Embedded Minimal Disks with Curvature Blow-up on a Line Segment

by
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Abstract

In this thesis we discuss some results concerning embedded minimal disks. We construct a sequence of compact embedded minimal disks in the unit ball in Euclidean 3-space whose boundaries are in the boundary of the ball and where the curvatures blow up at every point of a line segment of the vertical axis, extending from the origin. We further study the transversal structure of the minimal limit lamination and find removable singularities along the line segment and a non-removable singularity at the origin. This extends a result of Colding and Minicozzi where they constructed a sequence with curvatures blowing up only at the center of the ball, Dean's construction of a sequence with curvatures blowing up at a prescribed discrete set of points, and the classical case of the sequence of re-scaled helicoids with curvatures blowing up along the entire vertical axis.

Readers: Dr. William P. Minicozzi II (Advisor), Dr. Chikako Mese, Dr. Joel Spruck, Dr. Daniel Q. Naiman, Dr. David Audley.

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Finally, I dedicate this thesis to my grandfather, Towheed Ali, who has worked tirelessly and sacrificed throughout his life to ensure that his family had the opportunities that he and those of his generation couldn't have.

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Chapter 1

Introduction

In the study and classification of Minimal Surfaces an important question is what are the possible singular sets for limits of sequences of embedded minimal surfaces. The global problem in $\mathbf{R}^3$ is understood by results of Colding and Minicozzi [CM2]-[CM5], where all singularities are removable at least when the sequence is simply connected, and by the work of Meeks and Rosenberg in [MR] where they explain why the singular set is perpendicular to the limit foliation.

In contrast, for the local case in $\mathbf{R}^3$, Colding and Minicozzi in [CM1] prove the existence of a sequence of embedded minimal disks with boundaries in a sphere and with curvatures blowing up only at the center of the ball, where there is a non-removable singularity. A result of Dean, [BD], extends this example by constructing a sequence with curvatures blowing up at a prescribed discrete set of points. There is also the well known case of the sequence of re-scaled helicoids with curvature blowing up along the entire x_3-axis. Meeks and Weber [MW] construct singular sets that are properly embedded $C^{1,1}$-curves. Hoffman and White [HW], and Calle and Lee [CL] also give a variational way to compute examples.

In this thesis we construct a sequence of compact embedded minimal disks in the unit ball in $\mathbf{R}^3$ whose boundaries are in the boundary of the ball and where the curvatures blow up at every point of a line segment of the negative x_3-axis. This sequence converges to a minimal limit lamination. We study the transversal structure of the limit lamination and find a foliation by parallel planes in the lower hemisphere and a leaf in the upper hemisphere that spirals into the $\{x_3 = 0\}$ plane, such that there are removable singularities at every point along the line segment of the negative x_3-axis but the singularity at the origin cannot be removed.

We will follow the structure of Colding and Minicozzi's result in [CM1]. The key difference in our approach here is that we alter the domain and the Weierstrass data used in [CM1] to create not just one singularity converging to the origin, but a sequence of singularities that converges to a line segment extending from the origin. In addition, the construction of the limit lamination uses a convergence result from [CM5]. And a Bernstein-type theorem is used to obtain the foliation by parallel planes in the lower hemisphere of the unit ball.

Our main result is Theorem 1.0.2 below, which constructs our sequence of compact embedded minimal disks in the unit ball with boundaries in the boundary of the ball and describes the limit lamination. Theorem 1.0.1 first constructs a sequence of compact embedded minimal disks with the necessary curvature and boundary properties.

Theorem 1.0.1. *There exists a sequence of compact embedded minimal disks* $0 \in$

$M_N \subset \mathbf{R}^2 \times [-1/2, 1/2] \subset \mathbf{R}^3$ *(with boundary), each containing the vertical segment* $\{(0,0,t)\,|\,|t| \leq 1/2\} \subset M_N$, *with the following properties:*

(a) $\forall p \in \{(0,0,t)\,|\,-1/2 \leq t \leq 0\}$, $\lim_{N\to\infty} |A_{M_N}|^2(p) = \infty$.

(b) $M_N \setminus \{(0,0,t)|\,|t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ *for multi-valued graphs* $M_{1,N}$, $M_{2,N}$ *over the* $\{x_3 = 0\}$ *plane.*

(c) $\sup_N \sup_{M_N \setminus B_\delta} |A_{M_N}|^2 = C_\delta < \infty$ *for all* $\delta > 0$ *and some constant* C_δ *depending on* δ, *and where* B_δ *is a* δ*-neighborhood of* $\{(0,0,t)\,|\,-1/2 \leq t \leq 0\}$.

(d) The boundary ∂M_N *lies outside a fixed cylinder* $\{(x_1,x_2,x_3)|\,x_1^2+x_2^2 \leq r_0^2, -1/2 < x_3 < 1/2\}$ *where* r_0 *does not depend on* N. *Also, in each horizontal slice* $\{x_3 = t\} \cap M_N$, *for* $-1/2 \leq t \leq 0$ *(i.e. below the* $\{x_3 = 0\}$ *plane) the distance from the* x_3*-axis to* ∂M_N *goes to infinity as* $N \to \infty$.

Figure 1.1 shows a diagram of horizontal slices of $M_{1,N}$, $M_{2,N}$.

The boundary properties of the sequence in Theorem 1.0.1(d) allow us to intersect with a smaller ball that is contained in the cylinder, pass to a subsequence, and then scale to obtain the sequence in Theorem 1.0.2 below that has the same properties (a), (b) and (c). Theorem 1.0.2 further describes the convergence of this sequence to a limit lamination of the unit ball with singularities along the line segment $\{(0,0,t)\,|\,-1 \leq t \leq 0\}$.

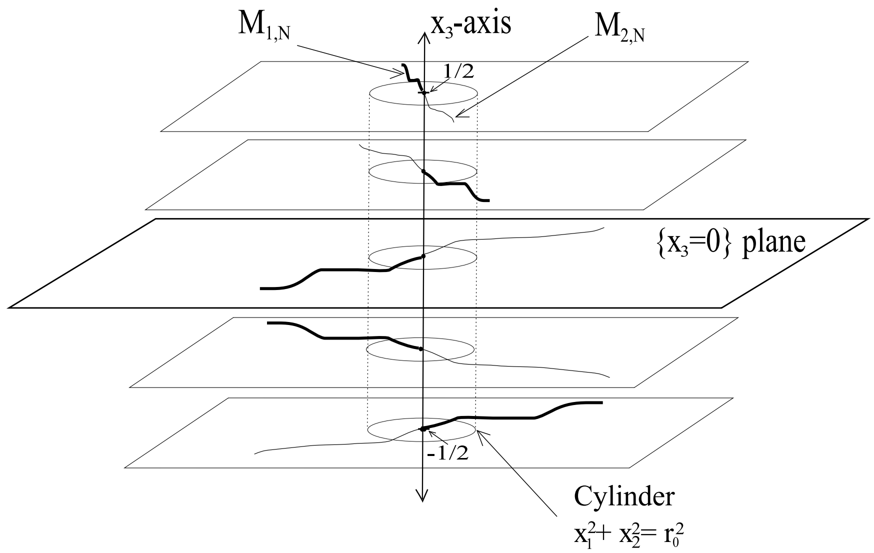

Figure 1.1: Horizontal slices of $M_N \setminus \{(0,0,t) | \, |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ in Theorem 1.0.1

Theorem 1.0.2. *There exists a sequence of compact embedded minimal disks* $0 \in \Sigma_N \subset B_1 \subset \mathbf{R}^3$ *with* $\partial \Sigma_N \subset \partial B_1$ *and each containing the vertical segment* $\{(0,0,t) \, | \, |t| \leq 1\} \subset \Sigma_N$ *with the following properties:*

(a) $\forall p \in \{(0,0,t) \, | \, -1 \leq t \leq 0\}$, $\lim_{N \to \infty} |A_{\Sigma_N}|^2(p) = \infty$.

(b) $\Sigma_N \setminus \{(0,0,t) | \, |t| \leq 1\} = \Sigma_{1,N} \cup \Sigma_{2,N}$ *for multi-valued graphs* $\Sigma_{1,N}$, $\Sigma_{2,N}$ *over the* $\{x_3 = 0\}$ *plane.*

(c) $\sup_N \sup_{\Sigma_N \setminus B_\delta} |A_{\Sigma_N}|^2 < \infty$ *for all* $\delta > 0$, *where* B_δ *is a* δ*-neighborhood of* $\{(0,0,t) \, | \, -1 \leq t \leq 0\}$.

This sequence of compact embedded minimal disks converges to a minimal lamination of $B_1 \setminus \{(0,0,t) \, | \, -1 \leq t \leq 0\}$ *consisting of a foliation by parallel planes of the lower hemisphere below* $\{x_3 = 0\}$ *and one leaf in the upper hemisphere,* Σ, *such that* $\Sigma \setminus \{x_3 - axis\} = \Sigma' \cup \Sigma''$, *where* Σ' *and* Σ'' *are multi-valued graphs, each of which* *spirals into* $\{x_3 = 0\}$*. This limit lamination has removable singularities along the line segment* $\{(0,0,t) \mid -1 \leq t \leq 0\}$ *of the negative* x_3*-axis but the singularity at the origin cannot be removed.*

Figure 1.2 shows a schematic picture of this limit lamination.

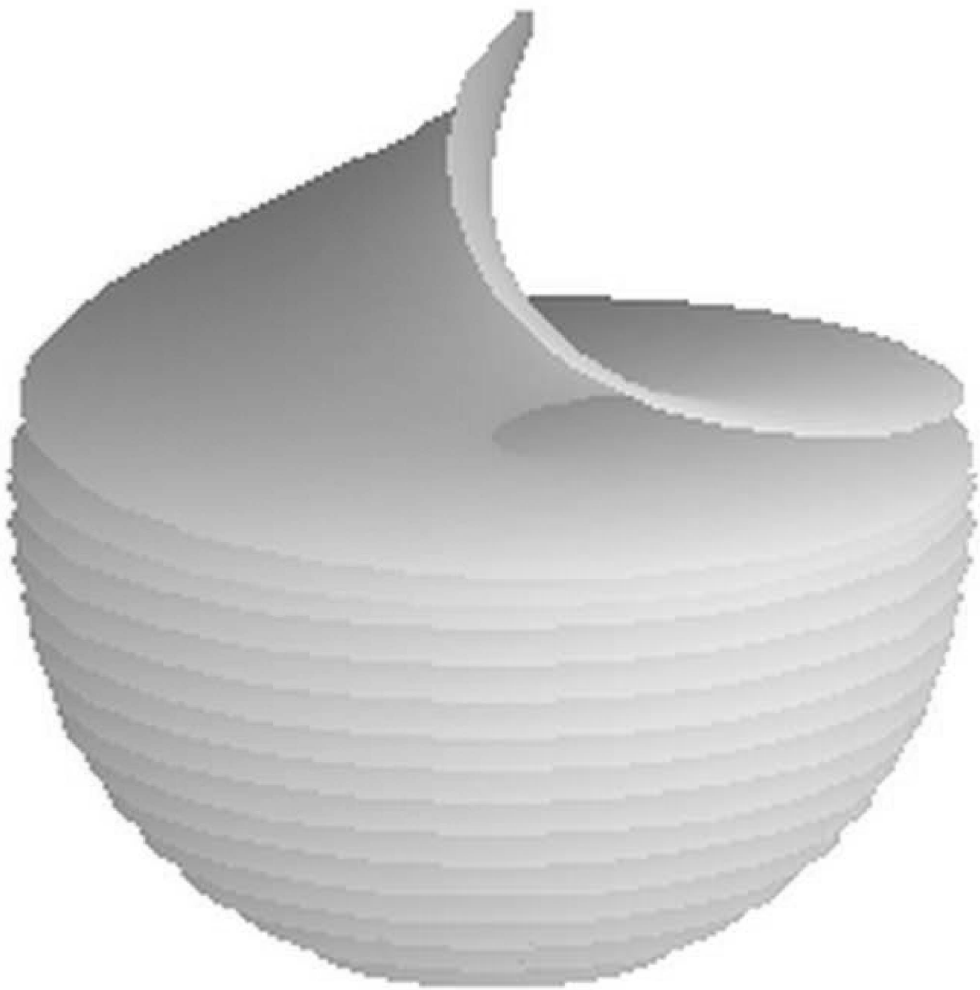

Figure 1.2: Schematic picture of the limit lamination in Theorem 1.0.2

Chapter 2

Background on Minimal Surfaces

We begin with some background on Minimal Surfaces, Minimal Laminations and the Weierstrass Representation for Minimal Surfaces.

2.1 Notation

In this thesis we will use standard (x_1, x_2, x_3) coordinates on $\mathbf{R}^3$ and $z = x + i\,y$ on $\mathbb{C}$. Given a function $f : \mathbb{C} \to \mathbb{C}^n$, we will use $\partial_x f$ and $\partial_y f$ to denote $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. Also, $\partial_z f = (\partial_x f - i\partial_y f)/2$.

For $p \in \mathbf{R}^3$ and $s > 0$, the ball of radius s in $\mathbf{R}^3$ will be denoted by $B_s(p)$. is oriented, we let $\mathbf{n}_\Sigma$ be the unit normal.

2.2 Minimal Surfaces

A minimal surface is defined as a surface that is a critical point of the area functional. As outlined in [CM8], Section 2, if $\Sigma \subset \mathbf{R}^3$ is a smooth orientable surface (possibly with boundary) with a unit normal n_Σ and if we let $\phi \in C_0^\infty(\Sigma)$ be a compactly supported function on Σ, then we can define the one-parameter variation:

$$\Sigma_{t,\phi} = \{x + t\phi(x)n_\Sigma(x) | x \in \Sigma\}.$$

The surface Σ is defined to be a minimal surface if and only if:

$$\left.\frac{d}{dt}\right|_{t=0} \text{Area}(\Sigma_{t,\phi}) = 0,\ \forall \phi \in C_0^\infty(\Sigma)$$

Now, the mean curvature of Σ is defined as $H = \kappa_1 + \kappa_2$ where κ_1, κ_2 are the principal curvatures of Σ. The definition of a Minimal Surface as a critical point of the area functional given above is equivalent to the statement that Σ is a minimal surface if and only if its mean curvature H is identically zero. This follows immediately from the first variation formula:

$$\left.\frac{d}{dt}\right|_{t=0} \text{Area}(\Sigma_{t,\phi}) = \int_\Sigma \phi H \, dA,$$

where the integration is with respect to dArea. Two other important quantities associated with smooth minimal surfaces in $\mathbf{R}^3$ are the sectional curvature (which is equal to the Gaussian curvature), $\mathrm{K}_\Sigma = \kappa_1\kappa_2$, and the norm squared of the second fundamental form, $|A_\Sigma|^2 = \kappa_1^2 + \kappa_2^2$. Thus, since for a minimal surface, $0 = H = \kappa_1 + \kappa_2$, we have that $|A_\Sigma|^2 = -2\,\mathrm{K}_\Sigma$.

2.3 Minimal Laminations and Foliations

A codimension one lamination of a 3−manifold, M^3 is a collection $\mathcal{L}$ of smooth disjoint connected surfaces (these surfaces are called leaves) such that $\cup_{\Gamma \in \mathcal{L}}$ is closed (See [CM8], Section 22). For each $x \in M$ there exists an open neighborhood U of x and a local coordinate chart, (U, Φ), where $\Phi(U) \subset \mathbf{R}^3$ such that in these coordinates the leaves in $\mathcal{L}$ pass through the chart in slices of the form $(\mathbf{R}^2 \times \{t\}) \cap \Phi(U)$. A foliation is a lamination where the union of the leaves is all of M. A minimal lamination is one where the leaves are (smooth) minimal surfaces.

2.4 The Weierstrass representation

Given a meromorphic function $g(z)$ and a holomorphic function $\phi(z)$, defined on a domain Ω, the Weierstrass representation of a (branched) conformal minimal immersion, $F : \Omega \to \mathbb{R}^3$, is given by (see [Os], Lemma 8.2):

$$F(z) = \mathrm{Re} \int_{\zeta \in \gamma_{z_0,z}} \left(\frac{1}{2}(1 - g^2(\zeta)), \frac{i}{2}(1 + g^2(\zeta)), g(\zeta) \right) \phi(\zeta)\, d\zeta, \tag{2.1}$$

where z_0 is a fixed base point in Ω and the integration is taken along a path $\gamma_{z_0,z}$ from z_0 to z in Ω. Different choices for z_0 change F by adding a constant. In this thesis we set the base point $z_0 = 0$. We also choose ϕ such that it has no zeros and g such that it has no poles or zeros in Ω and we choose Ω to be simply connected. These ensure that $F(z)$ does not depend on the choice of path $\gamma_{z_0,z}$ and that the differential dF, is non-zero (and this ensures that F is an immersion). We will need the following lemma that gives the differential of F (see [CM1]):

Lemma 2.4.1. *If F is given by (2.1) with $g(z) = e^{i\,(u(z)+iv(z))}$ and $\phi(z) = e^{-i\,(u(z)+iv(z))}$, then*

$$\partial_x F = (\sinh v\, \cos u, \sinh v\, \sin u, 1)\,, \tag{2.2}$$

$$\partial_y F = (\cosh v\, \sin u, -\cosh v\, \cos u, 0)\,. \tag{2.3}$$

Proof. We let $g(z) = \mathrm{e}^{i\,(u(z)+iv(z))}$ and $\phi(z) = \mathrm{e}^{-i\,(u(z)+iv(z))}$ in the Weierstrass representation above (2.1). This gives:

$$F(z) = (F_1, F_2, F_3) = \mathrm{Re} \int_{z\in\gamma_{z_0,z}} (f_1, f_2, f_3)\, dz,$$

where

$$f_1 = \frac{1}{2}\phi(1-g^2),\ f_2 = \frac{i}{2}\phi(1+g^2),\ f_3 = \phi g,$$

and F_1, F_2, F_3 are the coordinate functions of the image $F(z)$.

$\Longrightarrow$ $f_1 = \frac{1}{2}(e^{v-iu} - e^{-v+iu}), f_2 = \frac{i}{2}(e^{v-iu} + e^{-v+iu})$, and $f_3 = 1$, For $k \in \{1,2,3\}$, we let $F_k^* = \mathrm{Im} \int_{z\in\gamma_{z_0,z}} f_k\, dz$. Then since for each k, F_k, F_k^* are the real and imaginary parts of a holomorphic function, the Cauchy-Riemann equations apply:

$$\partial_x F_k = \partial_y F_k^*$$

$$\partial_y F_k = -\partial_x F_k^*$$

$$\implies \partial_x F_k - i\partial_y F_k = \frac{1}{2}\left(2\partial_x F_k - 2i\partial_y F_k\right)$$
$$= \frac{1}{2}\left(\left(\partial_x F_k + \partial_y F_k^*\right) - i\left(\partial_y F_k - \partial_x F_k^*\right)\right)$$
$$= \frac{1}{2}\left(\partial_x - i\partial_y\right)\left(F_k + iF_k^*\right)$$
$$= \partial_z\left(\left(\mathrm{Re}\int_{z\in\gamma_{z_0,z}} f_k\, dz\right) + i\left(\mathrm{Im}\int_{z\in\gamma_{z_0,z}} f_k\, dz\right)\right) = \partial_z\left(\int_{z\in\gamma_{z_0,z}} f_k\, dz\right)$$

$$\implies \partial_x F_k - i\partial_y F_k = f_k \tag{2.4}$$

$$\implies \partial_x F = (\partial_x F_1, \partial_x F_2, \partial_x F_3) = \mathrm{Re}\left(\partial_x F_1 - i\partial_y F_1, \partial_x F_2 - i\partial_y F_2, \partial_x F_3 - i\partial_y F_3\right)$$

$$= \mathrm{Re}\left(f_1, f_2, f_3\right) = \mathrm{Re}\left(\frac{1}{2}\phi(1-g^2), \frac{i}{2}\phi(1+g^2), \phi g\right) \tag{2.5}$$

$$= \mathrm{Re}\left(\tfrac{1}{2}(e^{v-iu} - e^{-v+iu}), \tfrac{i}{2}(e^{v-iu} + e^{-v+iu}), 1\right)$$
$$= (\sinh v \cos u, \sinh v \sin u, 1)$$

And similarly,

$$\partial_y F = \mathrm{Im}\left(f_1, f_2, f_3\right) = \mathrm{Im}\left(\frac{1}{2}\phi(1-g^2), \frac{i}{2}\phi(1+g^2), \phi g\right) \tag{2.6}$$

$= -\mathrm{Im}\left(\frac{1}{2}(e^{v-iu} - e^{-v+iu}), \frac{i}{2}(e^{v-iu} + e^{-v+iu}), 1\right)$

$= (\cosh v \sin u, -\cosh v \cos u, 0)$

□

The following lemma gives two important quantities associated with the Weierstrass Representation of Minimal Surfaces that we use in this thesis. The computations also confirm that the map defined by (2.1) is conformal and that the surface is minimal (see Osserman [Os] sections 8, 9).

Lemma 2.4.2. *The unit normal* $\boldsymbol{n}$ *and the Gauss curvature* $\boldsymbol{K}$ *of the surface defined by (2.1) are :*

$$\boldsymbol{n} = (2Re\{g\}, 2Im\{g\}, |g|^2 - 1)/(|g|^2 + 1) \tag{2.7}$$

$$\boldsymbol{K} = -\left[\frac{4|\partial_z g|}{|\phi|(1 + |g|^2)^2}\right]^2 \tag{2.8}$$

Proof. By (2.4), we have that for $k \in \{1, 2, 3\}$, $\partial_x F_k - i\partial_y F_k = f_k$, where

$$f_1 = \frac{1}{2}\phi(1 - g^2),\ f_2 = \frac{i}{2}\phi(1 + g^2),\ f_3 = \phi g$$

and F_1, F_2, F_3 are the coordinate functions of the image $F(z)$.

$\Longrightarrow\ 0 = \left(\frac{1}{2}\phi(1 - g^2)\right)^2 + \left(\frac{i}{2}\phi(1 + g^2)\right)^2 + (\phi g)^2 = \sum_{k=1}^{3} f_k^2$

$= \sum_{k=1}^{3}(\partial_x F_k - i\partial_y F_k)^2 = \sum_{k=1}^{3}\left((\partial_x F_k)^2 - (\partial_y F_k)^2 - 2i(\partial_x F_k)(\partial_y F_k)\right)$

$$\implies \text{Re}\sum_{k=1}^{3}\big((\partial_x F_k)^2 - (\partial_y F_k)^2 - 2i(\partial_x F_k)(\partial_y F_k)\big)$$
$$= \text{Im}\sum_{k=1}^{3}\big((\partial_x F_k)^2 - (\partial_y F_k)^2 - 2i(\partial_x F_k)(\partial_y F_k)\big) = 0$$
$\implies$

$$|\partial_x F|^2 = |\partial_y F|^2$$

$$(\partial_x F)\cdot(\partial_y F) = 0$$

Now, we also note that $\partial_x F_k - i\partial_y F_k = f_k$

$$\implies (\partial_x F_k)^2 + (\partial_y F_k)^2 = |f_k|^2$$
$$\implies |\partial_x F| = |\partial_y F| = \frac{1}{2}\sum_{k=1}^{3}\big((\partial_x F_k)^2 + (\partial_y F_k)^2\big) = \frac{1}{2}\sum_{k=1}^{3}|f_k|^2$$
$$= \left|\frac{1}{2}\phi(1-g^2)\right|^2 + \left|\frac{i}{2}\phi(1+g^2)\right|^2 + |\phi g|^2 = \frac{1}{4}|\phi|^2(1+|g|^2)^2$$
$\implies$ the first fundamental form of the surface defined by (2.1) is

$$G = (g_{ij}) = \begin{pmatrix} (\partial_x F)\cdot(\partial_x F) & (\partial_x F)\cdot(\partial_y F) \\ (\partial_y F)\cdot(\partial_x F) & (\partial_y F)\cdot(\partial_y F) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}|\phi|^2(1+|g|^2)^2 & 0 \\ 0 & \frac{1}{4}|\phi|^2(1+|g|^2)^2 \end{pmatrix}$$

Hence $g_{ij} = \lambda^2\delta_{ij}$ where $\lambda^2 = \frac{1}{4}|\phi|^2(1+|g|^2)^2$. This implies that the Weierstrass map (2.1) is conformal (see [Os] Section 4).

Now, by (2.5) and (2.6),

$$\partial_x F \times \partial_y F = \frac{1}{4}|\phi|^2(1+|g|^2)(2\text{Re}\{g\}, 2\text{Im}\{g\}, |g|^2 - 1)$$

$$\implies |\partial_x F \times \partial_y F| = \tfrac{1}{4}|\phi|^2(1+|g|^2)^2$$

$$\implies \mathbf{n} = \frac{\partial_x F \times \partial_y F}{|\partial_x F \times \partial_y F|} = (2\mathrm{Re}\{g\}, 2\mathrm{Im}\{g\}, |g|^2 - 1)/(|g|^2 + 1)$$

[Os], Section 9, computes the principal curvatures (the maximum and minimum of the normal curvatures with respect to **n** in the direction of a tangent T to the surface as the direction of T varies) to be

$$\kappa_1 = \left[\frac{4|\partial_z g|}{|\phi|(1+|g|^2)^2}\right], \ \kappa_2 = -\left[\frac{4|\partial_z g|}{|\phi|(1+|g|^2)^2}\right]$$

And hence, the Gauss curvature is:

$$\mathbf{K} = \kappa_1\kappa_2 = -\left[\frac{4|\partial_z g|}{|\phi|(1+|g|^2)^2}\right]^2$$

We also note that these expressions derived for κ_1 and κ_2 confirm that the mean curvature $H = (\kappa_1 + \kappa_2)/2 = 0$ and hence that the surface defined by the Weierstrass representation (2.1) is indeed minimal. □

Chapter 3

Curvature Blow-up in Sequences of Compact Embedded Minimal Disks in $\mathbf{R}^3$

In this Chapter we review some of the previous results on convergent sequences of compact embedded minimal disks in a ball of fixed radius in $\mathbf{R}^3$ with boundaries in the boundary of the ball. We examine the sets of points at which the curvatures blow up.

3.1 Rescaled Helicoids

The example of a sequence of rescaled helicoids, contained in the ball B_1 and with boundaries in the boundary of the ball demonstrates the case of curvature blowing up along the entire x_3-axis.

The helicoid is one of the classical minimal surfaces, discovered in 1776 by Meusnier. It is described as a double spiral staircase swept out by a horizontal line in $\mathbf{R}^3$ as it rotates with constant speed about the x_3-axis and moves along the axis with constant speed. It is shown in Figure 3.1 (from Matthias Weber's Minimal Surface Archive [MA]).

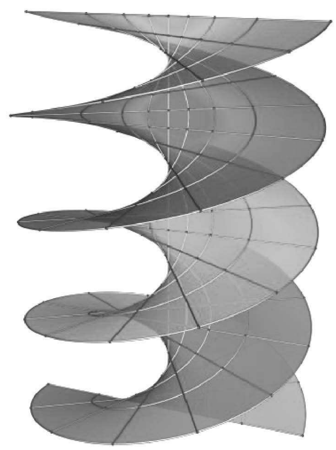

Figure 3.1: The Helicoid [MA]

The Weierstrass data for the Helicoid is $\phi(z) = e^{-iz}$, $g(z) = e^{iz}$ and the domain is $\mathbb{C}$. Therefore from (2.1), we obtain:

$$F(z) = \mathrm{Re} \int_{\zeta \in \gamma_{0,z}} \left(\frac{1}{2}(1 - e^{2i\zeta}), \frac{i}{2}(1 + e^{2i\zeta}), e^{i\zeta} \right) e^{-i\zeta}\, d\zeta$$

$$= \mathrm{Re} \int_{\zeta \in \gamma_{0,z}} (-\sinh(i\zeta), i\cosh(i\zeta), 1)\ d\zeta$$

$$= \mathrm{Re}(i\cosh(iz) - i, \sinh(iz), z)$$

$$= (\sinh y \sin x, \sinh y \cos x, x)$$

We note that the Helicoid is not graphical at the x_3-axis but away from the axis it is a multi-valued graph.

We first fix a helicoid $\Sigma = \{(\sinh y \sin x, \sinh y \cos x, x)\} \subset \mathbf{R}^3$. Then we consider a sequence of rescaled helicoids $\{\Sigma_a | \, \Sigma_a = a\Sigma\}$ with the parameter $a \in \mathbf{R}$, $a \to 0$. Figure 3.2 shows a diagram of a rescaled helicoid. (from Matthias Weber's Minimal Surface Archive [MA]).

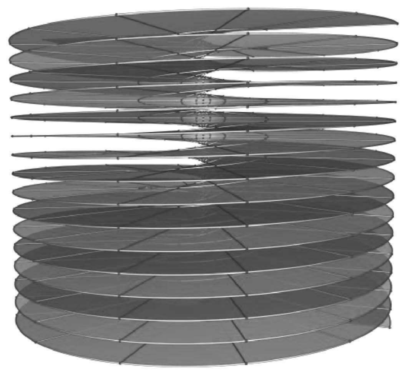

Figure 3.2: Rescaled Helicoid [MA]

Now $\Sigma_a = \{a(\sinh y \sin x, \sinh y \cos x, x)\} = F(\mathbb{C})$, where

$$F(z) = \mathrm{Re} \int_{\zeta \in \gamma_{0,z}} a \left(\frac{1}{2}(1 - e^{2i\zeta}), \frac{i}{2}(1 + e^{2i\zeta}), e^{i\zeta} \right) e^{-i\zeta} \, d\zeta$$

$$= \mathrm{Re} \int_{\zeta \in \gamma_{0,az}} \left(\frac{1}{2}(1 - e^{2i\zeta/a}), \frac{i}{2}(1 + e^{2i\zeta/a}), e^{i\zeta/a} \right) e^{-i\zeta/a} \, d\zeta,$$ by a change of variables.

Hence, the Weierstrass data for Σ_a is $\phi(z) = e^{-iz/a}$, $g(z) = e^{iz/a}$.

Then, from (2.7) and (2.8), we have that the curvature of Σ_a is:

$$\mathbf{K} = -\left[\frac{4|\partial_z e^{iz/a}|}{|e^{-iz/a}|(1 + |e^{iz/a}|^2)^2} \right]^2 = -\left[\frac{1}{a \cosh^2(y/a)} \right]^2$$

And the unit normal to Σ_a is:

$$\mathbf{n} = (2\mathrm{Re}\{e^{iz/a}\}, 2\mathrm{Im}\{e^{iz/a}\}, |e^{iz/a}|^2 - 1)/(|e^{iz/a}|^2 + 1)$$

$$= (2e^{-y/a}\cos(x/a), 2e^{-y/a}\cos(x/a), (e^{-2y/a} - 1))/(e^{-2y/a} + 1)$$

Therefore, at all points on the x_3-axis (i.e. where $y = 0$, since $a(\sinh y \sin x, \sinh y \cos x, x) = (0, 0, x) \iff y = 0$), we see that

$$\lim_{a \to 0} |\mathbf{K}| = \lim_{a \to 0} \left(\frac{1}{a}\right)^2 = \infty,$$

which implies that the curvature is blowing up at every point along the x_3-axis. However, at all points away from the x_3-axis (i.e. for all points in the domain $(x, y) \in \mathbb{C}$ where $|y| = \delta > 0$),

$$\lim_{a \to 0} |\mathbf{K}| = \lim_{a \to 0} \left[\frac{1}{a\cosh^2(\delta/a)}\right]^2 = 0,$$

In addition,

$$\lim_{a \to 0} \mathbf{n} = \lim_{a \to 0} (2e^{-\delta/a}\cos(x/a), 2e^{-\delta/a}\cos(x/a), (e^{-2\delta/a} - 1))/(e^{-2\delta/a} + 1) = (0, 0, -1)$$

This demonstrates that as $a \to 0$, $\{\Sigma_a\}$ converges away from the x_3-axis to a foliation of $\mathbf{R}^3$ by flat parallel planes with removable singularities at every point of the x_3-axis. By intersecting with the unit ball centered at the origin, we obtain a sequence $\{B_1 \cap$

$\Sigma_a\}$ of rescaled helicoids, contained in the unit ball B_1 and with boundaries in the boundary of the ball, such that the curvatures blow up at every point of the x_3-axis in the ball. The sequence converges to a foliation of the ball minus the x_3-axis by flat parallel planes with removable singularities at every point of the x_3-axis in the ball.

3.2 Curvature Blow-up at a single point

Colding and Minicozzi in [CM1] construct a sequence of compact embedded minimal disks in a unit ball in $\mathbf{R}^3$, centered at the origin, with boundaries in the boundary of the ball and where the curvatures blow up only at the origin.

A sequence of minimal immersions, $\{\Sigma_a\}$, for $0 < a < 1/2$ is constructed using the Weierstrass data, $\phi(z) = e^{-ih_a(z)}, g(z) = e^{ih_a(z)}$, in (2.1) where

$$h_a(z) = \frac{1}{a}\arctan\left(\frac{z}{a}\right),$$

and the corresponding domain $\Omega_a = \{(x,y)|\,|x| \leq 1/2, |y| \leq (x^2+a^2)^{(3/4)}/2\}$ is chosen such that the poles of $h(z)$, which are $\pm ia$, are outside Ω_a for all a. As $a \to 0$, the poles $\pm ia$ converge to the origin, but the domain also pinches off at the origin faster than the poles converge to the origin, so for all a, Σ_a is immersed. The properties of the sequence of minimal disks obtained via (2.1) with this choice of Weierstrass data result in curvature blowing up only at the origin, the surfaces being embedded for all a and their boundaries staying outside a cylinder of fixed radius. This allows intersection with a smaller ball and scaling to the unit ball B_1 to give a sequence of embedded minimal disks in B_1 with boundaries in the boundary ∂B_1. Colding and Minicozzi show that this sequence $\{\Sigma_a\}$ converges to a limit lamination of $B_1 \setminus \{0\}$ that consists of three leaves: $B_1 \cap \{x_3 = 0\} \setminus \{0\}$ and two embedded minimal disks Σ^+ and Σ^-. Σ^+ is in the upper hemisphere of B_1 and Σ^- is in the lower hemisphere. Both are multi-valued graphs away from the x_3-axis over the $\{x_3 = 0\}$ plane and both spiral into $\{x_3 = 0\}$. This spiralling from above and below results in $\{0\}$ being a non-removable singularity.

3.3 Curvature Blow-up at a discrete set of points

Brian Dean in [BD] constructs a sequence of compact embedded minimal disks in a ball in $\mathbf{R}^3$, centered at the origin, with boundaries in the boundary of the ball and where the curvatures blow up only at a prescribed discrete (finite) set of points on the x_3-axis.

A sequence of minimal immersions, $\{\Sigma_a\}$, for $0 < a < 1/2$ is constructed using the Weierstrass data, $\phi(z) = e^{-ih_a(z)}, g(z) = e^{ih_a(z)}$, in (2.1) where

$$h_a(z) = \sum_{j=1}^{n} \frac{1}{2^{j-1}a} \arctan\left(\frac{z - b_j}{a}\right),$$

where the b_j's are chosen so that $\{(0,0,b_j)\}$ is the prescribed discrete set of points on the x_3-axis at which curvature will blow up. The corresponding domain is chosen to be

$$\Omega_a = \bigcup_{j=1}^{n} \Omega_{a,j},$$

where for each j, $\Omega_{a,j}$ is similar to the domain used by Colding and Minicozzi in [CM1] but centered at the point $b_j \in \mathbb{C}$ instead of $0 \in \mathbb{C}$ and such that the union, Ω, is simply connected. Thus, the poles of $h_a(z)$, $\{b_j \pm ia\}$ stay outside of Ω_a. As $a \to 0$, Ω_a pinches off at the n points $\{b_j\}$ faster than the poles converge to these points. This results in curvature blowing up only at the n points $\{(0,0,b_j)\} \in \Sigma_a$ as $a \to 0$. By a similar analysis to the one used in [CM1], $\{\Sigma_a\}$ converges to a limit lamination of $B_1 \setminus \{(0,0,b_j) | \, 1 \leq j \leq n\}$ that consists of the punctured planes $B_1 \cap \{x_3 = b_j\} \setminus \{(0,0,b_j)\}$, and $(n+1)$ embedded minimal disks Σ^k that sit between and spiral into the appropriate planes $\{x_3 = b_j\}$. This spiralling from above and below results in each $\{(0,0,b_j)\}$ being a non-removable singularity.

Chapter 4

Embedded Minimal Disks with Curvature Blow-up on a Line Segment

We will now prove the main results in this thesis, Theorems 1.0.1 and 1.0.2.

4.1 Proof of the Main Theorems

We proceed to prove Theorem 1.0.1 by first constructing a family of minimal immersions $\{F_N\}$ with a specific choice of Weierstrass data $g(z) = e^{ih_N(z)}, \phi(z) = e^{-ih_N(z)}$, where $h_N(z) = u_N + iv_N$ and a corresponding domain $\Omega_N \subset \mathbb{C}$ to obtain $F_N(z)$ from (2.1) for each N.

We first define $\partial_z h_N(z)$ because it is this derivative that will be essential in determining the curvature properties required of our sequence of embedded minimal disks.

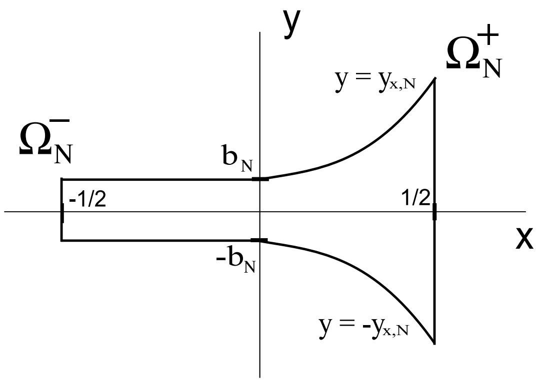

Figure 4.1: Diagram of the domain Ω_N

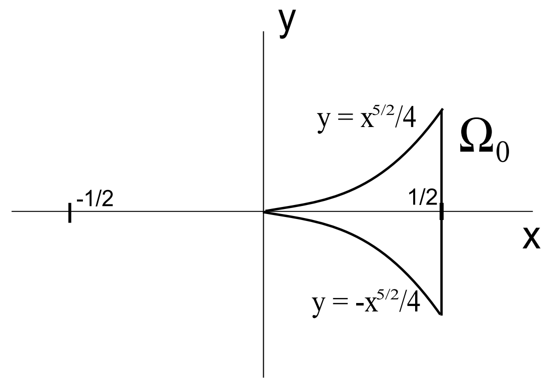

Figure 4.2: Diagram of the domain Ω_0

We let:

$$\partial_z h_N(z) = \frac{1}{2}\left[\frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(z+\frac{k}{N})^2 + (\frac{1}{N})^2]^2}\right],$$

for $N \geq 2$ on the domain $\Omega_N = \Omega_N^+ \cup \Omega_N^-$, where:

$$\Omega_N^+ = \left\{(x,y) \,\middle|\, |y| \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/4}}{4},\, 0 < x \leq 1/2\right\}$$

and

$$\Omega_N^- = \{(x,y) \mid |y| \leq b_N, -1/2 \leq x \leq 0\},$$

where $b_N = \dfrac{1}{4N^{5/2}}$. See Figure 4.1.

In this thesis, we will denote the upper and lower boundary of Ω_N as $y_{x,N}$ and $-y_{x,N}$ respectively. That is, on Ω_N^+, we set $y_{x,N} = \frac{(x^2+(\frac{1}{N})^2)^{5/4}}{4}$ and on Ω_N^-, we set $y_{x,N} = b_N$.

We note that for all N, $\partial_z h_N(z)$ is holomorphic on the domain Ω_N because the poles $\{\pm\frac{i}{N} - \frac{k}{N}\}$ for $0 \leq k \leq N$ lie outside the domain. Furthermore, these poles converge to the line segment $\{-1 \leq x \leq 0\}$ as $N \to \infty$.

Lemma 4.1.1 shows that there is a subsequence, $\{\partial_z h_{N_i}(z)\}$, that converges uniformly to a limit that we denote as:

$$\partial_z h(z) = \lim_{N_i \to \infty} \partial_z h_{N_i}(z)$$

on compact subsets of

$$\Omega_0 = \bigcap_N \Omega_N \setminus \{0\} = \left\{ (x, y) \,\middle|\, |y| \le \frac{x^{5/2}}{4}, 0 < x \le 1/2 \right\}$$

(See Figure 4.2). Therefore since for each N, $\partial_z h_N(z)$ is holomorphic on Ω_N, we have that $\partial_z h(z)$, and hence also $h(z)$, is holomorphic on Ω_0.

Lemma 4.1.1. *For $N \to \infty$, there is a subsequence $\{N_i\}$ such that $\{\partial_z h_{N_i}(z)\}$ converges uniformly on compact subsets of Ω_0.*

Proof. For every compact subset, $K \subset \Omega_0$, $\exists\, r_K > 0$ such that $\forall z \in K$, $\forall N, k > 0$,

$|(z + \frac{k}{N})^2 + (\frac{1}{N})^2| > r_K$, $|z^2 + (\frac{1}{N})^2| > r_K$

$$\implies |\partial_z h_N(z)| = \left| \frac{1}{2} \left[\frac{1}{[z^2 + (\frac{1}{N})^2]^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{[(z + \frac{k}{N})^2 + (\frac{1}{N})^2]^2} \right] \right|$$

$$< \frac{1}{2} \left[\frac{1}{r_K^2} + \frac{1}{r_K^2} \right] < \frac{1}{r_K^2}$$

$\implies$ $\{\partial_z h_N\}$ is a family of holomorphic functions , bounded on compact subsets of Ω_0

$\implies$ by Montel's theorem, there is a subsequence $\{\partial_z h_{N_i}(z)\}$ that converges uniformly on compact subsets of Ω_0 to a holomorphic limit. $\square$

Now, since for each N, $\partial_z h_N(z)$ is holomorphic on Ω_N, we integrate to obtain our Weierstrass data:

$$h_N(z) = \frac{N^2}{4}\left[N\arctan(Nz) + \frac{z}{z^2+(\frac{1}{N})^2} + \frac{1}{N}\sum_{k=1}^{N}\left(N\arctan(N(z+k/N)) + \frac{(z+k/N)}{(z+k/N)^2+(\frac{1}{N})^2}\right)\right] \tag{4.1}$$

And since $h_N(z)$ is also holomorphic on Ω_N, by the Cauchy-Riemann equations we have:

$$\partial_z h_N(z) = \partial_x u_N - i\partial_y u_N = \partial_y v_N + i\partial_x v_N$$

Therefore:

$$\begin{aligned}\partial_z h_N(z) &= \frac{1}{2}\left[\frac{1}{[z^2+(\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(z+\frac{k}{N})^2+(\frac{1}{N})^2]^2}\right] \\ &= \frac{1}{2}\Bigg[\frac{(x^2+(\frac{1}{N})^2-y^2)^2-4x^2y^2-4ixy(x^2+(\frac{1}{N})^2-y^2)}{([x^2+(\frac{1}{N})^2-y^2]^2+4x^2y^2)^2} \\ &+\frac{1}{N}\sum_{k=1}^{N}\Bigg(\frac{((x+k/N)^2+(\frac{1}{N})^2-y^2)^2-4(x+k/N)^2y^2}{([(x+k/N)^2+(\frac{1}{N})^2-y^2]^2+4(x+k/N)^2y^2)^2} \\ &-\frac{4i(x+k/N)y((x+k/N)^2+(\frac{1}{N})^2-y^2)}{([(x+k/N)^2+(\frac{1}{N})^2-y^2]^2+4(x+k/N)^2y^2)^2}\Bigg)\Bigg]\end{aligned}$$

$$\begin{aligned}\Longrightarrow\ \partial_y u_N = \frac{1}{2}\Bigg[&\frac{4xy(x^2+(\frac{1}{N})^2-y^2)}{([x^2+(\frac{1}{N})^2-y^2]^2+4x^2y^2)^2} \\ &+\frac{1}{N}\sum_{k=1}^{N}\frac{4(x+k/N)y((x+k/N)^2+(\frac{1}{N})^2-y^2)}{([(x+k/N)^2+(\frac{1}{N})^2-y^2]^2+4(x+k/N)^2y^2)^2}\Bigg]\end{aligned} \tag{4.2}$$

And

$$\partial_y v_N = \frac{1}{2}\left[\frac{(x^2+(\frac{1}{N})^2-y^2)^2-4x^2y^2}{([x^2+(\frac{1}{N})^2-y^2]^2+4x^2y^2)^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{((x+k/N)^2+(\frac{1}{N})^2-y^2)^2-4(x+k/N)^2y^2}{([(x+k/N)^2+(\frac{1}{N})^2-y^2]^2+4(x+k/N)^2y^2)^2}\right] \tag{4.3}$$

Now the main difficulty we encounter in the proof of Theorem 1.0.1 is showing that the immersions $F_N : \Omega_N \to \mathbf{R}^3$ are in fact embeddings.

The next Lemma gives this embeddedness result.

Lemma 4.1.2. *There exists $r_0 > 0$ (independent of N) such that $\forall (x, y) \in \Omega_N$,*

$$x_3(F_N(x, y)) = x\,. \tag{4.4}$$

(4.5)

The curve $F_N(x, \cdot) : [-y_{x,N}, y_{x,N}] \to \{x_3 = x\}$ is a graph in the $\{x_3 = x\}$ plane.

$$|F_N(x, \pm y_{x,N}) - F_N(x, 0)| > r_0 \text{ for all } N\,. \tag{4.6}$$

$$\text{In fact, for } x \le 0,\ |F_N(x, \pm y_{x,N}) - F_N(x, 0)| \to \infty \text{ as } N \to \infty \tag{4.7}$$

In Lemma 4.1.2, (4.4) shows that the horizontal slice of the image, $F_N(\Omega_N) \cap \{x_3 = t\}$, is the image of the vertical line $\{x = t\}$ in the domain (Ω_N). (4.5) shows that

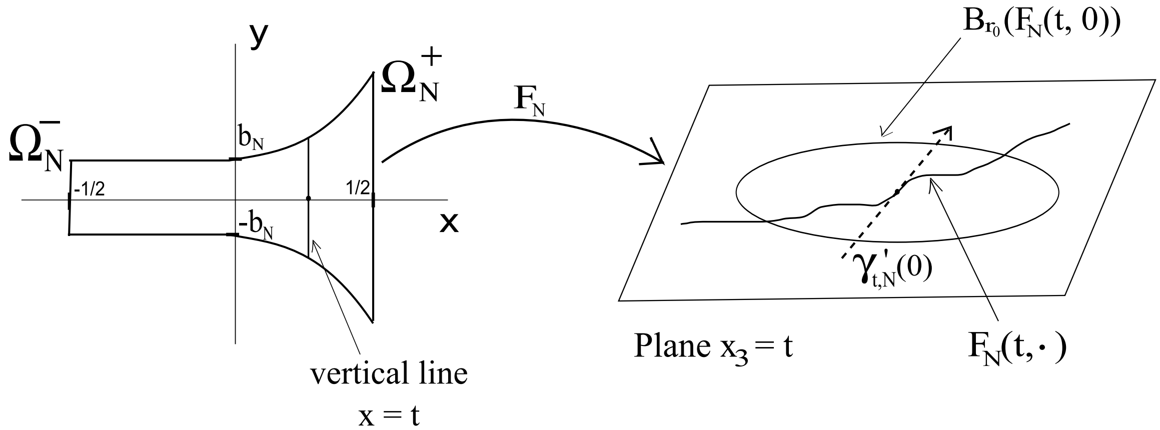

Figure 4.3: A horizontal slice of $F(\Omega_N)$ in Lemma 4.1.2

the image $F_N(\{x = t\} \cap \Omega_N)$ is a graph in the $\{x_3 = t\}$ plane over a line segment in that plane (see Figure 4.3). Together, these imply embeddedness. Also, (4.6) shows that there is some r_0 such that the boundary of the graph in (4.5) lies outside a disk $B_{r_0}(F_N(t, 0))$ for all N. And (4.7) shows that for all $x \leq 0$ (i.e. in the part of the image $F_N(\Omega_N)$ below the $\{x_3 = 0\}$ plane), these boundaries of the graph in (4.5) actually go to infinity as $N \to \infty$.

Proof. Since $z_0 = 0$ and the height differential is dz, (4.4) follows immediately from (2.1).

Now we prove (4.5) first for $0 < x \leq \frac{1}{2}$ (i.e. on Ω_N^+) and then for $-\frac{1}{2} \leq x \leq 0$ (i.e. on Ω_N^-)

4.2 Proof of (4.5) on Ω_N^+

We first note that on $\Omega_N^+ = \left\{(x,y) \,\Big|\, |y| \le \frac{(x^2+(\frac{1}{N})^2)^{5/4}}{4},\, 0 < x \le 1/2\right\}$,

$$y^2 \le \frac{(x^2+(\frac{1}{N})^2)^{5/2}}{16} \le \frac{x^2+(\frac{1}{N})^2}{16} \le \frac{(x+\frac{k}{N})^2+(\frac{1}{N})^2}{16}; \tag{4.8}$$

$$y^2 \le \frac{(x^2+(\frac{1}{N})^2)^{5/2}}{16} \le \frac{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^{5/2}}{16} \tag{4.9}$$

for all $k \ge 1$ since $0 < x \le 1/2$ and $N \ge 2 \implies (\frac{1}{N}) \le 1/2$.

Now using this in (4.2),

$$\begin{aligned}
|\partial_y u_N| &\le \frac{1}{2}\left[\frac{4x|y||x^2+(\frac{1}{N})^2-y^2|}{([x^2+(\frac{1}{N})^2-y^2]^2+4x^2y^2)^2}\right. \\
&\quad \left.+\frac{1}{N}\sum_{k=1}^{N}\frac{4(x+k/N)|y||(x+k/N)^2+(\frac{1}{N})^2-y^2|}{([(x+k/N)^2+(\frac{1}{N})^2-y^2]^2+4(x+k/N)^2y^2)^2}\right] \\
&\le \frac{1}{2}\left[\frac{4x|y|(x^2+(\frac{1}{N})^2)}{([\frac{15}{16}(x^2+(\frac{1}{N})^2)]^2)^2}\right. \\
&\quad \left.+\frac{1}{N}\sum_{k=1}^{N}\frac{4(x+k/N)|y|((x+k/N)^2+(\frac{1}{N})^2)}{([\frac{15}{16}((x+k/N)^2+(\frac{1}{N})^2)]^2)^2}\right] \\
&= 2\left(\frac{16}{15}\right)^4 |y| \left[\frac{x}{(x^2+(\frac{1}{N})^2)^3}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]
\end{aligned}$$

We set $y_{x,N} = \dfrac{(x^2+(\frac{1}{N})^2)^{5/4}}{4}$.

$$\implies \max_{|y|\le y_{x,N}} |u_N(x,y)-u_N(x,0)|$$

$$\le 2\left(\frac{16}{15}\right)^4 \left(\int_0^{y_{x,N}} t\,dt\right)\left[\frac{x}{(x^2+(\frac{1}{N})^2)^3}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 {y_{x,N}}^2 \left[\frac{x}{(x^2+(\frac{1}{N})^2)^3} + \frac{1}{N}\sum_{k=1}^{N} \frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 \frac{(x^2+(\frac{1}{N})^2)^{5/2}}{16} \left[\frac{x}{(x^2+(\frac{1}{N})^2)^3} + \frac{1}{N}\sum_{k=1}^{N} \frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 \frac{1}{16} \left[\frac{x(x^2+(\frac{1}{N})^2)^{5/2}}{(x^2+(\frac{1}{N})^2)^3} + \frac{1}{N}\sum_{k=1}^{N} \frac{(x+\frac{k}{N})((x+\frac{k}{N})^2+(\frac{1}{N})^2)^{5/2}}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]$$

$$= \left(\frac{16}{15}\right)^4 \frac{1}{16} \left[\frac{x}{(x^2+(\frac{1}{N})^2)^{\frac{1}{2}}} + \frac{1}{N}\sum_{k=1}^{N} \frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^{\frac{1}{2}}}\right]$$

$$\leq \left(\frac{16}{15}\right)^4 \frac{1}{16}[1+1] < 1 \tag{4.10}$$

We set $\gamma_{x,N}(y) = F_N(x,y) \implies \gamma'_{x,N}(y) = \partial_y F_N(x,y)$. We note that $v_N(x,0) = 0$ from (4.1) and that $\cos(1) > 1/2$.

Therefore (2.3) gives:

$$\langle \gamma'_{x,N}(y), \gamma'_{x,N}(0)\rangle = \cosh v_N(x,y)\cos(u_N(x,y) - u_N(x,0)) > \frac{1}{2}\cosh v_N(x,y) \tag{4.11}$$

Hence, by (4.11), the angle between $\gamma'_{x,N}(y)$ and $\gamma'_{x,N}(0)$ is always less than $\pi/2$. This gives us (4.5) for all $0 < x \leq \frac{1}{2}$ (i.e. on Ω_N^+)

Furthermore, this result holds uniformly in N because of the uniform bound in (4.10).

4.3 Proof of (4.5) on Ω_N^-

We recall that $\Omega_N^- = \{(x,y) \mid |y| \leq b_N, -1/2 \leq x \leq 0\}$, where $b_N = \dfrac{1}{4N^{5/2}}$

Now we note that on Ω_N^-, for $N \geq 2$,

$$y^2 \leq b_N^2 = \frac{1}{16N^5} \leq \frac{(x^2 + (\frac{1}{N})^2)^{5/2}}{16} \tag{4.12}$$

and

$$y^2 \leq b_N^2 = \frac{1}{16N^5} \leq \frac{((x + k/N)^2 + (\frac{1}{N})^2)^{5/2}}{16} \tag{4.13}$$

Also,

$$y^2 \leq b_N^2 = \frac{1}{16N^5} < \frac{x^2 + (\frac{1}{N})^2}{16} \tag{4.14}$$

and

$$y^2 \leq b_N^2 = \frac{1}{16N^5} < \frac{(x + k/N)^2 + (\frac{1}{N})^2}{16} \tag{4.15}$$

Now using these inequalities in (4.2),

$$|\partial_y u_N| \leq \frac{1}{2}\left[\frac{4x|y||x^2 + (\frac{1}{N})^2 - y^2|}{([x^2 + (\frac{1}{N})^2 - y^2]^2 + 4x^2y^2)^2}\right.$$
$$\left.+ \frac{1}{N}\sum_{k=1}^{N} \frac{4(x + k/N)|y||(x + k/N)^2 + (\frac{1}{N})^2 - y^2|}{([(x + k/N)^2 + (\frac{1}{N})^2 - y^2]^2 + 4(x + k/N)^2y^2)^2}\right]$$
$$\leq \frac{1}{2}\left[\frac{4x|y|(x^2 + (\frac{1}{N})^2)}{([\frac{15}{16}(x^2 + (\frac{1}{N})^2)]^2)^2} + \frac{1}{N}\sum_{k=1}^{N} \frac{4(x + k/N)|y|((x + k/N)^2 + (\frac{1}{N})^2)}{([\frac{15}{16}((x + k/N)^2 + (\frac{1}{N})^2)]^2)^2}\right]$$
$$= 2\left(\frac{16}{15}\right)^4 |y| \left[\frac{x}{(x^2 + (\frac{1}{N})^2)^3} + \frac{1}{N}\sum_{k=1}^{N} \frac{(x + \frac{k}{N})}{((x + \frac{k}{N})^2 + (\frac{1}{N})^2)^3}\right]$$

We recall that $y_{x,N} = b_N$

$$\Longrightarrow \max_{|y| \leq y_{x,N}} |u_N(x, y) - u_N(x, 0)|$$

$$
\begin{aligned}
&\leq 2\left(\frac{16}{15}\right)^4 \left(\int_0^{y_{x,N}} t\,dt\right)\left[\frac{x}{(x^2+(\frac{1}{N})^2)^3}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]\\
&=\left(\frac{16}{15}\right)^4 {y_{x,N}}^2\left[\frac{x}{(x^2+(\frac{1}{N})^2)^3}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]\\
&=\left(\frac{16}{15}\right)^4 b_N^2\left[\frac{x}{(x^2+(\frac{1}{N})^2)^3}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]\\
&\leq\left(\frac{16}{15}\right)^4 \frac{1}{16}\left[\frac{x(x^2+(\frac{1}{N})^2)^{5/2}}{(x^2+(\frac{1}{N})^2)^3}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})((x+\frac{k}{N})^2+(\frac{1}{N})^2)^{5/2}}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^3}\right]\\
&=\left(\frac{16}{15}\right)^4 \frac{1}{16}\left[\frac{x}{(x^2+(\frac{1}{N})^2)^{\frac{1}{2}}}+\frac{1}{N}\sum_{k=1}^{N}\frac{(x+\frac{k}{N})}{((x+\frac{k}{N})^2+(\frac{1}{N})^2)^{\frac{1}{2}}}\right]
\end{aligned}
$$

$$
\leq\left(\frac{16}{15}\right)^4 \frac{1}{16}[1+1]<1 \tag{4.16}
$$

And now we use the same argument we used to show (4.5) on Ω_N^+. We set $\gamma_{x,N}(y)=F_N(x,y)$. We note that $v_N(x,0)=0$ and $\cos(1)>1/2$.

Therefore (2.3) gives:

$$
\langle\gamma'_{x,N}(y),\gamma'_{x,N}(0)\rangle=\cosh v_N(x,y)\cos(u_N(x,y)-u_N(x,0))>\frac{1}{2}\cosh v_N(x,y) \tag{4.17}
$$

Hence, by (4.17), the angle between $\gamma'_{x,N}(y)$ and $\gamma'_{x,N}(0)$ is always less than $\pi/2$. This gives us (4.5) for all $-1/2\leq x\leq 0$ (i.e. on Ω_N^-). Furthermore, this result holds uniformly in N because of the uniform bound in (4.16).

Now we prove (4.6) first for $0<x\leq\frac{1}{2}$ (i.e. on Ω_N^+) and then for $-\frac{1}{2}\leq x\leq 0$ (i.e. on Ω_N^-).

4.4 Proof of (4.6) on Ω_N^+

We recall that $\Omega_N^+ = \left\{(x,y) \;\middle|\; |y| \le \frac{(x^2+(\frac{1}{N})^2)^{5/4}}{4},\ 0 < x \le 1/2\right\}$.

From (4.3), and by using (4.8) we have:

$$\begin{aligned}
\partial_y v_N = \frac{1}{2}\Bigg[&\frac{(x^2+(\frac{1}{N})^2-y^2)^2-4x^2y^2}{([x^2+(\frac{1}{N})^2-y^2]^2+4x^2y^2)^2} \\
&+\frac{1}{N}\sum_{k=1}^{N}\frac{((x+\frac{k}{N})^2+(\frac{1}{N})^2-y^2)^2-4(x+\frac{k}{N})^2y^2}{([(x+\frac{k}{N})^2+(\frac{1}{N})^2-y^2]^2+4(x+\frac{k}{N})^2y^2)^2}\Bigg] \\
&\ge \frac{1}{2}\Bigg[\frac{(\frac{15}{16}(x^2+(\frac{1}{N})^2))^2-4(x^2+(\frac{1}{N})^2)\frac{1}{16}(x^2+(\frac{1}{N})^2)}{([x^2+(\frac{1}{N})^2]^2+4(x^2+(\frac{1}{N})^2)\frac{1}{16}(x^2+(\frac{1}{N})^2))^2}+\frac{1}{N}\sum_{k=1}^{N} \\
&\Bigg(\frac{(\frac{15}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2))^2}{([(x+\frac{k}{N})^2+(\frac{1}{N})^2]^2+4((x+\frac{k}{N})^2+(\frac{1}{N})^2)\frac{1}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2))^2} \\
&-\frac{4((x+\frac{k}{N})^2+(\frac{1}{N})^2)\frac{1}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2)}{([(x+\frac{k}{N})^2+(\frac{1}{N})^2]^2+4((x+\frac{k}{N})^2+(\frac{1}{N})^2)\frac{1}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2))^2}\Bigg)\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{\frac{161}{256}(x^2+(\frac{1}{N})^2)^2}{\frac{25}{16}[x^2+(\frac{1}{N})^2]^4}+\frac{1}{N}\sum_{k=1}^{N}\frac{\frac{161}{256}((x+k/N)^2+(\frac{1}{N})^2)^2}{\frac{25}{16}[(x+k/N)^2+(\frac{1}{N})^2]^4}\right] \\
&= \frac{161}{800}\left[\frac{1}{[x^2+(\frac{1}{N})^2]^2}+\frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(x+k/N)^2+(\frac{1}{N})^2]^2}\right]
\end{aligned}$$

$$\Longrightarrow\ \partial_y v_N \ge \frac{161}{800[x^2+(\frac{1}{N})^2]^2} \tag{4.18}$$

We recall that $y_{x,N} = \dfrac{(x^2+(\frac{1}{N})^2)^{5/4}}{4}$

$$\Longrightarrow\ \min_{y_{x,N}/2\le|y|\le y_{x,N}} |v_N(x,y)| \ge \int_0^{y_{x,N}/2}\frac{161}{800[x^2+(\frac{1}{N})^2]^2}\,dt = \frac{161}{6400[x^2+(\frac{1}{N})^2]^{3/4}}$$

From (4.11), we have $\langle \gamma'_{x,N}(y), \gamma'_{x,N}(0)\rangle > \frac{1}{2}\cosh v_N(x,y)$.

Integrating this gives $\langle \gamma_{x,N}(y_{x,N}) - \gamma_{x,N}(0), \gamma'_{x,N}(0)\rangle > \int_{y_{x,N}/2}^{y_{x,N}} \frac{1}{2}\cosh(v_N(x,y))\,dy \geq \frac{1}{2}\frac{(x^2+(\frac{1}{N})^2)^{5/4}}{8}\cosh\left(\frac{161}{6400[x^2+(\frac{1}{N})^2]^{3/4}}\right)$

Now, since $|\gamma'_{x,N}(0)| = \cosh v_N(x,0) = 1$ and $\lim_{s\to 0} s^{5/4}\cosh\left(\frac{161}{6400 s^{3/4}}\right) = \infty$, this result and the analog for $\gamma_{x,N}(-y_{x,N})$ give our result on Ω_N^+ , (4.6), that

$$\forall x \in (0,1/2], \left| F_N(x, \pm\frac{(x^2+(\frac{1}{N})^2)^{5/4}}{4}) - F_N(x,0)\right| > r_1 \tag{4.19}$$

for some $r_1 > 0$ (independent of N) and all $N \geq 2$.

4.5 Proof of (4.6) on Ω_N^-

We recall that $\Omega_N^- = \{(x,y)\,|\,|y| \leq b_N, -1/2 \leq x \leq 0\}$, where $b_N = \dfrac{1}{4N^{5/2}}$

Now on Ω_N^-, from (4.3), and by using (4.14) and (4.15), we have:

$$\begin{aligned}
\partial_y v_N = \frac{1}{2}\Bigg[&\frac{(x^2+(\frac{1}{N})^2-y^2)^2-4x^2y^2}{([x^2+(\frac{1}{N})^2-y^2]^2+4x^2y^2)^2} \\
&+\frac{1}{N}\sum_{k=1}^{N}\frac{((x+\frac{k}{N})^2+(\frac{1}{N})^2-y^2)^2-4(x+\frac{k}{N})^2y^2}{([(x+\frac{k}{N})^2+(\frac{1}{N})^2-y^2]^2+4(x+\frac{k}{N})^2y^2)^2}\Bigg] \\
&\geq \frac{1}{2}\Bigg[\frac{(\frac{15}{16}(x^2+(\frac{1}{N})^2))^2-4(x^2+(\frac{1}{N})^2)\frac{1}{16}(x^2+(\frac{1}{N})^2)}{([x^2+(\frac{1}{N})^2]^2+4(x^2+(\frac{1}{N})^2)\frac{1}{16}(x^2+(\frac{1}{N})^2))^2}+\frac{1}{N}\sum_{k=1}^{N} \\
&\Bigg(\frac{(\frac{15}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2))^2}{([(x+\frac{k}{N})^2+(\frac{1}{N})^2]^2+4((x+\frac{k}{N})^2+(\frac{1}{N})^2)\frac{1}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2))^2} \\
&-\frac{4((x+\frac{k}{N})^2+(\frac{1}{N})^2)\frac{1}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2)}{([(x+\frac{k}{N})^2+(\frac{1}{N})^2]^2+4((x+\frac{k}{N})^2+(\frac{1}{N})^2)\frac{1}{16}((x+\frac{k}{N})^2+(\frac{1}{N})^2))^2}\Bigg)\Bigg]
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{\frac{161}{256}(x^2+(\frac{1}{N})^2)^2}{\frac{25}{16}[x^2+(\frac{1}{N})^2]^4} + \frac{1}{N}\sum_{k=1}^{N}\frac{\frac{161}{256}((x+k/N)^2+(\frac{1}{N})^2)^2}{\frac{25}{16}[(x+k/N)^2+(\frac{1}{N})^2]^4}\right]$$
$$= \frac{161}{800}\left[\frac{1}{[x^2+(\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(x+k/N)^2+(\frac{1}{N})^2]^2}\right]$$
$$\geq \frac{161}{800}\left[\frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(x+k/N)^2+(\frac{1}{N})^2]^2}\right]$$

Now $\forall x \in [-\frac{1}{2}, 0]$, $\forall N \geq 2$, $\exists t_x \in \mathbb{Z}$, $1 \leq t_x \leq N$ s.t. $-\frac{t_x}{N} < x \leq -\frac{t_x - 1}{N}$

$\implies x + \frac{t_x}{N} \leq \frac{1}{N}$

Hence $\forall -\frac{1}{2} \leq x \leq 0$,

$$\partial_y v_N \geq \frac{161}{800}\frac{1}{N}\frac{1}{[(x+\frac{t_x}{N})^2+(\frac{1}{N})^2]^2} \geq \frac{161}{800}\frac{1}{N}\frac{1}{[(\frac{1}{N})^2+(\frac{1}{N})^2]^2}$$

$$\implies \partial_y v_N \geq \frac{161}{3200}N^3 \tag{4.20}$$

$$\implies \min_{y_{x,N}/2 \leq |y| \leq y_{x,N}} |v_N(x,y)| \geq \int_0^{y_{x,N}/2} \frac{161}{3200}N^3\,dt = \frac{161}{3200}N^3\frac{y_{x,N}}{2} = \frac{161}{6400}N^3 b_N$$
$$= \frac{161}{6400}N^3\left(\frac{1}{4N^{5/2}}\right) = \frac{161}{25600}N^{1/2}$$

From (4.11), we have $\langle \gamma'_{x,N}(y), \gamma'_{x,N}(0)\rangle > \frac{1}{2}\cosh v_N(x,y)$

Integrating this gives

$$\langle \gamma_{x,N}(y_{x,N}) - \gamma_{x,N}(0), \gamma'_{x,N}(0)\rangle > \int_{y_{x,N}/2}^{y_{x,N}} \tfrac{1}{2}\cosh(v_N(x,y))\,dy \geq \frac{1}{2}\frac{b_N}{2}\cosh\left(\frac{161}{25600}N^{1/2}\right)$$

$$\implies \langle \gamma_{x,N}(y_{x,N}) - \gamma_{x,N}(0), \gamma'_{x,N}(0)\rangle > \frac{1}{16N^{5/2}}\cosh\left(\frac{161}{25600}N^{1/2}\right) \tag{4.21}$$

Now, since $|\gamma'_{x,N}(0)| = \cosh v_N(x,0) = 1$ and $\lim_{N\to\infty}\frac{1}{16N^{5/2}}\cosh\left(\frac{161}{25600}N^{1/2}\right) = \infty$, this result and the analog for $\gamma_{x,N}(-y_{x,N})$ give our result on Ω_N^- , (4.6), that

$$\forall x \in [-1/2, 0],\ |F_N(x, \pm b_N) - F_N(x, 0)| > r_2 \tag{4.22}$$

for some $r_2 > 0$ (independent of N) and all $N \geq 2$.

Hence, by choosing $r_0 = \min\{r_1, r_2\}$ given by (4.19) and (4.22), we have (4.6). Also by (4.21) we have the result (4.7) that for $x \leq 0$, $|F_N(x, \pm b_N) - F_N(x, 0)| \to \infty$ as $N \to \infty$.

□

Now we will prove the following corollary that gives us the embeddings F_N that we will use in the proof of Theorem 1.0.1.

Corollary 4.5.1. *Let r_0 be given by (4.6).*

(i) F_N is an embedding and $F_N(\Omega_N) \subset \mathbf{R}^2 \times [-1/2, 1/2] \subset \mathbf{R}^3$.

(ii) $F_N(t, 0) = (0, 0, t)$ for $|t| \leq 1/2$.

(iii) $F_N(\Omega_N) \backslash \{(0, 0, t) |\, |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ for multi-valued graphs $M_{1,N}$, $M_{2,N}$ over the $\{x_3 = 0\}$ plane.

Proof. (i) follows from (4.4) and (4.5).

We obtain (ii) by integrating (2.2) with respect to x, using the fact that $v_N(x, 0)$ is identically 0, and the fact that $F(0, 0) = (0, 0, 0)$ because of our choice of $z_0 = 0$ in (2.1). From (2.7), F_N is "vertical" (i.e. $\langle \mathbf{n}, (0, 0, 1) \rangle = 0$) exactly when $|g_N| = 1$. But since $g_N(z) = e^{i(u_N(z)+iv_N(z))}$, $|g_N(x, y)| = 1 \iff v_N(x, y) = 0$. Now for $x > 0$, by (4.18) and since $v_N(x,0) = 0$, we have $|v_N(x,y)| \geq \dfrac{161|y|}{800[x^2 + (\frac{1}{N})^2]^2}$. Similarly, for $x \leq 0$, by (4.20) and since $v_N(x,0) = 0$, we have $|v_N(x,y)| \geq \dfrac{161|y|}{3200}N^3$. Hence, for all x, $v_N(x,y) = 0 \iff y = 0$ and therefore $\langle \mathbf{n}, (0,0,1) \rangle = 0 \iff y = 0$.

Therefore by Corollary 4.5.1, (ii), the image of F_N is graphical away from the x_3-axis, giving us (iii).

$\square$

4.6 Proof of Theorem 1.0.1

Corollary 4.5.1 gives us a sequence of minimal embeddings $F_N : \Omega_N \to \mathbf{R}^2 \times [-1/2, 1/2] \subset \mathbf{R}^3$ with $F_N(t,0) = (0,0,t)$ for $|t| \leq 1/2$.

We let $M_N = F_N(\Omega_N)$.

4.6.1 Proof of Theorem 1.0.1 (a)

By using (2.8) with our Weierstrass data, $g(z) = \mathrm{e}^{i\,(u(z)+iv(z))}$ and $\phi(z) = \mathrm{e}^{-i\,(u(z)+iv(z))}$, we have that the curvature of F_N is given by

$$K_N(z) = \frac{-|\partial_z h_N|^2}{\cosh^4 v_N} \tag{4.23}$$

Therefore, if $|\partial_z h_N| \to \infty$ and for some constant $M > 0, \cosh^4 v_N < M$, then $K_N \to \infty$

Let $z \in [-1/2, 0]$.

$\forall N \geq 2,\ \exists t_z \in \mathbb{Z},\ 1 \leq t_z \leq N$ s.t. $-\dfrac{t_z}{N} < z \leq -\dfrac{t_z - 1}{N}$

$\implies z + \dfrac{t_z}{N} \leq \dfrac{1}{N}$

$$\implies \partial_z h_N(z) = \frac{1}{2}\left[\frac{1}{[z^2+(\frac{1}{N})^2]^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(z+\frac{k}{N})^2+(\frac{1}{N})^2]^2}\right]$$
$$\geq \frac{1}{2N}\frac{1}{[(z+\frac{t_z}{N})^2+(\frac{1}{N})^2]^2}$$
$$\geq \frac{1}{2N}\frac{1}{[(\frac{1}{N})^2+(\frac{1}{N})^2]^2} = N^3/8$$

$$\implies \lim_{N\to\infty} \partial_z h_N(z) = \infty$$

Now, we also note that $\forall x,\ -1/2 \leq x \leq 1/2,\ v_N(x,0) = 0$.

$\implies \cosh^4(v_N(x,0)) = 1$

Hence, we have curvature blowing up at all points of the line segment, $[-1/2, 0] \subset \Omega_N$.

This gives us that $\forall p \in \{(0,0,t) \mid -1/2 \leq t \leq 0\}$, $\lim_{N\to\infty} |A_{M_N}|^2(p) = \infty$.

4.6.2 Proof of Theorem 1.0.1 (b), (c) and (d)

Theorem 1.0.1 (b) follows immediately from Corollary 4.5.1(iii).

To prove Theorem 1.0.1 (c) we fix $\delta > 0$, and let B_δ be a δ-neighborhood of $\{(0,0,t) \mid -1/2 \leq t \leq 0\}$ that is cylindrically shaped (shown in Figure 4.4).

Then $\forall N$, $\forall\, p = (x_1, x_2, x_3) \in M_N$, where $x_3 > \delta$ (i.e. for all points of M_N that are more than a δ distance above the $x_3 = 0$ plane), by (4.4), $x = x_3 > \delta$, where $(x, y) \in \Omega_N^+$ such that $F_N(x,y) = (x_1, x_2, x_3)$.

Hence, since on Ω_N^+, $y^2 \leq \dfrac{x^2 + 1/N^2}{16} \leq \dfrac{(x+k/N)^2 + 1/N^2}{16}$, $|z^2 + (\dfrac{1}{N})^2|^2 =$

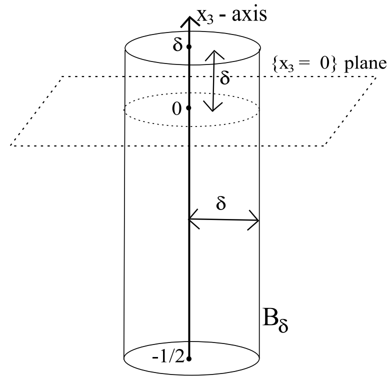

Figure 4.4: Diagram of B_δ, the δ-neighborhood of $\{(0,0,t)\,|\,-1/2 \leq t \leq 0\}$ in Theorem 1.0.1

$|(x+iy)^2+(1/N)^2|^2 = [x^2+(1/N)^2-y^2]^2+4x^2y^2$

$\geq (15/16)^2[x^2+(1/N)^2]^2+4x^2y^2 \geq (15/16)^2x^4 > (15/16)^2\delta^4$

Similarly, $|(z+k/N)^2+(\frac{1}{N})^2|^2 = |(x+k/N+iy)^2+(1/N)^2|^2$

$= [(x+k/N)^2+(1/N)^2-y^2]^2+4(x+k/N)^2y^2$

$\geq (15/16)^2[(x+k/N)^2+(1/N)^2]^2+4(x+k/N)^2y^2 \geq (15/16)^2x^4 > (15/16)^2\delta^4$

$$\Longrightarrow\ |\partial_z h_N(z)| = \left|\frac{1}{2}\left[\frac{1}{[z^2+(\frac{1}{N})^2]^2}+\frac{1}{N}\sum_{k=1}^{N}\frac{1}{[(z+\frac{k}{N})^2+(\frac{1}{N})^2]^2}\right]\right|$$

$$< \frac{1}{2}\left[\frac{1}{(15/16)^2\delta^4}+\frac{1}{(15/16)^2\delta^4}\right] < \frac{1}{(15/16)^2\delta^4}$$

This uniform bound of $|\partial_z h_N(z)|$ gives the curvature bound for all points of M_N that are at least a distance δ above the $\{x_3 = 0\}$ plane by (4.23). Let this curvature bound be $C_\delta^{(1)}$.

Now, at all points $p \in M_N$ that are at least a distance δ away from the x_3-axis (i.e. outside a cylinder about the x_3-axis of radius δ), Heinz's curvature estimate for graphs (11.7 in [Os]) applied to components of M_N over disks of radius $\delta/2$ in the $\{x_3 = 0\}$ plane, which are guaranteed to be graphs over the $\{x_3 = 0\}$ plane by Theorem 1.0.1(b), gives a uniform curvature bound, $C_\delta^{(2)}$.

Hence, $C_\delta = \max\{C_\delta^{(1)}, C_\delta^{(2)}\}$ is the uniform curvature bound in Theorem 1.0.1(c). Theorem 1.0.1(d) follows from (4.6) and (4.7).

4.7 Proof of Theorem 1.0.2

We will need the following lemma that gives us convergence.

Lemma 4.7.1. *Consider the sequence of embedded minimal disks* $\{M_N\}$ *given by Theorem 1.0.1. Let* $W = \mathcal{C} \cup H$ *where* $\mathcal{C}$ *is the cylinder* $\{(x_1, x_2, x_3) | \, x_1^2 + x_2^2 \leq r_0^2, -1/2 < x_3 < 1/2\}$ *with* r_0 *determined by Theorem 1.0.1(d) and* $H = \mathbf{R}^2 \times [-1/2, 0]$ *(a horizontal block of the half space below the* $\{x_3 = 0\}$ *axis).*

(a) $\{M_N\}$*, as a sequence of minimal laminations, has a subsequence that converges to a limit lamination on compact subsets of* W *away from* $\{(0,0,t)| -1/2 \leq t \leq 0\}$ *in the* C^α *topology for any* $\alpha < 1$*.*

(b) This subsequence of embedded minimal disks has a further subsequence $\{M_{N_i}\}$ *such that the leaves converge uniformly in the* C^k *topology for all* k*.*

Proof. To prove Lemma 4.7.1(a) we cover compact subsets K of $W \setminus \{(0,0,t)| -1/2 \leq t \leq 0\}$ with sufficiently small balls B_{r_K} (with radius r_K depending on the compact subset) such that the covering does not intersect $\{(0,0,t)| \, |t| \leq 1/2\}$. For each K, we take N in the sequence $\{M_N\}$ to be large enough (i.e. $N \geq N_K$ for some N_K

depending on K) such that ∂M_N is outside K. This ensures that in each ball B_{r_K}, the leaves of M_N in the ball have boundary contained in ∂B_{r_K}. Then we use the uniform curvature bound in Theorem 1.0.1(c), apply Proposition B.1 in [CM5] to each ball and pass to successive subsequences on each ball, to obtain a subsequence that we rename $\{M_N\}$ that converges to a lamination, $\mathcal{L}$, with minimal leaves on compact subsets of $W \setminus \{(0,0,t)| -1/2 \leq t \leq 0\}$ in the C^α topology for any $\alpha < 1$.

To prove Lemma 4.7.1(b) we consider the subsequence $\{M_N\}$ obtained above in Lemma 4.7.1(a) and we recall that by Theorem 1.0.1(b), $M_N \setminus \{(0,0,t)| \, |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ for multi-valued graphs $M_{1,N}$, $M_{2,N}$ over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane. We cover compact subsets, K, of $W \setminus \{(0,0,t)| \, |t| \leq 1/2\}$ with balls B_{r_K} in the same way as in the proof of Lemma 4.7.1(a) above, such that the covering does not intersect $\{(0,0,t)| \, |t| \leq 1/2\}$. This ensures that for all N, the leaves in the intersection with each ball B_{r_K} are graphical over a subdomain of the $\{x_3 = 0\} \setminus \{0\}$ punctured plane. Then since by Corollary 4.5.1 (i), for all N, $M_{1,N} \cup M_{2,N}$ has bounded maximum distance from the $\{x_3 = 0\}$ plane , we apply Corollary 16.7 in [GT] to each ball B_{r_K} to obtain uniform bounds (that are functions of r_K only) on the derivatives of all orders of the graphs of the leaves in $(M_N \setminus \{(0,0,t)| \, |t| \leq 1/2\}) \cap B_{r_K}$. Then, using standard compactness results and a diagonal argument whereby we pass to successive subsequences on each ball, we obtain a subsequence $\{M_{N_i}\}$ that converges uniformly in C^k for all k on compact subsets of $W \setminus \{(0,0,t)||t| \leq 1/2\}$.

□

Now, to prove Theorem 1.0.2, it is sufficient (by scaling) for us to show that there exists a sequence of compact embedded minimal disks $0 \in \Sigma_N \subset B_R \subset \mathbf{R}^3$ with $\partial\Sigma_N \subset \partial B_R$ for some $R > 0$. Theorem 1.0.1 gives us a sequence of minimal embeddings $M_N = F_N(\Omega_N) \subset \mathbf{R}^3$ with $F_N(t, 0) = (0, 0, t)$ for $|t| \leq 1/2$.

We set $R = \min\{r_0/2, 1/4\}$ where r_0 is given by Theorem 1.0.1(d) and we let $\Sigma_{N_i} = B_R \cap M_{N_i}$, where the sequence N_i is determined by Lemma 4.7.1. We rename this sequence $\{\Sigma_N\}$. From Theorem 1.0.1(d), we see that $\partial\Sigma_N \subset \partial B_R$. And from the properties satisfied by M_N in Theorem 1.0.1, Theorem 1.0.2 (a),(b) and (c) follow immediately.

Now, we note that Corollary 4.5.1 (iii) and the smooth convergence of the leaves in Lemma 4.7.1(b), give us that the limit minimal lamination $\mathcal{L}$ in the upper hemisphere of B_R consists of a leaf in the upper hemisphere, Σ, such that $\Sigma \setminus \{x_3-\text{axis}\} = \Sigma' \cup \Sigma''$, where Σ' and Σ'' are multi-valued graphs.

By (2.3) and (4.5), the horizontal slices $\{x_3 = x\} \cap \Sigma'$ and $\{x_3 = x\} \cap \Sigma''$ are graphs in the $\{x_3 = x\}$ plane over the line in the direction

$$\lim_{N\to\infty} \partial_y F_N(x, 0) = \lim_{N\to\infty} (\sin u_N(x, 0), -\cos u_N(x, 0), 0)\,. \tag{4.24}$$

We note that from (4.3), by the Cauchy-Riemann equations, $\forall N > 0$,

(4.25)

$$\partial_x u_N(x, 0) = \partial_y v_N(x, 0) = \frac{1}{2}\left[\frac{1}{(x^2 + (\frac{1}{N})^2)^2} + \frac{1}{N}\sum_{k=1}^{N}\frac{1}{((x + k/N)^2 + (\frac{1}{N})^2)^2}\right] > 0$$

$\Longrightarrow$ $u_N(x, 0)$ is monotonically increasing w.r.t. x, for each fixed N.

Therefore, for $0 < t < R$ the angle turned by the line in (4.24) for a change in x

from t to $2t$ is:

$$\lim_{N\to\infty} |u_N(2t,0) - u_N(t,0)| = \lim_{N\to\infty} \left| \int_t^{2t} \partial_x u_N(x,0)\, dx \right| \tag{4.26}$$

$$= \lim_{N\to\infty} \int_t^{2t} \frac{1}{2} \left[\frac{1}{(x^2 + (\frac{1}{N})^2)^2} + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{((x+k/N)^2 + (\frac{1}{N})^2)^2} \right] dx$$
$$\geq \lim_{N\to\infty} \int_t^{2t} \frac{1}{2} \frac{1}{(x^2 + (\frac{1}{N})^2)^2}\, dx$$
$$= \lim_{N\to\infty} \frac{N^2}{4} \left[\frac{x}{x^2 + (\frac{1}{N})^2} + N \arctan{(Nx)} \right]_t^{2t} = \frac{7}{48t^3}\,,$$

Hence we see that, for $0 < t < R$, $\{t < |x_3| < 2t\} \cap \Sigma'$ and $\{t < |x_3| < 2t\} \cap \Sigma''$ both contain an embedded S_t-valued graph where $S_t \geq \frac{7}{96\pi t^3} \to \infty$ as $t \to 0$. It follows that Σ' and Σ'' must both spiral into the $\{x_3 = 0\}$ plane. In addition, a Harnack inequality in Proposition II.2.12 in [CM2] gives a lower bound on the vertical separation of the sheets in both Σ' and Σ'', for each compact subset above the $\{x_3 = 0\}$ plane. This shows that the spiralling into the $\{x_3 = 0\}$ plane occurs with multiplicity one.

Finally, we show that the minimal lamination $\mathcal{L}$ in the lower hemisphere of $B_R \setminus \{(0,0,t) | -R \leq t \leq 0\}$ consists of a foliation by parallel planes.

We consider the sequence of embedded minimal disks M_N given by Theorem 1.0.1 and we recall that by Corollary 4.5.1 (iii), $M_N \setminus \{(0,0,t) | \, |t| \leq 1/2\} = M_{1,N} \cup M_{2,N}$ for multi-valued graphs $M_{1,N}$, $M_{2,N}$ over the $\{x_3 = 0\}$ plane.

For arbitrary $-\frac{1}{2} \leq t \leq 0$, fixed $j = 1$ or 2 and for all N we define $\Gamma_{j,N}(t)$ to be the component of $M_{j,N}$ that is contained between the planes $\{x_3 = t\}$ and $\{x_3 = t + \epsilon_N\}$ where ϵ_N is such that the tangent vector $\partial_y F_N(t,0)$ to $M_N \cap \{x_3 = t\}$ at the x_3

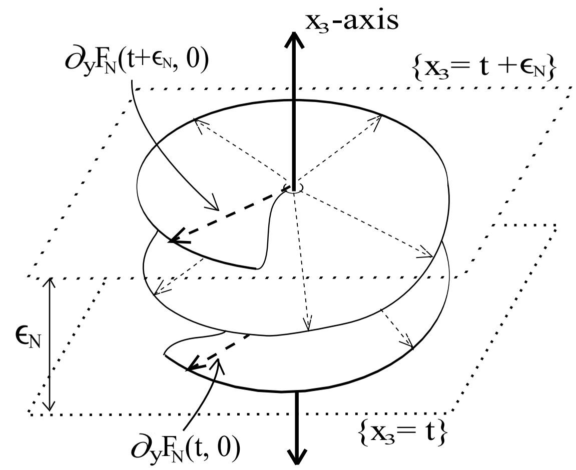

Figure 4.5: Diagram of $\Gamma_{j,N}(t)$

axis (recall that this intersection is a graph in the $\{x_3 = t\}$ plane over the line in the direction $\partial_y F_N(t,0) = (\sin u_N(t,0), -\cos u_N(t,0), 0)$ by (2.3) and (4.5)) turns through an angle of 4π to the direction of the tangent vector $\partial_y F_N(t+\epsilon_N, 0)$ to $M_N \cap \{x_3 = t+\epsilon_N\}$ at the x_3 axis, as x increases from t to $t+\epsilon_N$ (See Figure 4.5). This definition ensures that $\Gamma_{j,N}(t) = \{t \le x_3 \le t+\epsilon_N\} \cap M_{j,N}$ is a graph over the $\{x_3 = 0\} \setminus \{0\}$ punctured plane such that the level sets $M_{j,N} \cap \{x_3 = x\}$ sweep out an angle of magnitude between 3π and 5π for $t \le x \le t+\epsilon_N$.

Now we show that $\epsilon_N \to 0$ as $N \to \infty$.

For small s, the angle turned by the tangent vector $\partial_y F_N(x,0)$ at the x_3 axis for a change in x from t to $t+s$ is, by (4.25):

$$|u_N(t+s,0) - u_N(t,0)| = \left|\int_t^{t+s} \partial_x u_N(x,0)\,dx\right|$$

$$= \int_t^{t+s} \frac{1}{2}\left[\frac{1}{(x^2+(\frac{1}{N})^2)^2} + \frac{1}{N}\sum_{k=1}^{N} \frac{1}{((x+k/N)^2+(\frac{1}{N})^2)^2}\right] dx$$

$$\ge \frac{1}{2}\int_t^{t+s} \frac{1}{N}\sum_{k=1}^{N} \frac{1}{((x+k/N)^2+(\frac{1}{N})^2)^2}\,dx = \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} \frac{1}{((x'+k/N)^2+(\frac{1}{N})^2)^2}((t+s) - (t))$$ (for some $t \le x' \le t+s$ by the Mean Value Theorem)

$$\geq \frac{1}{2}\frac{1}{N}\frac{s}{((x'+c/N)^2+(\frac{1}{N})^2)^2} \text{ (for } 1 \leq c \leq N \text{ chosen so that } x'+c/N \leq 1/N)$$
$$\geq \frac{1}{2}\frac{1}{N}\frac{s}{((1/N)^2+(1/N)^2)^2} = \frac{1}{8}N^3 s$$

Therefore, for any s, $\{t \leq x_3 \leq t+s\} \cap M_{j,N}$ contains an embedded R_t-valued graph where $R_t \geq \frac{1}{16\pi}N^3 s \to \infty$ as $N \to \infty$. This means that since for all N, $\Gamma_{j,N}(t) = \{t \leq x_3 \leq t+\epsilon_N\} \cap M_{j,N}$ as defined above is at most 3-valued, $\epsilon_N \leq 48\pi/N^3 \to 0$ as $N \to \infty$.

Now we have that for each N, $\Gamma_{j,N}(t)$ is an embedded minimal graph over the $\{x_3 = 0\}$ plane by 1.0.1(b), the boundary of each horizontal slice of $\Gamma_{j,N}(t)$ tends to infinity by Theorem 1.0.1(d), and as we have shown above, $\Gamma_{j,N}(t) = \{t \leq x_3 \leq t+\epsilon_N\} \cap M_{j,N}$ is such that $\epsilon_N \to 0$ as $N \to \infty$. Therefore, by Lemma 4.7.1(b), a subsequence $\{\Gamma_{j,N_i}(t)\}$ converges uniformly on compact subsets in the C^k topology for all k to an entire minimal graph minus the point $\{(0,0,t)\}$. By a standard Bernstein type theorem, this limit graph must be a plane with a removable singularity at $\{(0,0,t)\}$. Since $-1/2 \leq t \leq 0$ was arbitrary we have that the limit lamination $\mathcal{L}$ below the $\{x_3 = 0\}$ plane is a foliation by planes parallel to $\{x_3 = 0\}$ with removable singularities along the negative x_3-axis. And by intersecting with B_R, we obtain the required result that the lamination of the lower hemisphere of $B_R \setminus \{(0,0,t) | -R \leq t \leq 0\}$ consists of a foliation by parallel planes, each with a removable singularity at the x_3-axis. The one exception is that the singularity at the origin is not removable because of the spiralling of the leaf in the upper hemisphere, Σ, into the $\{x_3 = 0\}$ plane.

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Vita

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