\varepsilon\log\lambda}\int \hat{a}(t/(\varepsilon\log\lambda))e^{it\lambda}e^{-itP}dt \tag{4.3}$$

we have

$$||a(\varepsilon \log \lambda(\lambda - P))f||_{L^{p'}(M)} \leq (\varepsilon \log \lambda)^{-1} \lambda^{2\delta(p)} ||f||_{L^p(M)}, \quad p < \frac{2(n+1)}{n+3}. \tag{4.4}$$

In fact, due to the non-negativity of $a(r)$ especially $a(r) \approx 1$ when r is near 0, we know that if we use the fact $a = (\sqrt{a})^2$ then a TT^* argument and the above estimate will immediately imply that

$$|| \sum_{|\lambda_j - \lambda| < 1/(\varepsilon \log \lambda)} E_j f ||_{L^2(M)} \leq C(\varepsilon \log \lambda)^{-\frac{1}{2}} \lambda^{\delta(p)} ||f||_{L^p(M)} \tag{4.5}$$

due to L^2 orthogonality.

We then proceed in the way that we proved the estimates on the local operator $\mathfrak{S}_{loc}(P)$, say breaking the t interval into one part when $t \leq 1$ and the other one when $1 \leq t \lesssim \log \lambda$ (c.f. (2.6) and (2.7)).

Now, if $\psi \in C^\infty(\mathbb{R}^1)$ is an even function and $\psi(r) = 1$ when $|r| > 2$ and $\psi(r) = 0$ when $|r| < 1$ then we claim that the operator defined as

$$b_\lambda(P) = \frac{1}{2\pi\varepsilon \log \lambda} \int (1 - \psi(t)) \hat{a}(t/(\varepsilon \log \lambda)) e^{i(\lambda - P)t} dt$$

satisfies the estimate in (4.4) when $p < 2(n+1)/(n+3)$. This can be proved very easily by the $L^p \to L^{p'}$ version of Lemma 2.3.2 and the fact that

$$|b_\lambda(\tau)| \le C(\varepsilon \log \lambda)^{-1}(1 + |\lambda - \tau|)^{-N}.$$

So we need only to consider

$$\frac{1}{2\pi\varepsilon \log \lambda} \int \psi(t)\hat{a}(t/(\varepsilon \log \lambda))e^{i(\lambda - P)t}dt. \tag{4.6}$$

To proceed, we need to replace the e^{-itP} in the integrand above by $\cos tP$ since we are going to use the latter's Huygens principle. Now, notice that since both ψ and a are even functions, the difference between the operator in the above formula and

$$\frac{1}{2\pi\varepsilon \log \lambda} \int \psi(t)\hat{a}(t/(\varepsilon \log \lambda))e^{i\lambda t} \cos tP dt$$

is a smoothing operator with size of $O(\lambda^{-N})$, as P is a positive operator. So we are reduced to proving that if we let $a_\lambda(P)$ denote

$$a_\lambda(P) = \int \psi(t)\hat{a}(t/\varepsilon \log \lambda)e^{i\lambda t} \cos tP dt$$

then

$$||a_\lambda(P)f||_{L^{p'}(M)} \le C\lambda^{2\delta(p)}||f||_{L^p(M)}, \quad p < 2(n+1)/(n+3). \tag{4.7}$$

We are going to use interpolation to prove (4.7). More specifically we want to prove that

$$||a_\lambda(P)||_{L^1\to L^\infty} \le C\log\lambda e^{c\varepsilon\log\lambda}\lambda^{(n-1)/2}, \tag{4.8}$$

in which c is a small number depending on the geometry of the manifold M and

$$||a_\lambda(P)||_{L^2\to L^2} \le C\log\lambda. \tag{4.9}$$

The second one is obvious since we have the fact that $\cos tP$ is a bounded L^2 operator (e^{itP} a unitary operator on $L^2(M)$). And we can see if we are able to prove (4.8) then by interpolation and the fact that $\log\lambda \in o(\lambda^\varepsilon)$ for an arbitrarily small $\varepsilon > 0$, we are able to show when $p < 2(n+1)/(n+3)$ then there is a number

$$\varepsilon(p) = (n+1)(\frac{1}{p} - \frac{n+3}{2(n+1)}) > 0$$

so that

$$||a_\lambda(P)||_{L^{p'}(M)} \le C\lambda^{2\delta(p)}\lambda^{\frac{2-p}{p}c\varepsilon-\varepsilon(p)}||f||_{L^p(M)} \le C\lambda^{2\delta(p)}||f||_{L^p(M)}$$

if we choose ε small enough according to $\varepsilon(p)$ and c. So (4.7) is proved.

Notice that due to the appearance of $\log\lambda$ we are not able to prove the end point estimate when $p = 2(n+1)/(n+3)$. In fact, by using Lemma 2.3.2 we showed in [2] that when $p = 2(n+1)/(n+3)$ we can only obtain the following bound

$$||a_\lambda f||_{L^{\frac{2(n+1)}{n-3}}(M)} \leq C\lambda^{\frac{n-1}{n+1}} \log\lambda ||f||_{L^{\frac{2(n+1)}{n+1}}(M)}$$

which is $\log\lambda$ worse than the (4.7).

Now let us prove (4.8), which derives from the original Bérard argument as follows. To prove this inequality, we shall use the fact that because of our assumption of non-positive sectional curvatures, there is a Poisson-like formula that relates the kernel of $\cos t\sqrt{-\Delta_g}$ to a periodic sum of wave kernels on the universal cover of (M, g), which is $\mathbb{R}^n$. If $p : \mathbb{R}^n \to M$ is a covering map, we shall let $\tilde{g}$ denote the pullback of g via p. If $\Delta_{\tilde{g}}$ denotes the associated Laplace-Beltrami operator on the universal cover $(\mathbb{R}^n, \tilde{g})$, the formula we require is

$$\big(\cos t\sqrt{-\Delta_g}\big)(x,y) = \sum_{\gamma\in\Gamma} \big(\cos t\sqrt{-\Delta_{\tilde{g}}}\big)(x, \gamma y), \quad x, y \in D, \tag{4.10}$$

where $D \subset \mathbb{R}^n$ is a fixed fundamental domain, which we identify with M via the covering map p, and Γ denotes the group of deck transformations for the covering. The latter is the group of homeomorphisms $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ for which $p = p \circ \gamma$, and γy denotes the image of y under γ. Note then that if $(\tilde{M}, \tilde{g}) = (\mathbb{R}^n, \tilde{g})$, then $M \simeq \tilde{M}/\Gamma$.

To use this formula let us first note that, by Huygens principle, $(\cos t\sqrt{-\Delta_{\tilde{g}}})(x,y) = 0$ if $d_{\tilde{g}}(x,y) > t$, with $d_{\tilde{g}}$ denoting the Riemannian distance with respect to the metric $\tilde{g}$. Consequently, since $\hat{a}(t) = 0$ for $|t| \geq 1$, in order to prove (4.8) it would be enough to show that for $x, y \in D$ we have

$$\sum_{\{\gamma\in\Gamma:\, d_{\tilde g}(x,\gamma y)\le\varepsilon\log\lambda\}} \left| \int_{-\infty}^{\infty} \psi(t)\hat a(t/\varepsilon\log\lambda) e^{it\lambda}\big(\cos t\sqrt{-\Delta_{\tilde g}}\big)(x,\gamma y)\,dt \right| \le C\log\lambda e^{c\varepsilon\log\lambda}\lambda^{(n-1)/2} \quad (4.11)$$

Since $(\mathbb{R}^n, \tilde g)$ has no conjugate points, following Bérard [1], we can use the Hadamard parametrix for large times to prove estimates like (4.11). We no longer have uniform bounds on the amplitudes as we did in (2.24) and (2.25) for short times. On the other hand, for $|t| \ge 1$, by writing the Fourier integral terms in the Hadamard parametrix in an equivalent way to those in [1] (cf. e.g., [13], Remark 1.2.5), we see that if $\tilde g$ is as above then there is a constant $c_0 = c_0(\tilde g) > 0$, which is independent of $T > 1$ so that

$$\big(\cos t\sqrt{-\Delta_{\tilde g}}\big)(x,y) = \sum_{\pm} \int_{\mathbb{R}^n} \alpha_{\pm}(t,x,y,|\xi|) e^{i\kappa(x,y)\cdot\xi} e^{\pm it|\xi|}\,d\xi + R(t,x,y), \quad (4.12)$$

where, as before, $\kappa(x,y)$ denotes the geodesic normal coordinates (now with respect to $\tilde g$) of x about y, but now the $0-$order symbol functions $\lambda_{\pm}$ in (2.24) satisfy

$$\left| \frac{d^j}{dt^j}\frac{d^k}{dr^k}\alpha_{\pm}(t,x,y,r) \right| \le A_{jk} e^{c_0 T}(1+r)^{-k},$$
$$\text{if } \ 1 \le |t| \le T, \ r > 0, \ \text{and} \ \ j,k \in \{0,1,2,\dots\}. \quad (4.13)$$

The remainder term can be taken to be continuous, but it also satisfies bounds that become exponentially worse with time:

$$|R(t,x,y)| \le Ae^{c_0T}, \quad \text{if } 1 \le |t| \le T. \tag{4.14}$$

To use these we shall require the following simple stationary phase lemma.

Lemma 4.1.2. *Suppose that* $\alpha(t,r) \in C^\infty(\mathbb{R}\times\mathbb{R}_+)$ *satisfies*

$$\alpha(t,r) = 0 \quad \textit{if} \;\; |t| \notin [1,T],$$

and for every $j = 0,1,2,\dots$ *and* $k = 0,1,2,\dots$

$$\left| \frac{d^j}{dt^j}\frac{d^k}{dr^k}\alpha(t,r)\right| \le A_{jk}e^{cT}(1+r)^{-k}, \tag{4.15}$$

for a fixed constant $c > 0$*. Then there is a constant* B *depending only on* n *and the size of finitely many of the constants in* (4.15) *so that*

$$\left| \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \alpha(t,|\xi|)e^{it\lambda\pm it|\xi|}e^{iv\cdot\xi}\, d\xi dt \right| \le BTe^{cT}\lambda^{\frac{n-1}{2}}, \quad \lambda, T > 1.$$

Proof. If $|v| \le 1/2$, then one can integrate by parts in the ξ-variable to see that better bounds hold where in the right side $\lambda^{\frac{n-1}{2}}$ is replaced by one. The result for $|v| \ge 1/2$ follows from (2.15) and (2.16) after noting that if $\hat{\alpha}(\tau,|\xi|)$ denotes the partial Fourier transform in the t-variable then our assumptions (4.15) and an integration by parts argument imply that

$$\int_{\mathbb{R}^n} |\hat{\alpha}(\lambda \pm |\xi|, |\xi|)|\,|\xi|^{-\frac{n-1}{2}}\,d\xi \le BTe^{cT}\lambda^{\frac{n-1}{2}}.$$

$\square$

We can now finish the proof of (4.11). If we use the lemma along with (4.12)-(4.14), we conclude that each term in the sum in the left side of (4.11) is bounded by

$$BTe^{c_0T}\lambda^{\frac{n-1}{2}}, \quad T = \varepsilon \log \lambda, \tag{4.16}$$

for some constant B when $\lambda \gg 1$. Furthermore, as noted by Bérard [1], the classical volume growth estimates imply that there are $O(e^{c_1T})$ nonzero terms in the sum in (4.11) for a fixed constant $c_1 = c_1(\tilde{g})$. Thus, we can choose a sufficiently large c to replace c_0 in (4.16) such that the whole sum is bounded by this quantity. So we obtain (4.11), and hence Theorem 1.2.9.

4.2 Torus $\mathbb{T}^n$

In this section we are going to prove Theorem 1.2.6 in a similar way as the non-positive curvature manifold case in the previous section. The main difference here is instead of the exponential type parametrix as in (4.12) we have now the classical Poisson formula, which can give better information about the spectral distributions for the torus.

By an argument similar to the one prior to (4.7), we need only to study that if we define an operator for $0 < \varepsilon(p) \le 1$ as follows:

$$a_\lambda(P) = \int \psi(t)\hat{a}(t/\lambda^{\varepsilon(p)})e^{it\lambda}\cos tP dt, \tag{4.17}$$

then we can have

$$||a_\lambda(P)f||_{L^{p'}(\mathbb{T}^n)} \le \lambda^{2\delta(p)}||f||_{L^p(\mathbb{T}^n)}, p \le \frac{2(n+1)}{n+3}. \tag{4.18}$$

Here as before $\psi(t)$ is a smooth function with support outside $(-4,4)$ and equals to one $|t| > 10$, and $\hat{a}(t)$ is supported in $(-1,1)$. Both functions are even, similar to the non-positive curvature manifold case.

As we have seen in the proof to Theorem 3.3.2, if we can find such a value of $\varepsilon(p)$ which satisfies (4.18), then any smaller positive number than $\varepsilon(p)$ will do as well. Therefore in the following paragraphs, we can simply focus on finding a largest possible $\varepsilon(p)$ based on an interpolation argument similar to the one used in previous section.

Now we are going to prove the following estimates to interpolate with:

$$||a_\lambda f||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \le C\lambda^{\frac{n-1}{n+1}}\lambda^{\varepsilon(p)}||f||_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}, \tag{4.19}$$

and

$$||a_\lambda f||_{L^\infty(\mathbb{T}^n)} \le C\lambda^{\frac{n-1}{2}}\lambda^{\varepsilon(p)\frac{n+1}{2}}||f||_{L^1(\mathbb{T}^n)}. \tag{4.20}$$

The first one is an easy application of Lemma 2.3.2 so we need only to prove the second one. Recall that if we identify $\mathbb{T}^n$ with its fundamental domain $Q = (-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$, then we have

$$\cos t\sqrt{-\Delta_{\mathbb{T}^n}}(x,y) = \sum_{j\in\mathbb{Z}^n} \cos t\sqrt{-\Delta_{\mathbb{R}^n}}(x-y+j), \quad x,y\in Q \tag{4.21}$$

then by the finite speed of propagation of $\cos tP$ we are reduced to estimating the size of the following integral

$$\begin{aligned} A_1(x,y) &= \sum_{\substack{j\in\mathbb{Z}^n\\ |j|\le\lambda^{\varepsilon(p)}}} \int\int e^{i(x-y+j)\cdot\xi}\psi(t)\hat{a}(t/\lambda^{\varepsilon(p)})e^{it\lambda}\cos t|\xi|dtd\xi \\ &= \sum_{\substack{j\in\mathbb{Z}^n\\ |j|\le\lambda^{\varepsilon(p)}\\ |x-y+j|\ge 1}} \int\int e^{i(x-y+j)\cdot\xi}\psi(t)\hat{a}(t/\lambda^{\varepsilon(p)})e^{it\lambda}\cos t|\xi|dtd\xi \\ &\quad + \sum_{\substack{j\in\mathbb{Z}^n\\ |j|\le\lambda^{\varepsilon(p)}\\ |x-y+j|<1}} \int\int e^{i(x-y+j)\cdot\xi}\psi(t)\hat{a}(t/\lambda^{\varepsilon(p)})e^{it\lambda}\cos t|\xi|dtd\xi \\ &= (I)+(II). \end{aligned} \tag{4.22}$$

Notice that (II) will disappear in odd dimension due to Huygens principle. Fortunately it does not cause any harm in even dimensions neither due to the following simple argument. In fact, due to our choice on the fundamental domain Q, there are only $O(2^n)$ many terms non-vanishing in (II), so by Euler's formula we need only to prove that

$$\int \int_{\mathbb{R}^n} e^{i(x-y+j)\cdot\xi}\psi(t)\hat{a}(t/\lambda^{\varepsilon(p)})e^{it(\lambda\pm|\xi|)}d\xi dt \in O(\lambda^{-N}), \quad |x-y+j|<1. \tag{4.23}$$

Integrating by parts with respect to t shows that we need only prove (4.23) when an extra cut-off function $\beta(|\xi|/\lambda)$ is inserted in the integrand, in which the function β is as in (2.8). Then since the support of the integrand is restricted in $|t|>4$ and $|x-y+j|<1$, another integration by parts in ξ variable completes the proof.

Now we notice that in (I) if we replace $\psi(t)$ by $1-\psi(t)$ we will end up with $O(2^n)$ many integrals like

$$\int_{\mathbb{R}^n} e^{i(x-y+j)\cdot\xi}\left(\Psi(\lambda-|\xi|)+\Psi(\lambda+|\xi|)\right)d\xi, \quad |x|>1,$$

in which $\Psi(r)$ is a Schwartz function. So using polar coordinates $\xi\to r\omega$ will immediately show that these integrals are in $O(\lambda^{\frac{n-1}{2}})$, which is better than (4.20). Thus, we need only to prove

$$\sum_{\substack{j\in\mathbb{Z}^n\\ |j|\le\lambda^{\varepsilon(p)}\\ |x-y+j|>1}} |\int_{\mathbb{R}^n} e^{i(x-y+j)\cdot\xi}\lambda^{\varepsilon(p)}a(\lambda^{\varepsilon(p)}(\lambda\pm|\xi|))d\xi| \le \lambda^{\frac{n-1}{2}+\varepsilon(p)\frac{n+1}{2}}. \tag{4.24}$$

This can be proved easily as follows: if again we use polar coordinates and the decay estimate of the Fourier transform of the spheres as in (2.15), (2.16), and a scale

$\xi \to \lambda\xi$, then each term in (4.24) is bounded by a constant multiple of

$$\lambda^{\varepsilon(p)}\lambda^{n}\lambda^{-\frac{n-1}{2}}(1+|j|)^{-\frac{n-1}{2}}\int\frac{ds}{(1+\lambda^{\varepsilon(p)+1}|s|)^{2}},$$

due to the fact that $a \in \mathcal{S}(\mathbb{R})$. Summing over $|j| < \lambda^{\varepsilon(p)}$ finishes the proof of (4.24), thus (4.20).

After proving (4.19) and (4.20) we can now carry out the interpolation. Let $0 \le t \le 1$ be determined by the following equation

$$t + (1-t)\frac{n+3}{2(n+1)} = \frac{1}{p}, p \le \frac{2(n+1)}{n+3}, \tag{4.25}$$

then the interpolation shows that we should have

$$||a_\lambda f||_{L^{p'}(\mathbb{T}^n)} \le \lambda^{\frac{n-1}{n+1}(1-t)+\varepsilon(p)(1-t)+\frac{n-1}{2}t+\varepsilon(p)\frac{n+1}{2}t}||f||_{L^p(\mathbb{T}^n)}, p \le \frac{2(n+1)}{n+3}. \tag{4.26}$$

So we need only to solve for a largest possible positive $\varepsilon(p)$ which satisfies (4.18) when $t > 0$, which is determined from the following equation:

$$\varepsilon(p) = \frac{\frac{n(n-1)t+n-1}{2(n+1)} - \frac{n-1}{n+1}(1-t) - \frac{n-1}{2}t}{\frac{n+1}{2}t + 1 - t}, \; t = \frac{2(n+1)}{n-1}\frac{1}{p} - \frac{n+3}{n-1}. \tag{4.27}$$

An elementary derivative test shows that this function is increasing when $t \in [0,1]$ and $\varepsilon(2(n+1)/(n+3)) = 0$, which coincides with our earlier interpolation estimate

(4.20) as the $(1,\infty)$ endpoint has a better bound. In particular when $p = 2n/(n+2)$ we have $\varepsilon(p) = 1/(n+1)$.

Now what we need is simply to compose the projections between $L^p \to L^2$ and $L^2 \to L^q$. Here we just need to choose the weaker estimates during the composition, say for a general $(1/p, 1/q)$ pair in the $\overline{AA'}$ admissible range we just choose $\varepsilon(p)$ when $(1/p, 1/q)$ is below the line of duality, and $\varepsilon(q')$ when it is above it. So not surprisingly the improvement is symmetric with respect to the line of duality. The closer the exponent $(1/p, 1/q)$ is to the middle point F in figure 1.3, the better improvement we can have.

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