CONCENTRATION AND VANISHING OF EIGENFUNCTIONS ON A MANIFOLD WITH BOUNDARY

by

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Abstract

The first part of this dissertation contains work from [1] concerning the concentration of Dirichlet eigenfunctions of the Laplacian on a compact two-dimensional Riemannian manifold with strictly geodesically concave boundary. We link three inequalities which bound the concentration in different ways. We also prove one of these inequalities, which bounds the L^p norms of the restrictions of eigenfunctions to broken geodesics. The second part of this dissertation contains work from [2] concerning the nodal sets of Dirichlet or Neumann eigenfunctions on a compact Riemannian manifold with boundary. We prove lower bounds for the size of the nodal sets.

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Chapter 1

Introduction

1.1 Manifolds without Boundary

We first review background on manifolds without boundary. Let (M, g) be a compact Riemannian manifold without boundary and let Δ be the Laplacian. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be eigenfunctions which form an orthonormal basis of $L^2(M)$. Let $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ be the corresponding eigenvalues, normalized so that $-\Delta\varphi_j = \lambda_j^2\varphi_j$. That is, the numbers λ_j are eigenvalues of the first-order operator $\sqrt{-\Delta}$. Note that we only consider eigenfunctions which are L^2-normalized.

1.1.1 $\mathbf{L^p}$ Norms

The L^p norms of the eigenfunctions measure their concentration on small sets. Sogge [27] proved the following theorem.

Theorem 1.1.1. *On a two-dimensional manifold, there is a constant C such that*

$$\|\varphi_j\|_{L^p(M)} \leq C(1 + \lambda_j)^{\delta(p)}$$

where

$$\delta(p) = \begin{cases} \frac{1}{4} - \frac{1}{2p} & \text{if } 2 \le p \le 6 \\ \frac{1}{2} - \frac{2}{p} & \text{if } 6 \le p \le \infty \end{cases}$$

On the round sphere $\mathbb{S}^2$, the spherical harmonics are eigenfunctions. Theorem 1.1.1 is sharp for these eigenfunctions, meaning

$$\limsup_{j\to\infty} \lambda_j^{-\delta(p)} \|\varphi_j\|_{L^p(M)} > 0 \tag{1.1}$$

A natural problem is to determine the manifolds and eigenfunctions for which (1.1) holds. For the ranges $p \ge 6$ and $2 < p \le 6$, the inequality (1.1) reflects different types of concentration. On $\mathbb{S}^2$, the zonal spherical harmonics have L^2 mass concentrated near two antipodal points and satisfy (1.1) for $p \ge 6$. Sogge and Zelditch [30] proved that if (1.1) holds for $p > 6$, then there must be a point x such that the set of geodesics which start at x and loop back through x has positive measure.

A central problem is to determine when (1.1) holds for $2 < p \le 6$. On $\mathbb{S}^2$, the highest weight spherical harmonics have L^2 mass concentrated near a geodesic and satisfy (1.1) for $2 \le p \le 6$. More generally, if M has a stable elliptic periodic geodesic γ, then Gaussian beams can be constructed along γ. These are approximate eigenfunctions which have Gaussian decay in directions transverse to the geodesic and satisfy (1.1) for $2 \le p \le 6$. In some cases, such as surfaces of revolution, these are actual eigenfunctions. Sogge, Toth, and Zelditch have conjectured that if (1.1) holds for $2 < p < 6$, then there must be a stable elliptic periodic geodesic on M, and the eigenfunctions must concentrate near this geodesic.

Let Π denote the set of geodesic segments which have length one. The following theorem also concerns the concentration of eigenfunctions near geodesics. It was proven by Burq, Gérard, and Tzvetkov [5]. Their work was motivated by Reznikov [21] who considered hyperbolic surfaces.

Theorem 1.1.2. *On a two-dimensional manifold, there is a positive constant C such that*

$$\sup_{\gamma\in\Pi}\|\varphi_j\|_{L^p(\gamma)} \le C(1+\lambda_j)^{\sigma(p)}$$

where

$$\sigma(p) = \begin{cases} \frac{1}{4} & \text{if } 2 \le p \le 4 \\ \frac{1}{2} - \frac{1}{p} & \text{if } 4 \le p \le \infty \end{cases}$$

The case $p = 2$ also follows from an earlier result of Tataru [33]. The following theorem connects Theorem 1.1.1 and Theorem 1.1.2. It is a combination of results due to Bourgain [3] and Sogge [29].

Theorem 1.1.3. *On a two-dimensional manifold, the following are equivalent:*

$$\limsup_{j\to\infty} \lambda_j^{-1/8}\|\varphi_j\|_{L^4(M)} > 0$$

$$\limsup_{j\to\infty}\sup_{\gamma\in\Pi} \lambda_j^{-1/4}\|\varphi_j\|_{L^2(\gamma)} > 0$$

Moreover, if γ is in Π and the eigenfunctions satisfy

$$\limsup_{j\to\infty} \lambda_j^{-1/4}\|\varphi_j\|_{L^2(\gamma)} > 0$$

then γ must be a segment of a periodic geodesic.

This theorem is a significant step towards determining the manifolds and eigenfunctions for which (1.1) holds. In particular, Sogge and Zelditch [32] used this theorem to show that (1.1) cannot hold for $2 < p < 6$ if M has non-positive curvature.

1.1.2 Nodal Sets

Define the nodal set

$$Z_j = \left\{ x \in M : \varphi_j(x) = 0, x \notin \partial M \right\}$$

Let n be the dimension of M and let $\mathcal{H}$ be the $(n-1)$-dimensional Hausdorff measure on M. We will prove lower bounds for $\mathcal{H}(Z_j)$. We use the notation $A \lesssim B$ to mean there is a positive constant C, independent of λ_j and φ_j, such that $A \leq CB$.

A conjecture of Yau states that

$$\lambda_j \lesssim \mathcal{H}(Z_j) \lesssim \lambda_j$$

For the real analytic case, this conjecture was proven by Donnelly and Fefferman [8].

For the smooth case, the conjecture remains open, except for the lower bound in two dimensions. This was established by Brüning [4]. See also the work of Savo [22], who studied the constant in Brüning's inequality. Until recently, the best known lower bound for higher dimensions was established by Han and Lin [13], who proved that there is a constant $c > 0$ such that

$$c^{-\lambda_j} \leq \mathcal{H}(Z_j)$$

In two dimensions, Donnelly and Fefferman [10] and Dong [7] proved that

$$\mathcal{H}(Z_j) \lesssim \lambda_j^{3/2}$$

The best known upper bound for higher dimensions was proven by Hardt and Simon [14], who showed that there is a constant $c > 0$ such that

$$\mathcal{H}(Z_j) \lesssim \lambda_j^{c\lambda_j}$$

Recently, there has been significant improvement for lower bounds in the case of a smooth manifold without boundary. Sogge and Zelditch [30] proved that

$$\lambda_j^{\frac{7-3n}{4}} \lesssim \mathcal{H}(Z_j)$$

This estimate was the first lower bound with a polynomial rate of decay for $n \geq 3$. This was improved by Colding and Minicozzi [6], who used a local argument to establish the following.

Theorem 1.1.4. *The nodal sets satisfy*

$$\lambda_j^{\frac{3-n}{2}} \lesssim \mathcal{H}(Z_j)$$

Mangoubi [19] used another local method to prove

$$\lambda_j^{3-n-\frac{1}{n}} \lesssim \mathcal{H}(Z_j)$$

Using global arguments, the bound in Theorem 1.1.4 was later obtained for $n \leq 5$ by Hezari and Wang [16], and then for all n by Hezari and Sogge [15]. Their proofs were based on the following theorem

Theorem 1.1.5. *The eigenfunctions satisfy*

$$\lambda_j^2 \int_M |\varphi_j| \, dV = 2 \int_{Z_j} |\nabla \varphi_j| \, dS$$

Here dV is the Riemannian volume measure and dS is the Riemannian surface measure on Z_j.

This identity had been proven by Sogge and Zelditch [30] and was inspired by earlier work of Dong [7]. Sogge and Zelditch [30] also showed that

$$\lambda_j^{-\frac{n-1}{4}} \lesssim \int_M |\varphi_j| \, dV \tag{1.2}$$

Hezari and Sogge [15] proved that

$$\int_{Z_j} |\nabla \varphi_j|^2 \, dS \lesssim \lambda_j^3 \tag{1.3}$$

and then used Theorem 1.1.5, (1.2), and (1.3) to obtain the bound in Theorem 1.1.4.

1.2 Manifolds with Boundary

We now assume that (M, g) is a Riemannian manifold with boundary. We also assume that the eigenfunctions φ_j satisfy Dirichlet boundary conditions, meaning φ_j vanishes on ∂M, or Neumann boundary conditions, meaning $\partial_\nu \varphi_j$ vanishes on ∂M. Here ν is the outward pointing unit normal vector on ∂M, and ∂_ν denotes the corresponding directional derivative.

1.2.1 $\mathbf{L^p}$ Norms

The presence of a boundary provides a significant difficulty, and eigenfunctions can be more concentrated near the boundary. Smith and Sogge [26] proved the following theorem.

Theorem 1.2.1. *On a two-dimensional manifold with boundary, for Dirichlet or Neumann boundary conditions, the eigenfunctions satisfy*

$$\|\varphi_j\|_{L^p(M)} \le C \lambda_j^{\alpha(p)} \tag{1.4}$$

where

$$\alpha(p) = \begin{cases} \frac{1}{3} - \frac{2}{3p} & \text{if } 2 \le p \le 8 \\ \frac{1}{2} - \frac{2}{p} & \text{if } 8 \le p \le \infty \end{cases}$$

On the two-dimensional flat unit disk $\{|x| \le 1\}$, there are whispering gallery modes which concentrate near the boundary. Grieser [12] showed that these eigenfunctions satisfy

$$\limsup_{j\to\infty} \lambda_j^{-\alpha(p)} \|\varphi_j\|_{L^p(M)} > 0 \tag{1.5}$$

for $2 \le p \le 8$. A natural problem is to determine the manifolds and eigenfunctions for which (1.5) holds.

The disk is an example of a manifold with strictly geodesically convex boundary. To define this property, let (Ω, g) be an n-dimensional Riemannian manifold without boundary, and let M be an n-dimensional submanifold with boundary. Let N be the complement of the interior of M. To say that the boundary of M is strictly geodesically convex at a point x means that every geodesic going through x tangent to ∂M has exactly first order contact with ∂M and stays in N near x. In this situation, we also say that the boundary of N is strictly geodesically concave at x. For example, if (Ω, g) is any Riemannian manifold, then a closed geodesic ball of sufficiently small radius has strictly geodesically convex boundary, and its complement has strictly geodesically concave boundary. In particular, on the round sphere $\mathbb{S}^2$, any closed geodesic ball with radius less than $\pi/2$ has strictly geodesically convex boundary, and any closed geodesic ball with radius between $\pi/2$ and π has strictly geodesically concave boundary.

If the boundary of a two-dimensional manifold is strictly geodesically convex at some point, then whispering gallery quasi-modes can be constructed. These are approximate eigenfunctions which satisfy (1.5) for $2 \le p \le 8$.

In contrast, if the boundary is strictly geodesically concave, then Theorem 1.4 can be significantly improved for $2 < p < 8$. The following theorem was proven by Grieser [12] and extended to higher dimensions by Smith and Sogge [25].

Theorem 1.2.2. *On a two-dimensional manifold with strictly geodesically concave boundary, for Dirichlet boundary conditions, there is a constant C such that*

$$\|\varphi_j\|_{L^p(M)} \leq C(1+\lambda_j)^{\delta(p)}$$

where

$$\delta(p) = \begin{cases} \frac{1}{4} - \frac{1}{2p} & \text{if } 2 \leq p \leq 6 \\ \frac{1}{2} - \frac{2}{p} & \text{if } 6 \leq p \leq \infty \end{cases}$$

The arguments of Grieser [12] and Smith and Sogge [25] require an understanding of solutions of the wave equation. In particular, singularities of solutions of the wave equation propagate along broken geodesics. These are curves which are geodesic away from the boundary and reflect off the boundary according to the reflection law for g. Broken geodesics that intersect the boundary transversally do not provide a significant difficulty. The difficulty arises when broken geodesics intersect the boundary tangentially. For a manifold with strictly geodescially concave boundary, these broken geodesics must intersect the boundary with exactly first order contact. These can be treated using the Melrose-Taylor parametrix [20]. Using this parametrix, we will prove the following, which extends Theorem 1.1.2. Let Π be the set of broken geodesic segments which have length one.

Theorem 1.2.3. *On a two-dimensional manifold with strictly geodesically concave boundary, for Dirichlet boundary conditions, the eigenfunctions satisfy*

$$\sup_{\gamma \in \Pi} \|\varphi_j\|_{L^p(\gamma)} \leq C(1+\lambda_j)^{\sigma(p)}$$

where

$$\sigma(p) = \begin{cases} \frac{1}{4} & \text{if } 2 \le p \le 4 \\ \frac{1}{2} - \frac{1}{p} & \text{if } 4 \le p \le \infty \end{cases}$$

A natural problem is to determine when Theorem 1.2.2 is sharp. We will prove the following, which is an analogue of Theorem 1.1.3 and connects Theorem 1.2.2 and Theorem 1.2.3.

Theorem 1.2.4. *On a two-dimensional manifold with strictly geodesically concave boundary, for Dirichlet boundary conditions, the following are equivalent:*

$$\limsup_{j\to\infty} \lambda_j^{-1/8} \|\varphi_j\|_{L^4(M)} > 0$$

$$\limsup_{j\to\infty} \sup_{\gamma\in\Pi} \lambda_j^{-1/4} \|\varphi_j\|_{L^2(\gamma)} > 0$$

Moreover, if γ is in Π and the eigenfunctions satisfy

$$\limsup_{j\to\infty} \lambda_j^{-1/4} \|\varphi_j\|_{L^2(\gamma)} > 0$$

then γ must be a segment of a periodic broken geodesic.

An open problem is to prove analogue of Theorem 1.1.2 for general manifolds with boundary. I expect that on a two-dimensional manifold with boundary, the eigenfunctions satisfy

$$\sup_{\gamma\in\Pi} \|\varphi_j\|_{L^p(\gamma)} \le C(1+\lambda_j)^{\beta(p)} \tag{1.6}$$

where

$$\beta(p) = \begin{cases} \frac{1}{3} & \text{if } 2 \le p \le 6 \\ \frac{1}{2} - \frac{1}{p} & \text{if } 6 \le p \le \infty \end{cases}$$

One approach to studying eigenfunctions on a manifold with boundary is to remove the boundary at the expense of losing regularity of the metric. If M is a Riemannian manifold with boundary, then we can attach two copies of M along the boundary ∂M. Then the Dirichlet and Neumann eigenfunctions as well as the metric on M can be extended to obtain eigenfunctions on the doubled manifold. However, in general, the extended metric will only be Lipschitz continuous. For a two-dimensional Lipschitz Riemannian manifold without boundary, Smith [23] proved that (1.4) holds for $2 \le p \le 6$ and $p = \infty$. Smith and Sogge [26] proved Theorem 1.2.1 for all $p \ge 2$, by looking at the singularities of the metric in the doubled manifold more precisely.

The techniques developed by Smith [23] may yield (1.6) for $2 \le p \le 4$. It may also be possible to prove (1.6) for a larger range of p by using the techniques of Smith and Sogge [26].

A possible analogue of Theorem 1.1.3 for two-dimensional manifolds with boundary might say that the following are equivalent:

$$\limsup_{j\to\infty} \lambda_j^{-1/6} \|\varphi_j\|_{L^4(M)} > 0$$

$$\limsup_{j\to\infty} \sup_{\gamma\in\Pi} \lambda_j^{-1/3} \|\varphi_j\|_{L^2(\gamma)} > 0$$

1.2.2 Nodal Sets

Define nodal sets

$$Z_j = \left\{ x \in M : \varphi_j(x) = 0, x \notin \partial M \right\}$$

Let $\mathcal{H}$ be the $(n-1)$-dimensional Hausdorff measure on M, where n is the dimension of M. For a real analytic Riemannian manifold with boundary, Donnelly and Fefferman [9] proved that, for Dirichlet or Neumann boundary conditions,

$$\lambda_j \lesssim \mathcal{H}(Z_j) \lesssim \lambda_j$$

For the smooth case, if $n = 2$, then the argument of Brüning [4] yields

$$\lambda_j \lesssim \mathcal{H}(Z_j)$$

For higher dimensions, we will prove the following, which extends Theorem 1.1.4.

Theorem 1.2.5. *For Neumann boundary conditions,*

$$\lambda_j^{\frac{5-2n}{3}} \lesssim \mathcal{H}(Z_j)$$

If $n \leq 3$, then for Dirichlet boundary conditions,

$$\lambda_j^{\frac{5-2n}{3}} \lesssim \mathcal{H}(Z_j)$$

If the boundary is strictly geodesically concave and $n \leq 4$, then for Dirichlet boundary conditions,

$$\lambda_j^{\frac{3-n}{2}} \lesssim \mathcal{H}(Z_j)$$

The proof is based on the method of Hezari and Sogge [15] and Sogge and Zelditch [31]. In particular, we will prove the following, which extends Theorem 1.1.5.

Theorem 1.2.6. *For Dirichlet or Neumann boundary conditions,*

$$\lambda_j^2 \int_M |\varphi_j| \, dV = \int_{\partial M} |\partial_\nu \varphi_j| \, dS + 2 \int_{Z_j} |\nabla \varphi_j| \, dS$$

where dV is the Riemannian volume measure and dS is the Riemannian surface measure on Z_j.

For Neumann boundary conditions, the first term on the right side is zero, and this identity is the same as the one in Theorem 1.1.5. For Dirichlet boundary conditions, the integral over ∂M is an additional obstacle and causes the argument to break down in higher dimensions.

Chapter 2

Concentration near a concave boundary

2.1 Introduction

In this chapter, we assume that (M, g) is two-dimensional with smooth strictly geodesically concave boundary. We assume the eigenfunctions φ_j satisfy Dirichlet boundary conditions. This chapter concerns the concentration of the eigenfunctions.

Recall that, for $p \geq 2$, Grieser [12] proved that

$$\|\varphi_j\|_{L^p(M)} \lesssim \lambda_j^{\delta(p)} \tag{2.1}$$

where

$$\delta(p) = \begin{cases} \frac{1}{4} - \frac{1}{2p} & \text{if } 2 \leq p \leq 6 \\ \frac{1}{2} - \frac{2}{p} & \text{if } 6 \leq p \leq \infty \end{cases}$$

One purpose of this chapter is to prove Theorem 1.2.3 which says that

$$\sup_{\gamma \in \Pi} \|\varphi_j\|_{L^p(\gamma)} \lesssim \lambda_j^{\sigma(p)}$$

where

$$\sigma(p) = \begin{cases} \frac{1}{4} & \text{if } 2 \leq p \leq 4 \\ \frac{1}{2} - \frac{1}{p} & \text{if } 4 \leq p \leq \infty \end{cases}$$

Note that in proving Theorem 1.2.3, it suffices to prove the case $p = 4$. The case $p = \infty$ follows from (2.1) since the eigenfunctions are continuous. Then interpolation will yield the cases $4 < p < \infty$, and Hölder's inequality will yield the cases $2 \leq p < 4$. Another upper bound for the L^2 norms over broken geodesics is given by the following corollary.

Corollary 2.1.1. *If γ is a broken geodesic of unit length, $p \geq 2$, and $\varepsilon > 0$, then there is a constant C_ε such that*

$$\|\varphi_j\|_{L^2(\gamma)} \leq C_\varepsilon \lambda_j^{\frac{1}{2p}} \|\varphi_j\|_{L^p(M)} + \varepsilon \lambda_j^{\frac{1}{4}}$$

For two-dimensional manifolds without boundary, Bourgain [3] gave a stronger version of this inequality, without the second term in the right side. Later, we will use his result and Theorem 1.2.3 to prove Corollary 2.1.1.

For γ in Π, define the neighborhoods

$$\mathcal{N}_j(\gamma) = \left\{ x \in M : d_g(x, \gamma) < \lambda_j^{-\frac{1}{2}} \right\}$$

Here d_g is the Riemannian distance function corresponding to g. We will prove the following theorem.

Theorem 2.1.2. *Assume Λ is large and fix $\varepsilon > 0$. There is a constant C_ε such that, for $\lambda_j \geq \Lambda$, the eigenfunctions satisfy*

$$\|\varphi_j\|_{L^4(M)}^4 \leq C_\varepsilon \lambda_j^{\frac{1}{2}} \sup_{\gamma \in \Pi} \|\varphi_j\|_{L^2(\mathcal{N}_j(\gamma))}^2 + \varepsilon \lambda_j^{\frac{1}{2}} + C$$

This extends a result of Sogge [29], who considered compact two-dimensional Riemannian manifolds without boundary. Corollary 2.1.1 and Theorem 2.1.2 imply the following result.

Corollary 2.1.3. *Let φ_{j_k} be a subsequence of eigenfunctions and let $2 < p < 6$. The following are equivalent:*

$$\limsup_{k\to\infty} \lambda_{j_k}^{-\delta(p)} \|\varphi_{j_k}\|_{L^p(M)} > 0 \tag{2.2}$$

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \|\varphi_{j_k}\|_{L^2(\mathcal{N}_{j_k}(\gamma))} > 0 \tag{2.3}$$

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \lambda_{j_k}^{-\frac{1}{4}} \|\varphi_{j_k}\|_{L^2(\gamma)} > 0 \tag{2.4}$$

If (2.2) holds for some p in the range $2 < p < 6$, then it holds for all such p, by (2.1) and interpolation. So to prove Corollary 2.1.3, it suffices to consider the case $p = 4$. In this case, (2.2) implies (2.3) by Theorem 2.1.2. It is clear that (2.3) implies (2.4), and (2.4) implies (2.2) by Corollary 2.1.1.

A related problem is to determine when a subsequence φ_{j_k} of eigenfunctions is quantum ergodic. To define this condition, let S^*M be the unit cosphere bundle. The eigenfunctions φ_j induce distributions U_j on S^*M defined by

$$U_j(a) = \Big\langle Op(a)\varphi_j, \varphi_j \Big\rangle$$

where $Op(a)$ is the pseudodifferential operator, for a fixed quantization, with complete symbol a. To say a subsequence φ_{j_k} of eigenfunctions is quantum ergodic means that the weak* limit of the distributions U_{j_k} is the normalized Liouville measure on S^*M. This definition is independent of the choice of quantization. In particular, this implies that the probability measures $|\varphi_{j_k}|^2\, dx$ converge weakly to the normalized Riemannian measure. In this case (2.3) cannot hold, so Corollary 2.1.3 implies the following.

Corollary 2.1.4. *Assume a subsequence φ_{j_k} of eigenfunctions is quantum ergodic. Then*

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \lambda_{j_k}^{-\frac{1}{4}} \|\varphi_{j_k}\|_{L^2(\gamma)} = 0$$

and for $2 < p < 6$,

$$\limsup_{k\to\infty} \lambda_{j_k}^{-\delta(p)} \|\varphi_{j_k}\|_{L^p(M)} = 0$$

Zelditch-Zworski [34] proved that if the billiard flow is ergodic, then there is a subsequence φ_{j_k} of density one which is quantum ergodic. A subsequence is of density one when

$$\lim_{\lambda\to\infty} \frac{\#\{\lambda_{j_k} \le \lambda\}}{\#\{\lambda_j \le \lambda\}} = 1$$

Their result demonstrates that the global dynamics of the billiard flow influence the concentration of eigenfunctions. Our last result also demonstrates this.

Proposition 2.1.5. *Fix a broken geodesic γ in M of unit length which is not contained in a periodic broken geodesic. Then*

$$\limsup_{j\to\infty} \lambda_j^{-\frac{1}{4}} \|\varphi_j\|_{L^2(\gamma)} = 0$$

That is, if Theorem 1.1 is sharp for a fixed broken geodesic, then it must be a segment of a periodic broken geodesic.

2.2 Reductions

The beginning of the proofs of Theorem 1.2.3 and Theorem 2.1.2 are similar so we begin both in this section. We can assume that M is a subset of a boundaryless compact two-dimensional Riemannian manifold (M_0, g). Let d_0 be the Riemannian distance function on M_0 corresponding to g and let Δ_0 be the Laplacian on M_0. Let $P_0 = \sqrt{-\Delta_0}$ and $P = \sqrt{-\Delta}$, where $\sqrt{-\Delta}$ is defined with respect to Dirichlet boundary conditions. For the rest of this chapter, we will assume $\lambda \geq 1$.

Fix a small $\delta > 0$, and choose a $\chi \in \mathcal{S}(\mathbb{R})$ with $\chi(0) = 1$ and $\hat{\chi}$ supported on a closed interval contained strictly inside of $(\frac{1}{2}\delta, \delta)$. Define the translations $\chi_\lambda(s) = \chi(s - \lambda)$. We will use the operators $\chi_\lambda(P)$ and $\chi_\lambda(P_0)$. Notice

$$\chi_\lambda(\lambda) = 1$$

Define $\rho_\lambda(s) = \chi_\lambda(s) + \chi_\lambda(-s)$. For large λ, we have

$$1/2 \leq |\rho_\lambda(\lambda)|$$

Define the set

$$H_\delta = \Big\{ x \in M : d_g(x, \partial M) \leq \delta \Big\}$$

and let E_δ be the complement of H_δ in M. To prove Theorem 1.2.3, it suffices to prove that

$$\|\rho_\lambda(P) f\|_{L^4(\gamma \cap E_\delta)} + \|\chi_\lambda(P) f\|_{L^4(\gamma \cap H_\delta)} \lesssim \lambda^{1/4} \|f\|_{L^2(M)} \tag{2.5}$$

We have the following analogue.

Theorem 2.2.1. *If γ is a smooth curve on M_0 of unit length, then*

$$\|\rho_\lambda(P_0) f\|_{L^4(\gamma)} + \|\chi_\lambda(P_0) f\|_{L^4(\gamma)} \lesssim \lambda^{1/4} \|f\|_{L^2(M_0)}$$

Burq-Gérard-Tzvetkov proved this inequality for χ_λ, and the inequality for ρ_λ follows easily from the following lemma, which we will prove later.

Lemma 2.2.2. *The kernel of $\chi_\lambda(-P_0)$ is uniformly bounded, independent of λ.*

Let Π_0 be the set of all unit length geodesics in M_0. Fix $r \in (0,1)$. For $\gamma \in \Pi_0$, define the neighborhoods

$$\mathcal{T}_\lambda(\gamma) = \left\{ x \in M_0 : d_0(x,\gamma) < r\lambda^{-1/2} \right\}$$

There is a constant Λ such that for any geodesic $\gamma \in \Pi_0$, there exists a fixed finite number of broken geodesics $\gamma_i \in \Pi$ such that $\mathcal{T}_{\lambda_j}(\gamma) \cap M \subset \bigcup \mathcal{N}_j(\gamma_i)$ for $\lambda_j \geq \Lambda$. By (2.1), we know $\|\varphi_j\|_{L^4(M)} \lesssim \lambda_j^{1/8}$, so to prove Theorem 2.1.2 it suffices to show that

$$\begin{aligned} \int_{E_\delta} |\rho_\lambda(P)f(x)|^2 |g(x)|^2\,dx + \int_{H_\delta} |\chi_\lambda(P)f(x)|^2 |g(x)|^2\,dx \leq \\ C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(M)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M)}\|g\|^2_{L^4(M)} \\ + C\|f\|^2_{L^2(M)}\|g\|^2_{L^2(M)} \end{aligned} \tag{2.6}$$

We have the following analogue.

Theorem 2.2.3. *Fix $\varepsilon > 0$. There is a constant C_ε such that*

$$\begin{aligned} \int_{M_0} |\rho_\lambda(P_0)f(x)|^2 |g(x)|^2\,dx + \int_{M_0} |\chi_\lambda(P_0)f(x)|^2 |g(x)|^2\,dx \leq \\ C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(M_0)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M_0)}\|g\|^2_{L^4(M_0)} \\ + C\|f\|^2_{L^2(M_0)}\|g\|^2_{L^2(M_0)} \end{aligned}$$

For $r = 1$, Sogge [29] proved this inequality for χ_λ. Moreover, the same proof shows this holds for smaller values of r as well, and the inequality for ρ_λ follows easily from Lemma 2.2.2.

Define projection operators Π_j on $L^2(M)$ by $\Pi_j f = \langle f, e_j \rangle e_j$. For f in $L^2(M)$,

$$\begin{aligned}\chi_\lambda(P)f &= \sum_{j=0}^{\infty} \chi_\lambda(\lambda_j)\Pi_j f \\ &= (2\pi)^{-1} \int \hat{\chi}(t) e^{-it\lambda} \sum_{j=0}^{\infty} e^{it\lambda_j} \Pi_j f \, dt \\ &= (2\pi)^{-1} \int \hat{\chi}(t) e^{-it\lambda} e^{itP} f \, dt\end{aligned}$$

Likewise,

$$\chi_\lambda(-P)f = (2\pi)^{-1} \int \hat{\chi}(t) e^{-it\lambda} e^{-itP} f \, dt$$

which yields

$$\rho_\lambda(P)f = \pi^{-1} \int \hat{\chi}(t) e^{-it\lambda} \cos(tP) f \, dt$$

Similarly, for f in $L^2(M_0)$,

$$\chi_\lambda(P_0)f = (2\pi)^{-1} \int \hat{\chi}(t) e^{-it\lambda} e^{itP_0} f \, dt \tag{2.7}$$

and

$$\rho_\lambda(P_0)f = \pi^{-1} \int \hat{\chi}(t) e^{-it\lambda} \cos(tP_0) f \, dt$$

If t is in supp $\hat{\chi}$, then

$$\Big(\cos(tP)f\Big)\Big|_{E_\delta} = \Big(\cos(tP_0)f\Big)\Big|_{E_\delta}$$

which implies that

$$\Big(\rho_\lambda(P)f\Big)\Big|_{E_\delta} = \Big(\rho_\lambda(P_0)f\Big)\Big|_{E_\delta} \tag{2.8}$$

For a broken geodesic γ on M of unit length, Theorem 2.2.1 yields

$$\|\rho_\lambda(P)f\|_{L^4(\gamma \cap E_\delta)} \lesssim \lambda^{1/4} \|f\|_{L^2(M)}$$

So to prove (2.5), it remains to prove

$$\|\chi_\lambda(P)f\|_{L^4(\gamma\cap H_\delta)} \lesssim \lambda^{1/4}\|f\|_{L^2(M)} \tag{2.9}$$

Similarly, Theorem 2.2.3 yields

$$\int_{E_\delta} |\rho_\lambda(P)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M)}\|g\|^2_{L^4(M)} + C\|f\|^2_{L^2(M)}\|g\|^2_{L^2(M)}$$

So to prove (2.6), it remains to prove

$$\int_{H_\delta} |\chi_\lambda(P)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M)}\|g\|^2_{L^4(M)} + C\|f\|^2_{L^2(M)}\|g\|^2_{L^2(M)} \tag{2.10}$$

It is equivalent to show (2.9) and (2.10) with $\chi_\lambda(P)e^{it_0P}f$ in place of $\chi_\lambda(P)f$ for some fixed t_0, because

$$\|e^{-it_0P}f\|_{L^2(M)} = \|f\|_{L^2(M)}$$

Adapting (2.2) gives

$$\chi_\lambda(\sqrt{-\Delta_g})e^{it_0P}f = (2\pi)^{-1}\int \hat{\chi}(t)e^{-it\lambda}e^{i(t+t_0)P}f\,dt$$

Before proceeding, we prove Lemma 2.2.2.

Proof of Lemma 2.2.2. If δ is small, we can apply a parametrix as follows. See, for example, Theorem 4.1.2 in Sogge [28]. In appropriately chosen coordinate charts, the operator

$\chi_\lambda(-P_0)$ is equal, modulo smoothing operators, to an operator with kernel

$$\iint \hat{\chi}(t) e^{i[\varphi_0(x,y,\xi) - tp_0(y,\xi) - t\lambda]} q(t,x,y,\xi)\, dt d\xi$$

Here φ_0 is smooth, p_0 is the principal symbol of P_0, and q is a symbol of type $(1,0)$ and order zero. Since $p_0(y,\xi) \sim |\xi|$ and $\lambda \geq 1$,

$$\Big|\frac{\partial}{\partial t}\Big(\varphi_0(x,y,\xi) - tp_0(y,\xi) - t\lambda\Big)\Big| = |p_0(y,\xi) + \lambda| \gtrsim 1 + |\xi|$$

An integration by parts argument shows that for any positive integer N,

$$\Big|\int \hat{\chi}(t) e^{i[\varphi_0(x,y,\xi) - tp_0(y,\xi) - t\lambda]} q(t,x,y,\xi)\, dt\Big| \leq C_N (1 + |\xi|)^{-N}$$

So the kernel of $\chi_\lambda(-P_0)$ is uniformly bounded, independent of λ. $\square$

We reduce the problem by following Smith and Sogge [25]. For an operator A from M_0 to $\mathbb{R} \times M_0$, define associated operators $I_\lambda(A)$ by

$$I_\lambda(A) f(x) = \int \hat{\chi}(t) e^{-it\lambda} A f(t,x)\, dt$$

Here we can identify operators from M to $\mathbb{R} \times M$ with operators from M_0 to $\mathbb{R} \times M_0$ whose kernels are supported in $M \times (\mathbb{R} \times M)$. Let E_g be the operator given by

$$E_g f(t,x) = \Big(e^{i(t+t_0)P} f\Big)(x)$$

Then we have

$$I_\lambda(E_g) = 2\pi\, \chi_\lambda(P) \circ e^{it_0 P}$$

We can rewrite (2.9) and (2.10) as

$$\|I_\lambda(E_g)f\|_{L^4(\gamma\cap H_\delta)} \lesssim \lambda^{1/4}\|f\|_{L^2(M_0)}$$

and

$$\int_{H_\delta} |I_\lambda(E_g)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M_0)}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M_0)}\|g\|^2_{L^4(M)} + C\|f\|^2_{L^2(M_0)}\|g\|^2_{L^2(M)}$$

It suffices to write E_g as a finite sum of operators, where for each operator A in the sum, $I_\lambda(A)$ satisfies

$$\|I_\lambda(A)f\|_{L^4(\gamma\cap H_\delta)} \lesssim \lambda^{1/4}\|f\|_{L^2(M_0)} \tag{2.11}$$

and

$$\int_{H_\delta} |I_\lambda(A)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M_0)}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M_0)}\|g\|^2_{L^4(M)} + C\|f\|^2_{L^2(M_0)}\|g\|^2_{L^2(M)} \tag{2.12}$$

If an operator A has a kernel $K(t,x,y)$ which is uniformly bounded over the region

$$\Big\{(t,x,y) : t\in \operatorname{supp}\,\hat\chi, x\in H_\delta, y\in M_0\Big\}$$

then the kernel of $I_\lambda(A)$ is uniformly bounded, independent of λ. In this case the estimates (2.11) and (2.12) are trivial. In particular, this applies when A is smoothing.

Since ∂M is strictly geodesically concave, there is a $c_0 > 0$ such that if $t_0 > 0$ is small then any unit speed broken geodesic γ with $d(\gamma(0), \partial M) \leq c_0 t_0^2$ must satisfy

$$d(\gamma(t), \partial M) \geq c_0 t_0^2$$

for $\frac{1}{2}t_0 \leq t \leq 4t_0$. Now define Ω to be the set of points y in M such that there is a unit speed broken geodesic γ with $\gamma(0) = y$ and $d(\gamma(t_0 + t), \partial M) \leq 2\delta$ for some $t \in [-\delta, \delta]$. We assume that $2\delta < c_0 t_0^2$ and $\delta < \frac{1}{2}t_0$, which implies $d(\omega, \partial M) \geq c_0 t_0^2$.

If the kernel of E_g has a singularity at (t, x, y) then there is a broken geodesic of length $t + t_0$ with endpoints at x and y. So there is a smooth function α with support in Ω such that the kernel of the operator

$$f \to E_g(1-\alpha)f$$

is smooth over the region $\{(t, x, y) : t \in \text{supp } \hat{\chi}, x \in H_\delta, y \in M_0\}$. This reduces the problem to only considering f with support in Ω.

Define an operator E_0 from M_0 to $\mathbb{R} \times M_0$ by

$$E_0 f(t, x) = \left(e^{i(t+t_0)P_0} f\right)(x)$$

Let $\mathcal{R}$ be an operator from M_0 to $\mathbb{R} \times \partial M$ given by

$$\mathcal{R}f = (E_0 f)\big|_{\mathbb{R} \times \partial M}$$

Let $\Box_g = \partial_t^2 - \Delta_g$ and $\Box_0 = \partial_t^2 - \Delta_0$. Let W be the forward solution operator of the Dirichlet problem for $\Box_g$, mapping data on $\mathbb{R} \times \partial M$ which vanish for $t \leq -t_0$ to functions on $\mathbb{R} \times M$. That is, the equation $u = Wh$ means u solves

$$\begin{cases} \Box_g u & = 0 \\ u & = 0 \quad \text{for } t \leq -t_0 \\ u|_{\mathbb{R}\times\partial M} & = h \end{cases}$$

Recall we are assuming $\delta < \frac{1}{2}t_0$. Now over $[\frac{1}{2}\delta, \delta] \times M$, for f supported in Ω,

$$E_g f = E_0 f - W\mathcal{R}_+ f$$

where $\mathcal{R}_+$ is $\mathcal{R}$ smoothly cutoff to t in $[-t_0, t_0]$.

We can break up the cotangent bundle of $\mathbb{R} \times \partial M$ into three time-independent conic regions. These are the elliptic and hyperbolic regions where the Dirichlet problem is elliptic and hyperbolic, respectively, and the glancing region which is the region between them. We can break up the identity operator into a sum of time-independent conic pseudodifferential cutoffs as

$$I = \Pi_e + \Pi_h + \Pi_g$$

where Π_e and Π_h are essentially supported strictly inside the elliptic and hyperbolic regions, respectively, and Π_g is essentially supported in a small conic set about the glancing region. Then over $[\frac{1}{2}\delta, \delta] \times M$,

$$E_g f = E_0 f - W\Pi_e\mathcal{R}_+ f - W\Pi_h\mathcal{R}_+ f - W\Pi_g\mathcal{R}_+ f$$

The operator $I_\lambda(E_0)$ is equal to $2\pi\,\chi_\lambda(P_0) \circ e^{it_0 P_0}$, so it satisfies (2.11) and (2.12) by Theorems 2.2.1 and 2.2.3.

The projection of any characteristic direction of $\Box_g$ onto $T^*(\mathbb{R}\times\partial M)$ is contained in the hyperbolic or glancing regions, so $W\Pi_e\mathcal{R}_+$ is smoothing. This implies that $I_\lambda(W\Pi_e\mathcal{R}_+)$

satisfies (2.11) and (2.12).

On the essential support of Π_h, we can solve the forward Dirichlet problem for $\Box_g$ locally, modulo smoothing operators, on an open set in $\mathbb{R} \times M_0$ around $\mathbb{R} \times \partial M$. This gives a positive constant t_1 and an operator $\tilde{W}$ from $\mathbb{R} \times \partial M$ to $\mathbb{R} \times M_0$ such that for any v supported by t in $[-t_1, t_1]$, we have that $\Box_0 \tilde{W} v$ is smooth over $[-2t_1, 2t_1] \times M_0$ and $(W - \tilde{W})\Pi_h v$ is smooth over $\mathbb{R} \times M$.

We can assume $t_0 \leq t_1$ and define operators J_1 and J_2 by

$$J_1 f = \left(\tilde{W} \Pi_h \mathcal{R}_+ f\right)\Big|_{t=-t_0}$$

$$J_2 f = (-\Delta_0)^{-1/2} \left(\left(\partial_t \tilde{W} \Pi_h \mathcal{R}_+ f\right)\Big|_{t=-t_0} \right)$$

These are Fourier integral operators of order zero associated to the relation of reflection about ∂M.

Define operators C_0 and S_0 from M_0 to $\mathbb{R} \times M_0$ by

$$C_0 f(t,x) = \Big(\cos\big((t+t_0)P_0\big) f\Big)(x)$$

and

$$S_0 f(t,x) = \Big(\sin\big((t+t_0)P_0\big) f\Big)(x)$$

We can write $W \Pi_h \mathcal{R}_+ f$, modulo smoothing operators, as $C_0 J_1 f + S_0 J_2 f$. By the L^2 continuity of J_1 and J_2, it remains to show that $I_\lambda(C_0)$ and $I_\lambda(S_0)$ satisfy (2.11) and (2.12). Define an operator $\tilde{E}_0$ from M_0 to $\mathbb{R} \times M_0$ by

$$\tilde{E}_0 f(t,x) = \left(e^{-i(t+t_0)P_0} f\right)(x)$$

Since $I_\lambda(E_0)$ satisfies (2.11) and (2.12), it suffices to show that the same is true for $I_\lambda(\tilde{E}_0)$.

This follows from Lemma 2.2.2, completing the argument for the term $W\Pi_h\mathcal{R}_+f$.

Now we break up Π_g into a finite sum of pseudodifferential cutoffs, each essentially supported in a suitably small conic neighborhood of a glancing ray. This breaks up $W\Pi_g\mathcal{R}_+f$ into a finite sum and the Melrose-Taylor parametrix [20] can be applied to each term. We will use coordinates for M_0, chosen so that M is given by $x_2 > 0$. Then each term in this sum can be written, modulo smoothing operators, in the form GKf, where K is a Fourier integral operator of order zero, compactly supported on both sides, and G is an operator from $\mathbb{R}^2$ to $\mathbb{R}^3$ with kernel

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi}\Big(A_+\big(\zeta(x,\xi)\big)a(x,\xi)+A_+'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)\frac{Ai}{A_+}\big(\zeta_0(\xi)\big)\,d\xi$$

The functions a and b are symbols of type $(1,0)$ and order $1/6$ and $-1/6$, respectively, and both are supported by x in a small ball about the origin and by ξ is in a small conic neighborhood of the ξ_1-axis. Also Ai is the Airy function, and A_+ is given by $A_+(z) = Ai(e^{-\frac{2}{3}\pi i}z)$. The function ζ_0 is defined by $\zeta_0(\xi) = -\xi_1^{-1/3}\xi_2$, and the phases θ and ζ are real, smooth, and homogeneous in ξ of degree 1 and $2/3$, respectively, with

$$\zeta\big((x_1,0),\xi\big) = \zeta_0(\xi) \quad \text{and} \quad \frac{\partial\zeta}{\partial x_2}\big((x_1,0),\xi\big) < 0 \tag{2.13}$$

Let $\langle\ ,\ \rangle_x$ be the inner product given by g. In the region $\zeta(x,\xi)\le 0$, the functions θ and ζ satisfy

$$\begin{cases} \xi_1^2 - \langle d_x\theta, d_x\theta\rangle_x + \zeta\langle d_x\zeta, d_x\zeta\rangle_x = 0 \\ \langle d_x\theta, d_x\zeta\rangle_x = 0 \end{cases} \tag{2.14}$$

Also, θ and ζ satisfy these equations to infinite order at $x_2 = 0$ in the region $\zeta(x,\xi) > 0$.

Fix a small $r > 0$ and define the set

$$S_r = \Big\{ x \in \mathbb{R}^2 : |x| \leq r, x_2 \geq 0 \Big\}$$

We identify S_r with a subset of M. For an operator A from $\mathbb{R}^2$ to $\mathbb{R}^3$, define associated operators $I_\lambda(A)$ by

$$I_\lambda(A)f(x) = \int \hat{\chi}(t) e^{-it\lambda} Af(t,x)\, dt$$

By the L^2 continuity of K it suffices to show that $I_\lambda(G)$ has the following properties. For a broken geodesic γ in S_r of unit length and for f with fixed compact support, we need to show that

$$\|I_\lambda(G)f\|_{L^4(\gamma)} \lesssim \lambda^{1/4} \|f\|_{L^2(\mathbb{R}^2)}$$

We also need to show that for any $\varepsilon > 0$, there is a constant C_ε such that for f with fixed compact support,

$$\int_{S_r} |I_\lambda(G)f(x)|^2 |g(x)|^2\, dx \leq C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma \in \Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$
$$+ \ \varepsilon \lambda^{1/4} \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^4(\mathbb{R}^2)} + \ C \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^2(\mathbb{R}^2)}$$

It suffices to write G as a finite sum of operators, where for each operator A in the sum and for f with fixed compact support, $I_\lambda(A)$ satisfies

$$\|I_\lambda(A)f\|_{L^4(\gamma)} \lesssim \lambda^{1/4} \|f\|_{L^2(\mathbb{R}^2)} \tag{2.15}$$

and

$$\int_{S_r} |I_\lambda(A)f(x)|^2 |g(x)|^2\,dx \le C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)} \tag{2.16}$$

If an operator A has a kernel $K(t,x,y)$ which is uniformly bounded over compact subsets of

$$\Big\{(t,x,y) : t \in \text{supp }\hat{\chi}, x\in S_r, y\in\mathbb{R}^2\Big\}$$

then the kernel of $I_\lambda(A)$ is uniformly bounded, independent of λ, over compact subsets of $S_r\times\mathbb{R}^2$. In this case the estimates (2.15) and (2.16) are trivial. In particular, this applies when A is smoothing.

Let ρ be a smooth function with $\rho(s) = 0$ for $s \ge -1$ and $\rho(s) = 1$ for $s \le -2$. Following Zworski [35], we break up G into $G_m + G_d$, where the kernel of G_m is

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi}\Big((\rho A_+)\big(\zeta(x,\xi)\big)a(x,\xi) + (\rho A_+)'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)\frac{Ai}{A_+}\big(\zeta_0(\xi)\big)\,d\xi$$

and the kernel of G_d is

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi} q(x,\xi)\,d\xi$$

Here we have

$$q(x,\xi) = \Big(\big((1-\rho)A_+\big)\big(\zeta(x,\xi)\big)a(x,\xi) + \big((1-\rho)A_+\big)'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)\frac{Ai}{A_+}\big(\zeta_0(\xi)\big) \tag{2.17}$$

We will refer to G_m as the main term and to G_d as the diffractive term.

Define an operator $\tilde{G}_m$ with kernel

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi}\Big((\rho A_+)\big(\zeta(x,\xi)\big)a(x,\xi) + (\rho A_+)'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)\,d\xi$$

Then to control $I_\lambda(G_m)$, it suffices to show that $I_\lambda(\tilde{G}_m)$ satisfies (2.15) and (2.16), because

$$|\frac{Ai}{A_+}(s)| \leq 2 \quad \text{for } s \in \mathbb{R}$$

By stationary phase,

$$\widehat{(\rho A_+)}(s) = 2\pi\, e^{i\frac{1}{3}s^3}\Psi_+(s)$$

where Ψ_+ is smooth and satisfies

$$\Big|\frac{d^k}{ds^k}\Psi_+(s)\Big| \leq C_k$$

Applying the Fourier inversion formula and changing variables gives

$$(\rho A_+)(\zeta) = \int e^{i(s\xi_1^{-2/3}\zeta + \frac{1}{3}s^3\xi_1^{-2})}\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}s)ds$$

Similarly,

$$(\rho A_+)'(\zeta) = \int e^{i(s\xi_1^{-2/3}\zeta + \frac{1}{3}s^3\xi_1^{-2})}s\xi_1^{-4/3}\Psi_+(\xi_1^{-2/3}s)ds$$

So the kernel of $\tilde{G}_m$ is

$$\iint e^{i[\theta(x,\xi)+t\xi_1+s\xi_1^{-2/3}\zeta(x,\xi)+\frac{1}{3}s^3\xi_1^{-2}-y\cdot\xi]}\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}s)\Big(a(x,\xi)+s\xi_1^{-2/3}b(x,\xi)\Big)\,dsd\xi$$

Here the symbol

$$\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}s)\Big(a(x,\xi)+s\xi_1^{-2/3}b(x,\xi)\Big)$$

is of type $(2/3, 1/3)$ and order $-1/2$ on $\mathbb{R}^2_x \times \mathbb{R}^3_{s,\xi}$. Let ψ_0 be the function

$$\psi_0(x,t,\xi,s) = \theta(x,\xi) + t\xi_1 + s\xi_1^{-2/3}\zeta(x,\xi) + \frac{1}{3}s^3\xi_1^{-2}$$

We need to prove the following.

Lemma 2.2.4. *Fix $B \in S^{-1/2}_{2/3,1/3}(\mathbb{R}^2_x \times \mathbb{R}^3_{s,\xi})$ supported by x in a small neighborhood of the origin and ξ in a small conic neighborhood of the ξ_1-axis. Define an operator V_B with kernel*

$$\iint e^{i\psi_0(x,t,\xi,s)-iy\cdot\xi} B(x,\xi,s)\, dsd\xi$$

Then for any broken geodesic γ in S_r of unit length and for f with fixed compact support, the operators $I_\lambda(V_B)$ satisfy

$$\|I_\lambda(V_B)f\|_{L^4(\gamma)} \lesssim \lambda^{1/4}\|f\|_{L^2(\mathbb{R}^2)} \tag{2.18}$$

Also for any $\varepsilon > 0$ and for f with fixed compact support, there is a constant C_ε such that the operators $I_\lambda(V_B)$ satisfy

$$\int_{S_r} |I_\lambda(V_B)f(x)|^2|g(x)|^2\, dx \le C_\varepsilon \lambda^{1/2}\|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)} \tag{2.19}$$

We have seen that the estimates for the main term will follow from Lemma 2.2.4. Before proving Lemma 2.2.4, we will show that it also implies the estimates for the diffractive term. First, we will show that for x in S_r and for ξ in a small conic neighborhood of the ξ_1-axis, the symbol $q(x,\xi)$ defined by (2.17) can be written as

$$q(x,\xi) = h\big(x,\xi,\zeta(x,\xi)\big) \tag{2.20}$$

where

$$\left|\partial^\alpha_\xi \partial^j_\zeta \partial^k_{x_1}\partial^\ell_{x_2} h(x,\xi_1,\zeta)\right| \le C_{\alpha,j,k,\ell}\, \xi_1^{1/6-|\alpha|+2\ell/3} e^{-cx_2^{3/2}\xi_1 - \frac{1}{2}|\zeta|^{3/2}}$$

for some $c > 0$. Fix $\varepsilon > 0$. Then

$$\left|\partial_\zeta^k\big((1-\rho)A_+\big)(\zeta)\right| \leq C_{\varepsilon,k}\, e^{(\frac{2}{3}+\varepsilon)|\zeta|^{3/2}}$$

If ε is small, then it suffices to show that, in the region $\zeta(x,\xi) \geq -2$,

$$\frac{Ai}{A_+}\big(\zeta_0(\xi)\big) = H\big(x, \xi_1, \zeta(x,\xi)\big)$$

where

$$\left|\partial_{\xi_1}^m \partial_\zeta^j \partial_{x_1}^k \partial_{x_2}^\ell H(x,\xi_1,\zeta)\right| \leq C_{m,j,k,\ell}\, \xi_1^{-m+2\ell/3} e^{-cx_2^{3/2}\xi_1 - (\frac{4}{3}-\varepsilon)|\zeta|^{3/2}} \tag{2.21}$$

By (2.13), there is a $c > 0$ such that

$$\zeta_0(\xi) \geq \zeta(x,\xi) + cx_2\xi_1^{2/3}$$

In the region $\zeta(x,\xi) \geq -2$, the asymptotics of the Airy functions now yield

$$\left|\left(\frac{Ai}{A_+}\right)^{(m)}\big(\zeta_0(\xi)\big)\right| \leq C_{\varepsilon,m} e^{-cx_2^{3/2}\xi_1 - (\frac{4}{3}-\varepsilon)|\zeta(x,\xi)|^{3/2}} \tag{2.22}$$

Define a new variable

$$\tau(x,\xi) = \xi_1^{1/3}\zeta(x,\xi)$$

When $x_2 = 0$, we have $\tau = -\xi_2$. It follows that we can write $\xi_2 = \sigma(x,\xi_1,\tau)$, where σ is homogeneous of degree 1 in (ξ_1,τ). Now we define

$$H(x,\xi_1,\zeta) = \frac{Ai}{A_+}\big(-\xi_1^{-1/3}\sigma(x,\xi_1,\xi_1^{1/3}\zeta)\big)$$

To prove (2.21) it suffices to show that

$$\left|\partial_{\xi_1}^m \partial_\tau^j \partial_{x_1}^k \partial_{x_2}^\ell \frac{Ai}{A_+}\big(-\xi_1^{-1/3}\sigma(x,\xi_1,\tau)\big)\right| \le C_{m,j,k,\ell}\, \xi_1^{-m-j+2\ell/3} e^{-cx_2^{3/2}\xi_1-(\frac{4}{3}-\varepsilon)|\tau|^{3/2}\xi_1^{-1/2}} \tag{2.23}$$

If $x_2=\tau=0$, then $\sigma(x,\xi_1,\tau)=0$. So the homogeneity of σ implies that

$$\left|\partial_{\xi_1}^m \partial_\tau^j \partial_{x_1}^k \big(-\xi_1^{-1/3}\sigma(x,\xi_1,\tau)\big)\right| \le C_{m,j,k}(x_2\xi_1^{2/3}+\xi_1^{-1/3}|\tau|)\xi_1^{-m-j}$$

Together with (2.22), this implies (2.23) when $\ell=0$. It also follows for other values of ℓ because differentiating with respect to x_2 in (2.23) is similar to multiplying by a symbol of type $(1,0)$ and order $2/3$. Then (2.21) follows.

Now we can write the Fourier transform of $h(x,\xi,\zeta)$ in the ζ-variable as

$$\int e^{-is\zeta} q_0(x,\xi,\zeta)\,d\zeta = 2\pi\, e^{i\frac{1}{3}s^3} w(x,\xi,s)$$

where, for any $N>0$,

$$\left|\partial_\xi^\alpha \partial_s^j \partial_{x_1}^k \partial_{x_2}^\ell w(x,\xi,s)\right| \le C_{\alpha,j,k,\ell}\, \xi_1^{1/6-|\alpha|+2\ell/3} e^{-cx_2^{3/2}\xi_1}(1+s)^{-N}$$

Applying the Fourier inversion formula and changing variables gives

$$q_0(x,\xi,\zeta) = \int e^{i(s\xi_1^{-2/3}\zeta+\frac{1}{3}s^3\xi_1^{-2})}\xi_1^{-2/3} w(x,\xi,\xi_1^{-2/3}s)\,ds$$

Now we can write the kernel of G_d as

$$\iint e^{i\psi_0(x,t,\xi,s)-iy\cdot\xi} c(x,\xi,s)\,ds d\xi$$

where c is supported by x in a small ball and by ξ in a small conic neighborhood of the ξ_1 axis and satisfies

$$\left|\partial_\xi^\alpha \partial_s^j \partial_{x_1}^k \partial_{x_2}^\ell c(x,\xi,s)\right| \le C_{\alpha,j,k,\ell}\, \xi_1^{-1/2-|\alpha|-2j/3+2\ell/3} e^{-cx_2^{3/2}\xi_1}(1+\xi_1^{-2/3}s)^{-N}$$

for any $N > 0$. In particular,

$$x_2^j \partial_{x_2}^k c(x,\xi,s) \in S_{2/3,1/3}^{-1/2+2(k-j)/3}(\mathbb{R}_{x_1} \times \mathbb{R}^3_{\xi,s})$$

uniformly over x_2.

Let v be in $C_0^\infty(\mathbb{R}^2)$ have small support and satisfy $c(x,\xi,s) = v(x)c(x,\xi,s)$. Then we have

$$c(x,\xi,s) = v(x)c(x_1,0,\xi,s) + \int_0^{x_2} v(x)\partial_{x_2} c(x_1,\sigma,\xi,s)\, d\sigma$$

So we can write $G_d = A_d + B_d$ where the kernel of A_d is

$$\iint e^{i\psi_0(x,t,\xi,s)-iy\cdot\xi} v(x)c(x_1,0,\xi,s)\, ds d\xi$$

The symbol $v(x)c(x_1,0,\xi,s)$ is of type $(2/3,1/3)$ and order $-1/2$. So $I_\lambda(A_d)$ satisfies (2.15) and (2.16) by Lemma 2.2.4.

The kernel of $I_\lambda(B_d)$ is

$$\int_0^{x_2} \iiint \hat{\chi}(t) e^{-it\lambda+i\psi_0(x,t,\xi,s)-iy\cdot\xi} v(x)\partial_{x_2} c(x_1,\sigma,\xi,s)\, ds d\xi dt d\sigma$$

Let β be a smooth function supported in $[1/3,3]$ with $\beta = 1$ on $[1/2,2]$. Define operators B_λ with kernels

$$\int_0^{x_2} \iint e^{i\psi_0(x,t,\xi,s)-iy\cdot\xi} \beta\Big(\frac{\xi_1}{\lambda}\Big) v(x)\partial_{x_2} c(x_1,\sigma,\xi,s)\, ds d\xi d\sigma$$

The kernel of $I_\lambda(B_\lambda)$ is

$$\int_0^{x_2} \iiint \hat{\chi}(t) e^{-it\lambda + i\psi_0(x,t,\xi,s) - iy\cdot\xi} \beta\Big(\frac{\xi_1}{\lambda}\Big) v(x) \partial_{x_2} c(x_1, \sigma, \xi, s)\, ds d\xi dt d\sigma$$

Since $\partial_t \psi_0 = \xi_1$, an integration by parts argument shows that $I_\lambda(B_d)$ differs from $I_\lambda(B_\lambda)$ by an operator whose kernel is uniformly bounded, independent of λ. So it suffices to prove $I_\lambda(B_\lambda)$ satisfies (2.15) and (2.16). Let

$$P_{\sigma,\lambda}(x,\xi,s) = v(x)\beta\Big(\frac{\xi_1}{\lambda}\Big)\partial_{x_2} c(x_1,\sigma,\xi,s)$$

Then

$$|I_\lambda(B_\lambda) f| \le \int \Big| \iiiint \hat{\chi}(t) e^{-it\lambda + i\psi_0(x,t,\xi,s) - iy\cdot\xi} P_{\sigma,\lambda}(x,\xi,s) f(y)\, dy ds d\xi dt \Big| d\sigma$$

Define operators $B_{\sigma,\lambda}$ by

$$B_{\sigma,\lambda} f(t,x) = \iiint e^{i\psi_0(x,t,\xi,s) - iy\cdot\xi} \lambda^{-2/3}(1 + \lambda^{4/3}\sigma^2) P_{\sigma,\lambda}(x,\xi,s) f(y)\, dy ds d\xi$$

By Minkowski's integral inequality and Hölder's inequality,

$$\|I_\lambda(B_\lambda) f\|_{L^2(\gamma)} \lesssim \sup_\sigma \|I_\lambda(B_{\sigma,\lambda}) f\|_{L^2(\gamma)} \tag{2.24}$$

Also

$$\int_{S_r} |I_\lambda(B_\lambda) f(x)|^2 |g(x)|^2\, dx \lesssim \sup_\sigma \int_{S_r} |I_\lambda(B_{\sigma,\lambda}) f(x)|^2 |g(x)|^2\, dx \tag{2.25}$$

The amplitudes

$$\lambda^{-2/3}(1 + \lambda^{4/3}\sigma^2) P_{\sigma,\lambda}(x,\xi,s)$$

are symbols of type $(2/3, 1/3)$ and order $-1/2$ over $\mathbb{R}^2_x \times \mathbb{R}^3_{\xi,s}$, uniformly in σ and λ. By Lemma 2.2.4, the operators $I_\lambda(B_{\sigma,\lambda})$ satisfy (2.15) and (2.16), uniformly in σ. Then $I_\lambda(B_\lambda)$ satisfies (2.15) and (2.16) because of (2.24) and (2.25). So Lemma 2.2.4 will imply the estimates for the diffractive term.

To prove Lemma 2.2.4, note that V_B is a Fourier integral operator of type $(2/3, 1/3)$ and order zero associated to the canonical relation $\mathcal{C}$ given by

$$\mathcal{C} = \Big\{ \Big(x, t, \nabla_x \psi_0(x,t,\xi,s), \xi_1; \nabla_\xi \psi_0(x,t,\xi,s), \xi \Big) : \zeta(x,\xi) = -s^2 \xi_1^{-4/3} \}$$

Let $\mathcal{C}_0$ be the restriction of $\mathcal{C}$ to $t = 0$. It was shown in the proof of Lemma A.2 of Smith-Sogge [24] that $\mathcal{C}_0$ is the graph of a canonical transformation.

The projection of $\mathcal{C}$ onto $T^*(\mathbb{R}^3_{x,t})$ is contained in the characteristic variety of $\Box_0$, because of (2.14). So the canonical relation $\mathcal{C} \circ \mathcal{C}_0^{-1}$ is the flowout, under the bicharacteristic flow of $\Box_0$, of a conical subset of the diagonal at $t = 0$. By the Lax construction, $\mathcal{C} \circ \mathcal{C}_0^{-1}$ can be parametrized by a phase function

$$\varphi(t, x, \xi) - y \cdot \xi$$

where φ satisfies

$$\varphi(0, x, \xi) = x \cdot \xi \quad \text{and} \quad \frac{\partial \varphi}{\partial t} = p_0\Big(x, \frac{\partial \varphi}{\partial x}\Big) \tag{2.26}$$

Here p_0 is the principal symbol of P_0, that is

$$p_0(x,\xi) = \sqrt{\sum g^{jk}(x) \xi_j \xi_k}$$

Since $\varphi(t,x,\xi) - y\cdot\xi$ parametrizes $\mathcal{C}\circ\mathcal{C}_0^{-1}$, it follows that for small t,

$$y = \varphi'_\xi(t,x,\xi) \quad \text{implies} \quad t = d_0(x,y) \tag{2.27}$$

Now let J_0 and K_0 be Fourier integral operators of order zero, compactly supported on both sides, associated to the canonical relations $\mathcal{C}_0^{-1}$ and $\mathcal{C}_0$, respectively, such that $V_B\circ J_0\circ K_0$ differs from V_B by a smoothing operator. To prove Lemma 2.2.4, we need to show that $I_\lambda(V_B\circ J_0\circ K_0)$ satisfies (2.15) and (2.16). By the L^2 continuity of K_0, it suffices to show instead that $I_\lambda(V_B\circ J_0)$ satisfies (2.15) and (2.16). Here $V_B\circ J_0$ is a Fourier integral operator of type $(2/3,1/3)$ and order zero, associated to the canonical relation $\mathcal{C}\circ\mathcal{C}_0^{-1}$. So its kernel, modulo smoothing operators, is of the form

$$\int e^{i[\varphi(t,x,\xi)-y\cdot\xi]}a(t,x,\xi)\,d\xi$$

where a is a symbol of type $(2/3,1/3)$ and order zero on $\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi$. To show $I_\lambda(V_B\circ J_0)$ satisfies (2.16), it now suffices to prove the following lemma.

Lemma 2.2.5. *Fix $a\in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi)$, supported by x in a small neighborhood of S_r. Define an operator U_a by*

$$U_a f = \iint e^{i\varphi(t,x,\xi)-iy\cdot\xi}a(t,x,\xi)f(y)\,d\xi dy$$

For any $\varepsilon>0$ there is a constant C_ε such that for f with fixed compact support,

$$\int_{S_r}|I_\lambda(U_a)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(\mathbb{R}^2)}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)}$$

We will prove this lemma in Section 2.3. This will complete the proof of Theorem

2.1.2. The next lemma will show that $I_\lambda(V_B \circ J_0)$ satisfies (2.15).

Lemma 2.2.6. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$, supported by x in a small neighborhood of S_r. Define an operator U_a by*

$$U_a f = \iint e^{i\varphi(t,x,\xi)-iy\cdot\xi} a(t,x,\xi) f(y)\, d\xi dy$$

For any broken geodesic γ in S_r of unit length, and for f with fixed compact support,

$$\|I_\lambda(U_a) f\|_{L^4(\gamma)} \lesssim \lambda^{1/4} \|f\|_{L^2(\mathbb{R}^2)}$$

We will prove Lemma 2.2.6 in the Section 2.4. This will complete the proof of Theorem 1.2.3.

2.3 End of Proof of Theorem 2.1.2

To prove Theorem 2.1.2, it remains to prove Lemma 2.2.5. This will be a consequence of the following variant. To state it, let $\eta(x,y)$ be in $C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ be supported by x and y in a small neighborhood of S_r satisfying $\frac{1}{2}\delta \le d_0(x,y) \le \delta$. Also assume $\eta(x,y) = 1$ when x is in a small neighborhood of S_r and $d_0(x,y)$ is in an open neighborhood of the support of $\hat{\chi}$.

Lemma 2.3.1. *Fix $b \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,y} \times \mathbb{R}^2_\xi)$. Define an operator T_b by*

$$T_b f = \iint e^{i\varphi(t,x,\xi)-iy\cdot\xi} \eta(x,y) b(t,y,\xi) f(y)\, d\xi dy$$

For any $\varepsilon > 0$ there is a constant C_ε such that for f with fixed compact support,

$$\int_{S_r} |I_\lambda(T_b)f(x)|^2 |g(x)|^2 \, dx \leq C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma \in \Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$
$$+ \ \varepsilon \lambda^{1/4} \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^4(\mathbb{R}^2)} + \ C \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^2(\mathbb{R}^2)}$$

Using Lemma 2.3.1, we can prove Lemma 2.2.5.

Proof of Lemma 2.2.5. Fix a symbol $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$. We may assume that $\big(1 - \eta(x,y)\big)a(t,x,\xi)$ vanishes on a neighborhood of the set

$$\Sigma_0 = \Big\{(t,x,y,\xi) : t = d_0(x,y)\Big\}$$

We can make this assumption because $I_\lambda(U_a)$ only depends on t in the support of $\hat{\chi}$. The kernel of U_a is

$$\int e^{i\varphi(t,x,\xi) - iy\cdot\xi} a(t,x,\xi) \, d\xi$$

Define an operator D_a with kernel

$$\int e^{i\varphi(t,x,\xi) - iy\cdot\xi} \eta(x,y) a(t,x,\xi) \, d\xi$$

Define a set

$$\Sigma = \Big\{(t,x,y,\xi) : \varphi'_\xi(t,x,\xi) - y = 0\Big\}$$

By (2.27), the set Σ is contained in Σ_0. So the symbol $\big(1 - \eta(x,y)\big)a(t,x,\xi)$ vanishes on a neighborhood of Σ. By Proposition 1.2.4 of Hörmander [17], the difference between U_a and D_a is smoothing.

At $t = 0$, the determinant of the matrix $[\varphi''_{\xi_i x_j}]$ is 1. So if δ is small, then on the support of a we can apply the implicit function theorem to the equation

$$\varphi'_\xi(t,x,\xi) - y = 0$$

Specifically, we can use a partition of unity to break up a into a finite sum $a = \sum a_j$, so that there are functions $\psi_j(t,y,\xi)$ that are homogeneous in ξ of degree zero such that, on the support of a_j, the set Σ is given by

$$x = \psi_j(t,y,\xi)$$

Define $b_0 \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,y} \times \mathbb{R}^2_\xi)$ by

$$b_0(t,y,\xi) = \sum a_j\big(t, \psi_j(t,y,\xi), \xi\big)$$

Define an operator T_0 with kernel

$$\eta(x,y) \int e^{i\varphi(t,x,\xi) - iy\cdot\xi} b_0(t,y,\xi)\, d\xi$$

The difference between U_a and T_0 is an operator with kernel

$$\eta(x,y) \int e^{i\varphi(t,x,\xi) - iy\cdot\xi} \big(a(t,x,\xi) - b_0(t,y,\xi)\big)\, d\xi$$

The symbol $a(t,x,\xi) - b_0(t,y,\xi)$ vanishes on Σ, and the phase $\varphi(t,x,\xi) - y\cdot\xi$ is non-degenerate. It follows from Proposition 1.2.5 of Hörmander [17] that we can write this kernel in the form

$$\eta(x,y) \int e^{i\varphi(t,x,\xi) - iy\cdot\xi} a_0(t,x,y,\xi)\, d\xi$$

where a_0 is a symbol of order $-1/3$ and type $(2/3, 1/3)$.

Iterating this argument yields symbols $b_k(t,y,\xi)$ of order $-k/3$ and type $(2/3,1/3)$.

These symbols are such that if T_m is the operator with kernel

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}\sum_{k=0}^{m} b_k(t,y,\xi)\,d\xi$$

then the difference between U_a and T_m has a kernel of the form

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}a_m(t,x,y,\xi)\,d\xi$$

where a_m is a symbol of order $-(m+1)/3$ and type $(2/3,1/3)$. Let b be a symbol in $S^0_{2/3,1/3}(\mathbb{R}^3_{t,y}\times\mathbb{R}^2_\xi)$ with $b\sim\sum_{k=0}^{\infty}b_k$. Let T_b be the operator with kernel

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}b(t,y,\xi)\,d\xi$$

Then the difference between U_a and T_b is smoothing, so Lemma 2.2.5 will follow from Lemma 2.3.1. □

The following lemma gives a suitable description of the kernel of $I_\lambda(T_b)$. This description is sufficiently similar to the one used in Sogge [29], so that the same argument will yield Lemma 2.3.1.

Lemma 2.3.2. *Fix $b\in S^0_{2/3,1/3}(\mathbb{R}^3_{t,y}\times\mathbb{R}^2_\xi)$. The kernel of $I_\lambda(T_b)$ is of the form*

$$\lambda^{1/2}e^{-i\lambda d_0(x,y)}A_\lambda(x,y)+R_\lambda(x,y) \tag{2.28}$$

Here the functions R_λ are uniformly bounded, independent of λ, and the functions A_λ are in $C^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ satifying

$$|\partial_x^\alpha\partial_y^\beta A_\lambda|\le C_{\alpha,\beta}\lambda^{|\beta|/3}$$

Also the functions A_λ are supported by x and y in a small neighborhood of S_r satisfying $\frac{1}{2}\delta\le d_0(x,y)\le\delta$.

Proof. The kernel of $I_\lambda(T_b)$ is

$$\iint e^{i\varphi(t,x,\xi)-iy\cdot\xi-it\lambda}\hat{\chi}(t)\eta(x,y)b(t,y,\xi)\,d\xi dt$$

By (2.26),

$$\varphi(t,x,\xi)=x\cdot\xi+tp_0(x,\xi)+Q(t,x,\xi)$$

where Q is homogeneous of degree 1 in the ξ-variable. Also, for $k=0,1,2$ we have

$$|\partial_t^k\partial_\xi^\alpha Q|\le C_{k,\alpha}t^{2-k}|\xi|^{1-|\alpha|} \tag{2.29}$$

Let β be a smooth function with $\beta(\xi)=1$ when $|\xi|\in[C_0^{-1},C_0]$ and $\beta(\xi)=0$ when $|\xi|\notin[(2C_0)^{-1},2C_0]$, for some constant C_0. If C_0 is large and δ is small, then on the support of

$$\Big(1-\beta\Big(\frac{\xi}{\lambda}\Big)\Big)\hat{\chi}(t)\eta(x,y)b(t,y,\xi)$$

we have

$$\Big|\frac{\partial}{\partial t}\Big(\varphi(t,x,\xi)-y\cdot\xi-t\lambda\Big)\Big|\;\gtrsim\;p_0(x,\xi)+\lambda\;\gtrsim\;1+|\xi|$$

So for any positive integer N,

$$\int e^{i\varphi(t,x,\xi)-iy\cdot\xi-it\lambda}\Big(1-\beta\Big(\frac{\xi}{\lambda}\Big)\Big)\hat{\chi}(t)\eta(x,y)b(t,y,\xi)\,dt\le C_N(1+|\xi|)^{-N}$$

This implies that the difference between the kernel of $I_\lambda(T_b)$ and

$$\iint e^{i\varphi(t,x,\xi)-iy\cdot\xi-it\lambda}\beta\Big(\frac{\xi}{\lambda}\Big)\hat{\chi}(t)\eta(x,y)b(t,y,\xi)\,d\xi dt \tag{2.30}$$

is bounded uniformly in λ.

Now it suffices to show that (2.30) can be written as in (2.28). After changing variables

(2.30) becomes

$$\lambda^2 \iint e^{i\lambda\Phi(t,x,y,\xi)} p_\lambda(t,x,y,\xi)\, d\xi dt$$

where the phase is

$$\Phi(t,x,y,\xi) = \varphi(t,x,\xi) - y\cdot\xi - t$$

and the amplitude is

$$p_\lambda(t,x,y,\xi) = \beta(\xi)\hat{\chi}(t)\eta(x,y)b(t,y,\lambda\xi)$$

Here p_λ is smooth and compactly supported with

$$|\partial_t^k \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma p_\lambda| \le C_{k,\alpha,\beta,\gamma}\lambda^{(k+|\beta|+|\gamma|)/3}$$

To apply stationary phase, the Hessian of Φ, with respect to the (t,ξ)-variables, must be non-degenerate on the support of p_λ. First note that its determinant is homogeneous of degree -1 in the ξ-variable. We have

$$\Phi(t,x,y,\xi) = (x-y)\cdot\xi + tp_0(x,\xi) - t + Q(t,x,\xi)$$

We can compute explicitly the Hessian of

$$(x-y)\cdot\xi + tp_0(x,\xi) - t$$

with respect to the (t,ξ)-variables. Its determinant is

$$-\frac{t}{p_0(x,\xi)}\det g^{jk}$$

Now it follows from (2.29) that the determinant of the Hessian of Φ, with respect to the

(t, ξ)-variables, is

$$-\frac{t}{p_0(x,\xi)} \det g^{jk} + t^2 q(t,x,y,\xi)$$

where q is a smooth function, homogeneous of degree -1 in the ξ-variable. So if δ is small, then the Hessian of Φ, with respect to the (t, ξ)-variables, is non-degenerate on the support of p_λ.

The critical points of Φ, with respect to the (t, ξ)-variables, are the solutions of

$$\varphi'_\xi(t,x,\xi) = y \quad \text{and} \quad \varphi'_t(t,x,\xi) = 1$$

We can use the implicit function theorem at any critical point. By using a partition of unity and abusing notation, we can assume that there are smooth functions $t(x, y)$ and $\xi(x, y)$, such that if δ is small, then on the support of p_λ, the critical points are given by

$$\big(t(x,y), x, y, \xi(x,y)\big)$$

Because of (2.27), we have $t(x, y) = d_0(x, y)$. Applying Euler's homogeneity relation $\varphi = \varphi'_\xi \cdot \xi$ yields

$$\Phi\big(t(x,y), x, y, \xi(x,y)\big) = -t(x,y) = -d_0(x,y)$$

So Lemma 2.3.2 follows from the following stationary phase lemma. □

Lemma 2.3.3. *Consider the oscillatory integrals*

$$J_\lambda(x,y) = \int_{\mathbb{R}^3} e^{i\lambda\Psi(x,y,z)} q_\lambda(x,y,z)\, dz$$

where Ψ is a smooth real function and the amplitudes q_λ are smooth with fixed compact support and satisfy

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma q_\lambda| \le C_{\alpha,\beta,\gamma} \lambda^{(|\beta|+|\gamma|)/3}$$

Assume that on the support of the symbols q_λ, the Hessian of Ψ with respect to the z-variable is non-degenerate and the solutions of $\Psi'_z(x,y,z) = 0$ are given by $(x, y, z(x, y))$ where $z(x, y)$ is a smooth function. Then

$$\left|\partial_x^\alpha \partial_y^\beta \left(e^{-i\lambda\Psi(x,y,z(x,y))} J_\lambda(x,y)\right)\right| \le C_{\alpha,\beta}\lambda^{-3/2+|\beta|/3}$$

This lemma is similar to Corollary 1.1.8 in Sogge [28], which dealt with symbols q_λ with derivatives bounded independent of λ. Essentially the same proof as in Sogge [28] yields Lemma 2.3.3, and then Lemma 2.3.2 follows. We can now obtain Lemma 2.3.1 by using the argument in Sogge [29].

Argument from Sogge [29]

To finish the proof of Lemma 2.3.1 it suffices to show that for any $\varepsilon > 0$ there is a constant C_ε such that

$$(2.31)\quad \int_{S_r} \left|\lambda^{1/2} \int e^{-i\lambda d_0(x,y)} A_\lambda(x,y) f(y)\, dy\right|^2 |g(x)|^2\, dx \\ \le \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

By using a partition of unity and abusing notation, we can assume there are points x_0 and y_0 with x_0 in S_r and $\delta/2 \le d_0(x_0, y_0) \le \delta$ such that A_λ is supported by x in a small neighborhood $\mathcal{N}_x$ of x_0 and y in a small neighborhood $\mathcal{N}_y$ of y_0. In particular, we assume that $\mathcal{N}_x$ and $\mathcal{N}_y$ are, respectively, contained in $B(x_0, \delta/5)$ and $B(y_0, \delta/5)$, the geodesic balls of radius $\delta/5$ around x_0 and y_0, respectively.

We will work in Fermi normal coordinates $(\sigma, \tau)_F$ about γ_0, the geodesic going through x_0 which is orthogonal to the geodesic connecting x_0 and y_0. These coordinates are well defined on $B(x_0, 2\delta)$ if δ is small enough. These coordinates are such that γ_0 is given by a vertical line parallel to the τ-axis, and the geodesics which intersect γ_0 orthogonally are given by horizontal lines parallel to the σ-axis. Also x_0 lies on the negative σ-axis and y_0 on the positive σ-axis. Now it suffices to prove

$$\int \Big(\int_{S_r} \Big| \lambda^{1/2} \int e^{-i\lambda d_0(x,(\sigma,\tau)_F)} A_\lambda\big(x,(\sigma,\tau)_F\big) f(\sigma,\tau)\, d\tau \Big|^2 |g(x)|^2\, dx \Big)\, d\sigma \\ \leq \varepsilon \lambda^{1/4} \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^4(\mathbb{R}^2)} + C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

This will follow if we show

$$\text{(2.32)} \quad \int_{S_r} \Big| \lambda^{1/2} \int e^{-i\lambda d_0(x,(\sigma,\tau)_F)} A_\lambda\big(x,(\sigma,\tau)_F\big) h(\tau)\, d\tau \Big|^2 |g(x)|^2\, dx \\ \leq \varepsilon \lambda^{1/4} \|h\|^2_{L^2(\mathbb{R})} \|g\|^2_{L^4(\mathbb{R}^2)} + C_\varepsilon \lambda^{1/2} \|h\|^2_{L^2(\mathbb{R})} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

where C_ε is independent of σ. To simplify the notation, we will only prove this for a fixed value of σ, which we may take to be zero by relabeling the coordinates. The argument will also yield the uniformity in σ. Note that after relabeling, we can assume that the point $(0,0)_F$ is in $\mathcal{N}_y$. Then $x_0 = (-\sigma_0, 0)_F$ where $\sigma_0 > \delta/4$.

We take a smooth bump function $\eta \in C_0^\infty(\mathbb{R})$ supported in $[-1,1]$ and satisfying $\sum_{j\in\mathbb{Z}} \eta(\tau - j) = 1$. Define

$$\eta_{\lambda,j}(\tau) = \eta(\lambda^{1/2}\tau - j)$$

Let

$$z_j = z_j(\lambda, x, h) = \lambda^{1/2} \int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,j}(\tau) A_\lambda\big(x,(0,\tau)_F\big) h(\tau)\, d\tau$$

Then for $N = 1, 2, 3, \ldots,$

$$\begin{aligned}\Big|\sum_{j,k\in\mathbb{Z}} z_j z_k\Big| &\le \Big|\sum_{|j-k|>N} z_j z_k\Big| + \Big|\sum_{|j-k|\le N} z_j z_k\Big| \\ &\le \Big|\sum_{|j-k|>N} z_j z_k\Big| + \sum_{|j-k|\le N} \frac{1}{2}\Big(|z_j|^2 + |z_k|^2\Big) \\ &\le \Big|\sum_{|j-k|>N} z_j z_k\Big| + (2N+1)\sum_{j\in\mathbb{Z}} |z_j|^2\end{aligned}$$

This means that

$$\begin{aligned}(2.33)\quad &\Big|\lambda^{1/2}\int e^{-i\lambda d_0(x,(0,\tau)_F)} A_\lambda\big(x,(0,\tau)_F\big)h(\tau)\,d\tau\Big|^2 \\ &\le \Big|\lambda\iint e^{-i\lambda[d_0(x,(0,\tau)_F)+d_0(x,(0,\tau')_F)]} B_{N,\lambda}(x,\tau,\tau')h(\tau)h(\tau')\,d\tau d\tau'\Big| \\ &\quad + (2N+1)\sum_{j\in\mathbb{Z}}\lambda\Big|\int e^{-i\lambda d_0(x,(0,\tau)_F)}\eta_{\lambda,j}(\tau)A_\lambda\big(x,(0,\tau)_F\big)h(\tau)\,d\tau\Big|^2\end{aligned}$$

where

$$B_{N,\lambda}(x,\tau,\tau') = \sum_{|j-k|>N} \eta_{\lambda,j}(\tau)A_\lambda\big(x,(0,\tau)_F\big)\eta_{\lambda,k}(\tau')A_\lambda\big(x,(0,\tau')_F\big)$$

We will prove

$$\begin{aligned}(2.34)\quad &\Big\|\lambda\iint e^{-i\lambda[d_0(x,(0,\tau)_F)+d_0(x,(0,\tau')_F)]} B_{N,\lambda}(x,\tau,\tau')h(\tau)h(\tau')\,d\tau d\tau'\Big\|_{L^2_x(S_r)} \\ &\lesssim \lambda^{1/4}N^{-1/2}\|h\|^2_{L^2(\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}(2.35)\quad &\int_{S_r}\lambda\Big|\int e^{-i\lambda d_0(x,(0,\tau)_F)}\eta_{\lambda,j}(\tau)A_\lambda\big(x,(0,\tau)_F\big)H(\tau)\,d\tau\Big|^2|g(x)|^2\,dx \\ &\lesssim \lambda^{1/2}\|H\|^2_{L^2(\mathbb{R})}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}\end{aligned}$$

Let $\chi_{\lambda,j}$ be the characteristic function of supp $\eta_{\lambda,j}$. Then (2.35) will yield

$$(2.36)\quad \sum_{j\in\mathbb{Z}}\int_{S_r}\lambda\Big|\int e^{-i\lambda d_0(x,(0,\tau)_F)}\eta_{\lambda,j}(\tau)A_\lambda\big(x,(0,\tau)_F\big)h(\tau)\,d\tau\Big|^2\,dx$$

$$\lesssim \sum_{j\in\mathbb{Z}}\lambda^{1/2}\|h\chi_{\lambda,j}\|^2_{L^2(\mathbb{R})}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

$$\lesssim \lambda^{1/2}\|h\|^2_{L^2(\mathbb{R})}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

Then (2.33), (2.34), and (2.36) will yield (2.32). So it remains to prove (2.34) and (2.35).

The inequality (2.34) will be a consequence of the following lemma.

Lemma 2.3.4. *Let $B_\lambda(x,\tau,\tau')$ be a smooth function over $\mathbb{R}^4$ with $|\partial_x^\alpha B_\lambda|\le C_\alpha$ and assume B_λ vanishes unless $|x|\le\delta_0$ and $|\tau-\tau'|\le\delta_0$. Assume that $\mu(x,\tau)$ is a real smooth function over $\mathbb{R}^3$ satisfying the Carleson-Sjölin condition on the support of the amplitudes B_λ, that is*

$$\det\begin{pmatrix}\mu''_{x_1\tau} & \mu''_{x_2\tau}\\ \mu'''_{x_1\tau\tau} & \mu'''_{x_2\tau\tau}\end{pmatrix}\neq 0$$

If $\delta_0>0$ is sufficiently small, then

$$(2.37)\quad \Big\|\iint_{|\tau-\tau'|\ge N\lambda^{-1/2}} e^{i\lambda[\mu(x,\tau)+\mu(x,\tau')]}B_\lambda(x,\tau,\tau')F(\tau,\tau')\,d\tau d\tau'\Big\|^2_{L^2_x(S_r)}$$

$$\lesssim \lambda^{-3/2}N^{-1}\|F\|^2_{L^2(\mathbb{R}^2)}$$

Moreover, if the C_α are fixed and δ_0 is sufficiently small, this estimate is uniform over all functions B_λ which satisfy the hypotheses.

It is well known that the function $\mu(x,\tau)=-d_0(x,(0,\tau)_F)$ satisfies the Carleson-Sjölin condition. So Lemma 2.3.4 will imply (2.34).

Proof of Lemma 2.3.4. Let $\Upsilon(x,\tau,\tau')=\mu(x,\tau)+\mu(x,\tau')$. Then the determinant of the mixed Hessian of Υ satisfies

$$\Big|\det\Big(\frac{\partial^2\Upsilon}{\partial x\partial(\tau,\tau')}\Big)(x,\tau,\tau')\Big| = \mu''_{x_1\tau}(x,\tau)\mu''_{x_2\tau'}(x,\tau') - \mu''_{x_1\tau'}(x,\tau')\mu''_{x_2\tau}(x,\tau)$$

By the Carleson-Sjölin condition, the τ' derivative of this function is nonzero on the diagonal $\tau = \tau'$. This implies that

$$\Big|\det\Big(\frac{\partial^2\Upsilon}{\partial x\partial(\tau,\tau')}\Big)\Big| \geq c|\tau - \tau'|$$

for some $c > 0$ on the support of the amplitudes B_λ, if δ_0 is small. We use the change of variables

$$u = (\tau - \tau', \tau + \tau')$$

Since $|du/d(\tau,\tau')| = 2$, we obtain

$$\Big|\det\Big(\frac{\partial^2\Upsilon}{\partial x\partial u}\Big)\Big| \geq c|u_1|$$

Now Υ is an even function in the u_1-variable, so it is a smooth function of u_1^2. We can make another change of variables

$$v = (\frac{1}{2}u_1^2, u_2).$$

Then $|dv/du| = |u_1|$, so

$$\Big|\det\Big(\frac{\partial^2\Upsilon}{\partial x\partial v}\Big)\Big| \geq c$$

This implies that if v and $\tilde{v}$ are close then

$$\Big|\nabla_x[\Upsilon(x,v) - \Upsilon(x,\tilde{v})]\Big| \geq c'|v - \tilde{v}|$$

for some $c' > 0$. Since Υ is smooth as a function of x and v,

$$\left|\partial_x^\alpha[\Upsilon(x,v) - \Upsilon(x,\tilde{v})]\right| \leq C'_\alpha |v - \tilde{v}|$$

Now if we define

$$K_\lambda(v,\tilde{v}) = \int_{S_r} B_\lambda(x,\tau,\tau')\overline{B_\lambda(x,\tilde{\tau},\tilde{\tau}')}e^{i\lambda[\Upsilon(x,v)-\Upsilon(x,\tilde{v})]}\,dx$$

then for $j = 1, 2, 3, \ldots$, integrating by parts yields

$$|K_\lambda(v,\tilde{v})| \leq C_j(1 + \lambda|v - \tilde{v}|)^{-2j} \tag{2.38}$$

For $a, b \geq 0$,

$$(1 + 2a)(1 + b) \leq 2\Big(1 + (a^2 + b^2)^{1/2}\Big)^2$$

If we set $a = \lambda|v_1 - \tilde{v}_1|$ and $b = \lambda|v_2 - \tilde{v}_2|$, then (2.38) becomes

$$|K_\lambda(v,\tilde{v})| \leq C'_j(1 + \lambda|(u_1^2 - \tilde{u}_1^2|)^{-j}(1 + \lambda|u_2 - \tilde{u}_2|)^{-j} \tag{2.39}$$

Let $E_{N,\lambda}$ be the characteristic function of the set

$$\{(u,\tilde{u}) \in \mathbb{R}^4 : |u_1|, |\tilde{u}_1| \geq N\lambda^{-1/2}\}$$

Then the left side of (2.37) equals

$$\iint E_{N,\lambda}(u,\tilde{u})K_\lambda(u,\tilde{u})F(u)\overline{F(\tilde{u})}\,du d\tilde{u}$$

By Hölder's inequality, it remains to prove that

$$\Big\| \int E_{N,\lambda}(u,\tilde u) K_\lambda(u,\tilde u) F(u)\, du \Big\|_{L^2_{\tilde u}(\mathbb{R}^2)} \lesssim \lambda^{-3/2} N^{-1} \|F\|_{L^2(\mathbb{R}^2)}$$

This will follow from Young's inequality, if we show that

$$\sup_{\tilde u} \int_{|u_1| \geq N\lambda^{-1/2}} |K_\lambda(u,\tilde u)|\, du \lesssim \lambda^{-3/2} N^{-1}$$

and

$$\sup_{u} \int_{|\tilde u_1| \geq N\lambda^{-1/2}} |K_\lambda(u,\tilde u)|\, d\tilde u \lesssim \lambda^{-3/2} N^{-1}$$

Because of (2.39), both of these inequalities will follow if we check that

$$\sup_{c_1, c_2 \in \mathbb{R}} \int_{w_1 \geq N\lambda^{-1/2}} (1 + \lambda |w_1^2 - c_1|)^{-2} (1 + \lambda |w_2 - c_2|)^{-2}\, dw \lesssim \lambda^{-3/2} N^{-1} \tag{2.40}$$

By changing variables,

$$\sup_{c_2 \in \mathbb{R}} \int (1 + \lambda |w_2 - c_2|)^{-2}\, dw_2 = \lambda^{-1} \int (1 + |\tilde w_2|)^{-2}\, d\tilde w_2 \lesssim \lambda^{-1} \tag{2.41}$$

If we set $z = w_1^2$, then $dw_1 = \frac{1}{2} z^{-1/2} dz$, so we also have

$$\begin{aligned} \sup_{c_1 \in \mathbb{R}} \int_{w_1 \geq N\lambda^{-1/2}} &(1 + \lambda |w_1^2 - c_1|)^{-2}\, dw_1 \\ &= \frac{1}{2} \sup_{c_1 \in \mathbb{R}} \int_{z \geq N^2 \lambda^{-1}} (1 + \lambda |z - c_1|)^{-2} z^{-1/2}\, dz \\ &\leq \lambda^{1/2} N^{-1} \sup_{c_1 \in \mathbb{R}} \int (1 + \lambda |z - c_1|)^{-2}\, dz \\ &\leq \lambda^{-1/2} N^{-1} \int (1 + |\tilde z|)^{-2}\, d\tilde z \lesssim \lambda^{-1/2} N^{-1} \end{aligned} \tag{2.42}$$

Now (2.41) and (2.42) yield (2.40), completing the proof of Lemma 2.3.4. □

So we have proven (2.34), and it remains to show (2.35). To simplify the notation, we will only prove this for $j = 0$. The argument will also show that (2.35) holds for all j in $\mathbb{Z}$, uniformly.

Let $p = (0,0)_F$. Let T be the tangent plane at p. The exponential map is a diffeomorphism from a ball of radius 2δ in T to $B\big(p, 2\delta\big)$ if δ is small. Let κ be the inverse function. We will identify T with $\mathbb{R}^2$ in such a way that the Riemannian metric on T agrees with the Euclidean metric on $\mathbb{R}^2$. We can make this identification in such a way that $\exp_p(\sigma, 0) = (\sigma, 0)_F$ for all σ. Let κ_1 and κ_2 denote the component functions of κ, so that $\kappa = (\kappa_1, \kappa_2)$. The inequality (2.35) will be a consequence of the following lemma.

Lemma 2.3.5. *Let $\psi(x,\tau) = -d_0\big(x, (0,\tau)_F\big)$ and let ρ_λ be functions in $C_0^\infty(\mathbb{R}^3)$ satisfying*

$$|\partial_\tau^m \rho_\lambda(x,\tau)| \le C_m \lambda^{m/2} \tag{2.43}$$

and

$$\text{supp } \rho_\lambda \subset \Big\{(x,\tau) : |\tau| \le \lambda^{-1/2}, x \in \mathcal{N}_x, (0,\tau)_F \in \mathcal{N}_y\Big\} \tag{2.44}$$

Assume q_k are points in $\mathcal{N}_x$ satisfying

$$\Big|\frac{\kappa_2(q_k)}{|\kappa(q_k)|} - \frac{\kappa_2(q_\ell)}{|\kappa(q_\ell)|}\Big| \ge c\lambda^{-1/2}|k-\ell| \tag{2.45}$$

with $c > 0$, when $|k-\ell| \ge 2$. If $\mathcal{N}_x$ is sufficiently small, then

$$\lambda^{1/2} \int \Big|\sum_k e^{i\lambda\psi(q_k,\tau)} \rho_\lambda(q_k,\tau) p_k\Big|^2 d\tau \lesssim \sum |p_k|^2 \tag{2.46}$$

This estimate is uniform over different choices of the points q_k.

To see that Lemma 2.3.5 implies (2.35), let $\kappa_r(x)$ and $\kappa_\theta(x)$ be the polar coordinates of $\kappa(x)$ with $\kappa_\theta(x)$ in $[0, 2\pi)$. These functions are well defined and smooth on $\mathcal{N}_x$. Define

$$\rho_\lambda(x,\tau) = \eta_{\lambda,0}(\tau) A_\lambda\big(x, (0,\tau)_F\big)$$

Then (2.43) and (2.44) hold. Define the sets

$$V_k = \Big\{ x \in \mathcal{N}_x : \lambda^{-1/2} k \le \kappa_\theta(x) < \lambda^{-1/2}(k+1) \Big\}$$

We have

$$\begin{aligned} \int_{S_r} \lambda \Big| \int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,0}(\tau) A_\lambda\big(x,(0,\tau)_F\big) H(\tau)\, d\tau \Big|^2 |g(x)|^2 \, dx \\ \le \sum_k \lambda \Big\| \int e^{i\lambda\psi(x,\tau)} \rho_\lambda(x,\tau) H(\tau)\, d\tau \Big\|^2_{L^\infty_x(V_k)} \|g\|^2_{L^2(V_k)} \\ \le \sup_\ell \|g\|^2_{L^2(V_\ell)} \sum_k \lambda \Big\| \int e^{i\lambda\psi(x,\tau)} \rho_\lambda(x,\tau) H(\tau)\, d\tau \Big\|^2_{L^\infty_x(V_k)} \end{aligned}$$

If $\mathcal{N}_x$ is small, then each V_ℓ is contained in $\mathcal{T}_\lambda(\gamma_\ell)$ for some $\gamma_\ell \in \Pi_0$. In fact, each γ_ℓ can be chosen to go through p. This yields

$$\sup_\ell \|g\|^2_{L^2(V_\ell)} \le \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

Now to prove (2.35), it remains to show that

$$\sum_k \lambda^{1/2} \Big\| \int e^{i\lambda\psi(x,\tau)} \rho_\lambda(x,\tau) H(\tau)\, d\tau \Big\|^2_{L^\infty_x(V_k)} \lesssim \|H\|^2_{L^2(\mathbb{R})}$$

It suffices to check that for any choice of points q_k in V_k,

$$\sum_k \lambda^{1/2} \Big| \int e^{i\lambda\psi(q_k,\tau)} \rho_\lambda(q_k,\tau) H(\tau)\, d\tau \Big|^2 \lesssim \|H\|^2_{L^2(\mathbb{R})}$$

and that this holds uniformly over different choices of q_k. By duality, this inequality is equivalent to (2.46). To apply Lemma 2.3.5, we still need to check that any choice of points q_k in S_k satisfies (2.45). If $\mathcal{N}_x$ and $\mathcal{N}_y$ are sufficiently small, then $\kappa_\theta(\mathcal{N}_x)$ is contained in $[2\pi/3, 4\pi/3]$. When $|j-k| \geq 2$, we then have

$$\Big|\frac{\kappa_2(q_j)}{|\kappa(q_j)|} - \frac{\kappa_2(q_k)}{|\kappa(q_k)|}\Big| = \Big|\sin\big(\kappa_\theta(q_j)\big) - \sin\big(\kappa_\theta(q_k)\big)\Big| \geq \frac{1}{2}\Big|\kappa_\theta(q_j) - \kappa_\theta(q_k)\Big| \geq \frac{1}{4}\lambda^{-1/2}|j-k|$$

This is (2.45), so Lemma 2.3.5 will imply (2.35).

Proof of Lemma 2.3.5. We can write

$$\psi(x,\tau) = \psi(x,0) + \tau\partial_\tau\psi(x,0) + r(x,\tau)$$

where

$$|r(\tau,x)| \leq C_0|\tau|^2 \quad |\partial_\tau r(\tau,x)| \leq C_1|\tau|$$

and for $m = 2, 3, \ldots$

$$|\partial_\tau^m r(\tau,x)| \leq C_m$$

Fix x in $\mathcal{N}_x$ and let Θ be the geodesic sphere of radius $|\kappa(x)|$ around x. By Gauss' lemma, $\kappa(x)$ is normal to $\kappa(\Theta)$. Define a function G from $\mathbb{R}^2$ to $\mathbb{R}$ by

$$G(u) = -d_0(x, \exp_p(u))$$

Then $\kappa(\Theta)$ is a level set of G, so $\nabla G(0)$ is normal to $\kappa(\Theta)$. That is, $\nabla G(0)$ is a multiple of $\kappa(x)$. Define a curve c in T by $c(t) = t\kappa(x)$. Then $G(c(t)) = (t-1)|\kappa(x)|$ for t near 0, so $\nabla G(0)\cdot\kappa(x) = |\kappa(x)|$. Since $\nabla G(0)$ is a multiple of $\kappa(x)$, this implies that

$$\nabla G(0) = \frac{\kappa(x)}{|\kappa(x)|}$$

This yields

$$\partial_\tau \psi(x,0) = \nu \cdot \frac{\kappa(x)}{|\kappa(x)|}$$

where

$$\nu = \partial_\tau \kappa\big((0,\tau)_F\big)\Big|_{\tau=0}$$

That is, ν is the pushforward under κ of $\partial/\partial\tau$ at p. It must be transverse to the pushforward under κ of $\partial/\partial\sigma$ at p, whose second component is zero. So the second component of ν is nonzero. By (2.45),

$$\Big|\partial_\tau \psi(q_k,0) - \partial_\tau \psi(q_\ell,0)\Big| \geq c'\lambda^{-1/2}|j-k|$$

for some $c' > 0$ when $|k-\ell| \geq 2$.

Now define

$$P_\lambda(q_k,q_\ell,\tau) = \rho_\lambda(q_k,\tau)\overline{\rho_\lambda(q_\ell,\tau)}e^{i\lambda[\psi(q_k,0)+r(q_k,\tau)]}e^{-i\lambda[\psi(q_\ell,0)+r(\ell,\tau)]}$$

Then $P_\lambda(q_k,q_\ell,\tau)$ vanishes when $|\tau| \geq \lambda^{-1/2}$ and satisfies

$$\Big|\partial_\tau^m P_\lambda(q_k,q_\ell,\tau)\Big| \leq C_m\lambda^{m/2}$$

The left side of (2.46) is equal to

$$\lambda^{1/2}\sum_{k,\ell} p_k\overline{p_\ell}\Big(\int e^{i\tau\lambda[\partial_\tau\psi(q_k,0)-\partial_\tau\psi(q_\ell,0)]}P_\lambda(q_k,q_\ell,\tau)\,d\tau\Big)$$

We integrate by parts twice to control this by

$$\sum_{k,\ell}|p_k p_\ell|(1+|k-\ell|)^{-2} \lesssim \sum_{k,\ell}(|p_k|^2+|p_\ell|^2)(1+|k-\ell|)^{-2} \lesssim \sum_k |p_k|^2$$

This completes the proof of Lemma 2.3.5, and now Theorem 2.1.2 follows. □

2.4 End of Proof of Theorem 1.2.3

To complete the proof of Theorem 1.2.3, it remains to prove Lemma 2.2.6. This will be a consequence of the following variant. To state it, recall that $\eta(x,y)$ is in $C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and is supported by x and y in a small neighborhood of S_r satisfying $\frac{1}{2}\delta \le d_0(x,y) \le \delta$. Also $\eta(x,y) = 1$ when x is in a small neighborhood of S_r and $d_0(x,y)$ is in an open neighborhood of the support of $\hat{\chi}$.

Lemma 2.4.1. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$. Define an operator D_a by*

$$D_a f = \iint e^{i\varphi(t,x,\xi)-iy\cdot\xi}\eta(x,y)a(t,x,\xi)f(y)\,d\xi dy$$

For any smooth curve Γ in S_r of unit length, and for f with fixed compact support,

$$\|I_\lambda(D_a)f\|_{L^4(\Gamma)} \lesssim \lambda^{1/4}\|f\|_{L^2(\mathbb{R}^2)}$$

Using Lemma 2.4.1, we can now prove Lemma 2.2.6.

Proof of Lemma 2.2.6. Fix a symbol $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$. We may assume that

$$\big(1-\eta(x,y)\big)a(t,x,\xi)$$

vanishes on a neighborhood of the set

$$\Sigma_0 = \{(t,x,y,\xi) : t = d_0(x,y)\}$$

We can make this assumption because $I_\lambda(U_a)$ only depends on t in the support of $\hat{\chi}$. The kernel of U_a is

$$\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}a(t,x,\xi)\,d\xi$$

Define a set

$$\Sigma = \Big\{(t,x,y,\xi) : \varphi'_\xi(t,x,\xi) - y = 0\Big\}$$

Define an operator D_a with kernel

$$\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}\eta(x,y)a(t,x,\xi)\,d\xi$$

By (2.27), the set Σ is contained in Σ_0. So the symbol $\big(1-\eta(x,y)\big)a(t,x,\xi)$ vanishes on a neighborhood of Σ. By Proposition 1.2.4 of Hörmander [17], the difference between U_a and D_a is smoothing, so it suffices to show that $I_\lambda(D_a)$ satisfies (2.15). Any broken geodesic γ in S_r can be broken up into a fixed finite number of segments which are smooth curves, so this will follow from Lemma 2.4.1. □

The next lemma will give a suitable description of the kernel of $I_\lambda(D_a)$. This description is sufficiently similar to the one used in Burq-Gérard-Tzvetkov [5], so that the same argument will yield Lemma 2.4.1.

Lemma 2.4.2. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi)$. The kernel of $I_\lambda(D_a)$ is of the form*

$$\lambda^{1/2}e^{-i\lambda d_0(x,y)}A_\lambda(x,y) + R_\lambda(x,y) \tag{2.47}$$

where R_λ is uniformly bounded in λ and A_λ is in $C^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ and satisfies

$$|\partial_x^\alpha\partial_y^\beta A_\lambda| \le C_{\alpha,\beta}\lambda^{|\alpha|/3}$$

Also A_λ is supported by x and y in a small neighborhood of S_r satisfying

$$\delta/2 \le d_0(x,y) \le \delta$$

Lemma 2.4.2 follows from essentially the same proof as Lemma 2.3.2. Now we can follow the argument in Burq-Gérard-Tzvetkov [5] to finish the proof of Lemma 2.4.1.

Argument from Burq-Gérard-Tzvetkov [5]

Let T_λ be the operator with kernel

$$\lambda^{1/2} e^{-i\lambda d_0(x,y)} A_\lambda(x,y)$$

We will complete the proof of Lemma 2.4.1 by showing that for any smooth curve Γ in S_r of unit length,

$$\|T_\lambda f\|_{L^4(\Gamma)} \lesssim \lambda^{1/4} \|f\|_{L^2(\mathbb{R}^2)} \tag{2.48}$$

By using a partition of unity and abusing notation, we can assume there is a point x_0 in S_r such that A_λ is supported by x in the geodesic ball $B(x_0, c_0\delta)$ of radius $c_0\delta$ around x_0, where $c_0 > 0$ is small. Then there are small constants $c_2 > c_1 > 0$ such that A_λ is supported by y in the geodesic annulus $B(x_0, c_2\delta) \setminus B(x_0, c_1\delta)$.

Let T be the tangent plane at x_0. We will use geodesic polar coordinates (ρ, ω) for the y-variable, with ω a unit vector in T and $\rho > 0$, so that $y = \exp_{x_0}(\rho\omega)$. Then we can write

$$(T_\lambda f)(x) = \int_{c_1\delta}^{c_2\delta} (T_\lambda^\rho f_\rho)(x)\, d\rho$$

with

$$(T_\lambda^\rho f)(x) = \lambda^{1/2} \int_{S^1} e^{-i\lambda d_{0,\rho}(x,\omega)} A_{\lambda,\rho}(x,\omega) f(\omega)\, d\omega$$

Here

$$d_{0,\rho}(x,\omega) = d_0(x,y), \quad f_\rho(\omega) = f(y), \quad \text{and} \quad A_{\lambda,\rho}(x,\omega) = J(\rho,\omega)A_\lambda(x,y)$$

where J is a smooth function satisfying $J(\rho,\omega) = \rho$ when $c_1\delta \le \rho \le c_2\delta$.

If we can prove the uniform estimates

$$\|T_\lambda^\rho f\|_{L^4(\Gamma)} \lesssim \lambda^{1/4}\|f\|_{L^2(S^1)} \tag{2.49}$$

then (2.48) will follow, because we will have

$$\|T_\lambda f\|_{L^4(\Gamma)} \le \int_{c_1\delta}^{c_2\delta} \|T_\lambda^\rho f_\rho\|_{L^4(\Gamma)}\, d\rho \lesssim \lambda^{1/4} \int_{c_1\delta}^{c_2\delta} \|f_\rho\|_{L^2(S^1)}\, d\rho \lesssim \lambda^{1/4}\|f\|_{L^2(\mathbb{R})}$$

So it suffices to prove (2.49). By duality, (2.49) is equivalent to

$$\|(T_\lambda^\rho)^* f\|_{L^2(S^1)} \lesssim \lambda^{1/4}\|f\|_{L^{4/3}(\Gamma)} \tag{2.50}$$

We will prove

$$\|T_\lambda^\rho (T_\lambda^\rho)^* f\|_{L^4(\Gamma)} \lesssim \lambda^{1/2}\|f\|_{L^{4/3}(\Gamma)} \tag{2.51}$$

This will imply (2.50), because

$$\|(T_\lambda^\rho)^* f\|_{L^2(S^1)}^2 = \int_\Gamma T_\lambda^\rho (T_\lambda^\rho)^* f(s)\overline{f(s)}\, ds \le \|T_\lambda^\rho (T_\lambda^\rho)^* f\|_{L^4(\Gamma)}\|f\|_{L^{4/3}(\Gamma)} \lesssim \lambda^{1/2}\|f\|_{L^{4/3}(\Gamma)}^2$$

So it suffices to prove (2.51). Assume $x(t)$ parametrizes Γ by arc length with domain

$0 \leq t \leq 1$. The kernel of $T^\rho_\lambda (T^\rho_\lambda)^*$ is

$$K^\rho_\lambda(t,\tau) = \lambda \int_{S^1} e^{-i\lambda[d_{0,\rho}(x(t),\omega) - d_{0,\rho}(x(\tau),\omega)]} A_{\lambda,\rho}(x(t),\omega)\overline{A_{\lambda,\rho}(x(\tau),\omega)}\, d\omega$$

By making a linear change of variables, we may assume that $g_{ij}(x_0) = \delta^{ij}$. Then we have the following lemma, which we will use to control K^ρ_λ.

Lemma 2.4.3. *If $\rho > 0$ is small and ω is in S^1, then*

$$-\nabla_x d_{0,\rho}(x_0,\omega) = \omega \tag{2.52}$$

Proof. Let Θ be the geodesic sphere of radius ρ around $y = \exp_{x_0}(\rho\omega)$. By Gauss' lemma, the vector ω is normal to Θ at x_0. Define a function G by

$$G(x) = d_{0,\rho}(x,\omega)$$

Then Θ is a level set of G, so $\nabla G(x_0)$ is normal to Θ at x_0. That is, $\nabla G(x_0)$ is a multiple of ω. Let c be the geodesic satisfying $c(0) = x_0$ and $c'(0) = \omega$. Then for small s,

$$G\big(c(s)\big) = \rho - s$$

So $\nabla G(x_0) \cdot \omega = -1$. Since $\nabla G(x_0)$ is a multiple of ω, this implies that $\nabla G(x_0) = -\omega$, which is (2.52). □

Using Lemma 2.4.3, we can prove the following lemma.

Lemma 2.4.4. *There is a $\delta_0 > 0$ such that if $|t - \tau| < \delta_0$, then*

$$|K^\rho_\lambda(t,\tau)| \lesssim \lambda(1 + \lambda|t-\tau|)^{-1/2}$$

Proof. Define

$$K_\lambda^\rho(x,x') = \lambda \int_{S^1} e^{-i\lambda[d_{0,\rho}(x,\omega)-d_{0,\rho}(x',\omega)]} A_{\lambda,\rho}(x,\omega)\overline{A_{\lambda,\rho}(x',\omega)}\, d\omega$$

Since Γ is smooth and parametrized by arc length, it suffices to show that

$$|K_\lambda^\rho(x,x')| \lesssim \lambda(1+\lambda|x-x'|)^{-1/2} \tag{2.53}$$

We can write

$$d_{0,\rho}(x,\omega) - d_{0,\rho}(x',\omega) = (x-x')\cdot\Psi_{0,\rho}(x,x',\omega)$$

where

$$\Psi_{0,\rho}(x,x',\omega) = \int_0^1 \nabla_x d_{0,\rho}\big(x'+s(x-x'),\omega\big)\, ds$$

For σ in S^1, define

$$\Phi_{0,\rho}(x,x',\sigma,\omega) = \sigma\cdot\Psi_{0,\rho}(x,x',\omega)$$

Now when $x \neq x'$,

$$d_{0,\rho}(x,\omega) - d_{0,\rho}(x',\omega) = |x-x'|\Phi_{0,\rho}(x,x',\sigma_{x,x'},\omega)$$

where

$$\sigma_{x,x'} = \frac{x-x'}{|x-x'|}$$

If we define

$$J_\mu^\rho(x,x',\sigma) = \int_{S^1} e^{-i\mu\Phi_{0,\rho}(x,x',\sigma,\omega)} A_{\lambda,\rho}(x,\omega)\overline{A_{\lambda,\rho}(x',\omega)}\, d\omega \tag{2.54}$$

then it suffices to show that

$$|J^{\rho}_{\mu}(x, x', \sigma)| \lesssim (1 + \mu)^{-1/2} \tag{2.55}$$

Parametrize S^1 by

$$\omega(\theta) = (\cos\theta, \sin\theta)$$

for θ in $[0, 2\pi)$. Write

$$\sigma = (\cos\alpha, \sin\alpha)$$

where α is in $[0, 2\pi)$. Then by Lemma 2.4.3,

$$\Phi_{0,\rho}(x_0, x_0, \sigma, \omega(\theta)) = -\sigma \cdot \omega(\theta) = -\cos(\theta - \alpha)$$

So we have

$$\partial_\theta \Phi_{0,\rho}(x_0, x_0, \sigma, \omega(\theta)) = \sin(\theta - \alpha)$$

and

$$\partial^2_\theta \Phi_{0,\rho}(x_0, x_0, \sigma, \omega(\theta)) = \cos(\theta - \alpha)$$

There are relatively open sets A and B, with $A \cup B = [0, 2\pi)$, such that for θ in A,

$$|\partial_\theta \Phi_{0,\rho}(x_0, x_0, \sigma, \omega(\theta))| \geq c_A$$

and for θ in B,

$$|\partial^2_\theta \Phi_{0,\rho}(x_0, x_0, \sigma, \omega(\theta))| \geq c_B$$

Here c_A and c_B are positive constants. By continuity, if δ is sufficiently small and x, x'

are in $B(x_0, c_0\delta)$, then for θ in A,

$$|\partial_\theta \Phi_{0,\rho}(x, x', \sigma, \omega(\theta))| \geq c_A/2 \tag{2.56}$$

and for θ in B

$$|\partial_\theta^2 \Phi_{0,\rho}(x, x', \sigma, \omega(\theta))| \geq c_B/2 \tag{2.57}$$

By using a partition of unity on S^1 and abusing notation, it suffices to prove (2.55) in two cases. In the first case, we assume that (2.56) holds on the support of the amplitude in (2.54). This case can be handled by integrating by parts, which yields much stronger bounds than in (2.55). In the second case, we assume that (2.57) holds on the support of the amplitude in (2.54). This case can be handled by using stationary phase, which yields (2.55). □

Now we can use Lemma 2.4.4 and the Hardy-Littlewood fractional integration inequality to obtain

$$\|T_\lambda^\rho (T_\lambda^\rho)^* f\|_{L^4(\gamma)} \lesssim \left\| \int_0^1 \lambda(1+\lambda|t-\tau|)^{-1/2} f\big(x(\tau)\big)\, d\tau \right\|_{L^4(0,1)} \lesssim \lambda^{1/2} \|f\|_{L^{4/3}(\gamma)}$$

This is (2.51), so we have proven Lemma 2.4.1. Now Theorem 1.2.3 follows.

2.5 Proof of Corollary 2.1.1

Fix $\delta > 0$. Recall the set

$$H_\delta = \Big\{ x \in M : d(x, \partial M) \leq \delta \Big\}$$

and recall that E_δ is the complement of H_δ in M. Also recall that we are assuming M is a subset of a compact Riemannian manifold (M_0, g) and that Δ_0 is the Laplacian on M_0. If $\delta > 0$ is small enough, then we can break up γ into $\gamma \cap E_\delta$ and $\gamma \cap H_\delta$, where $\gamma \cap H_\delta$ is a broken geodesic with length at most $c_0 \delta^{1/2}$ for some fixed constant $c_0 > 0$. This is because the boundary is strictly geodesically concave. We can use Hölder's inequality and Theorem 1.2.3 to control $\|\varphi_j\|_{L^2(\gamma\cap H_\delta)}$. This gives

$$\|\varphi_j\|_{L^2(\gamma\cap H_\delta)} \lesssim \delta^{1/8} \|\varphi_j\|_{L^4(\gamma\cap H_\delta)} \lesssim \delta^{\frac{1}{8}} \lambda_j^{\frac{1}{4}} \tag{2.58}$$

Choose $\chi \in \mathcal{S}(\mathbb{R})$ with $\chi(0) = 1$ and $\hat{\chi}$ supported on a closed interval contained strictly inside of $(\frac{1}{2}\delta, \delta)$. Define $\chi_\lambda(s) = \chi(s - \lambda)$ and $\rho_\lambda(s) = \chi_\lambda(s) + \chi_\lambda(-s)$. For large λ, we have

$$1/2 \leq |\rho_\lambda(\lambda)|$$

To control $\|\varphi_j\|_{L^2(\gamma\cap E_\delta)}$ we will use the following inequality.

Theorem 2.5.1. *Let $p \geq 2$ and assume δ is small. If γ is a unit length geodesic on M_0 and $\lambda \geq 1$, then there is a constant C_δ independent of the choice of γ such that*

$$\|\rho_\lambda(\sqrt{-\Delta_0}) f\|_{L^2(\gamma)} + \|\chi_\lambda(\sqrt{-\Delta_0}) f\|_{L^2(\gamma)} \leq C_\delta \lambda^{\frac{1}{2p}} \|f\|_{L^p(M_0)}$$

Bourgain [3] proved this inequality for χ_λ, and the inequality for ρ_λ follows easily from Lemma 2.2.2. Recall (2.8), which says

$$(\rho_\lambda(\sqrt{-\Delta_g}) f)\big|_{\gamma\cap E_\delta} = (\rho_\lambda(\sqrt{-\Delta_0}) f)|_{\gamma\cap E_\delta}$$

for f in $L^2(M)$. So Theorem 2.5.1 yields

$$\|\varphi_j\|_{L^2(\gamma\cap E_\delta)} \leq C_\delta \lambda_j^{\frac{1}{2p}} \|\varphi_j\|_{L^p(M)} \tag{2.59}$$

Now if δ is sufficiently small, Corollary 2.1.1 follows from (2.58) and (2.59).

2.6 Proof of Proposition 2.1.5

For sufficiently small $\delta > 0$, we can break up γ into $\gamma \cap E_\delta$ and $\gamma \cap H_\delta$, where $\gamma \cap H_\delta$ is a broken geodesic with length at most $c_0 \delta^{1/2}$ for some fixed constant $c_0 > 0$. This is because the boundary is strictly geodesically concave. By Hölder's inequality and Theorem 1.2.3,

$$\limsup_{j\to\infty} \lambda_j^{-1/4} \|\varphi_j\|_{L^2(\gamma\cap H_\delta)} \lesssim \limsup_{j\to\infty} \lambda_j^{-1/4} \delta^{1/8} \|\varphi_j\|_{L^4(\gamma\cap H_\delta)} \lesssim \delta^{1/8}$$

Now it suffices to prove

$$\limsup_{j\to\infty} \lambda_j^{-1/4} \|\varphi_j\|_{L^2(\gamma\cap E_\delta)} = 0$$

By breaking up $\gamma \cap E_\delta$ into pieces and abusing notation, we may assume that γ is a geodesic in M with $d_g(\gamma, \partial M) \geq \delta$ and moreover, that γ is of length L where L is small and may depend on δ. With these assumptions, we can follow the proof by Sogge [29] for the boundaryless version of this problem, making only very minor modifications.

The proof will make use of Fermi normal coordinates about γ. These coordinates are well-defined on some neighborhood W of γ. In this coordinate system, γ becomes $\{(s, 0) : s \in [0, L]\}$ and the metric satisfies

$$g_{ij}(s, 0) = \delta^{ij}$$

In the Fermi coordinates, the principal symbol p_0 of P_0 satisfies

$$p\big((s, 0), \xi\big) = |\xi|$$

Let $\psi \in C_0^\infty(M)$ be supported strictly inside $W \cap E_{\delta/2}$ with $\psi = 1$ on γ. Let A, B_1, and

B_2 be pseudodifferential operators of order zero with symbols satisfying

$$\psi(x) = A(x,\xi) + B_1(x,\xi) + B_2(x,\xi)$$

In the Fermi coordinates, assume that A is supported outside a conic neighborhood of the ξ_1-axis, B_1 is essentially supported in a conic neighborhood of the positive ξ_1-axis, and B_2 is essentially supported in a conic neighborhood of the negative ξ_1-axis. We also assume that $Af = 0$ if f is supported in $H_{\delta/2}$.

Fix a positive integer N and a real-valued $\chi \in \mathcal{S}(\mathbb{R})$ with $\chi(0) = 1$. Assume the support of $\hat{\chi}$ is strictly inside $(-1/2, 1/2)$. Define $\chi_{N,\lambda}(s) = \chi(N(s-\lambda))$ and

$$\rho_\lambda(s) = \chi(\delta(s-\lambda)) + \chi(\delta(-s-\lambda))$$

Then

$$\chi_{N,\lambda}(\lambda) = 1$$

and for large λ,

$$1/2 \leq |\rho_\lambda(\lambda)|$$

Let $B = B_1 + B_2$. It suffices to show

$$\|A\rho_\lambda(P)f\|_{L^2(\gamma)} + \|B\chi_{N,\lambda}(P)f\|_{L^2(\gamma)} \leq CN^{-1/2}\lambda^{1/4}\|f\|_{L^2(M)} + C_N\|f\|_{L^2(M)}$$

We have

$$A\rho_\lambda(P)f = (\delta\pi)^{-1}\int \hat{\chi}(t/\delta)e^{-it\lambda}A\cos(tP)f\,dt$$

Note the support of the integrand is strictly inside $(-\delta/2, \delta/2)$.

The operator U defined by $Uf(t,x) = \cos(tP_0)f(x)$ is a Fourier integral operator from

M_0 to $M_0 \times \mathbb{R}$. Its canonical relation is

$$\Big\{(x, t, \xi, \tau; y, \eta) : (x, \xi) = \Phi_t(y, \eta), \pm\tau = p_0(x, \xi)\Big\}$$

where $\Phi_t : T^*M_0 \to T^*M_0$ is the geodesic flow on the cotangent bundle of M_0. The operator V defined by $Vf(t, x) = \big(\cos(tP_0)f\big)\big|_\gamma(x)$ is a Fourier integral operator from M_0 to $\gamma \times \mathbb{R}$. Using the Fermi normal coordinates, we can write its canonical relation as

$$\mathcal{C} = \Big\{\big((s, 0), t, \xi_1, \tau; y, \eta\big) : \big((s, 0), \xi\big) = \Phi_t(y, \eta), \pm\tau = |\xi|\Big\}$$

Then the projection from $\mathcal{C}$ to $T^*(\gamma \times \mathbb{R})$ is given by the map

$$(s, t, \xi) \to (s, t, \xi_1, |\xi|)$$

This has surjective differential away from $\xi_2 = 0$.

If $|t| < \delta/2$, then by our assumptions on A,

$$A\big(\cos(tP)f\big) = A\big(\cos(tP_0)f\big)$$

Define an operator by

$$f \to \big(A(\cos(tP_0)f)\big)\big|_\gamma$$

This is a non-degenerate Fourier integral operator of order zero, because A is supported away from the ξ_1-axis. This implies that

$$\int |\hat{\chi}(t/\delta)| \, \|A\big(\cos(tP)f\big)\|_{L^2(\gamma)} dt \lesssim \|f\|_{L^2(M)}$$

which yields

$$\|A\rho_\lambda(P)f\|_{L^2(\gamma)} \lesssim \|f\|_{L^2(M)}$$

It remains to control the operators χ_λ^{N,B_j} defined by

$$\chi_\lambda^{N,B_j} f = B_j \circ \chi(N(P-\lambda))f = N^{-1} \int \hat{\chi}(t/N)e^{-it\lambda}\Big(B_j \circ e^{itP}\Big) f dt$$

Define an operator V_j by

$$V_j f(t,x) = \Big((B_j \circ e^{itP} \circ B_j^*)f\Big)(x)$$

Fix a distribution u supported in the interior of M. Assume that (t, x, τ, ξ) is in the wave front set of $V_j u$. Then (x, ξ) is in the essential support of B_j, and for some (y, η) in the essential support of B_j, there is a broken geodesic Γ satisfying $\Gamma(0) = y$, $\Gamma'(0) = \eta$, $\Gamma(t) = x$ and $\Gamma'(t) = \xi$. Since γ is not contained in a periodic broken geodesic, the cutoffs ψ and B_j can be chosen with sufficiently small supports so that $V_j u$ is a smooth function over $2L \le |t| \le N+1$. That is, the operator V_j is smoothing over the region $2L \le |t| \le N+1$.

Define an operator U_j by

$$U_j f(t,x) = \Big((B_j \circ e^{itP_0} \circ B_j^*)f\Big)(x)$$

Then the operator $V_j - U_j$ is smoothing over the region $|t| \le 10L$, if L is small.

Let T be the operator $f \to (\chi_\lambda^{N,B_j} f)\big|_\gamma$. We want to show that

$$\|Tf\|_{L^2(M)} \le (CN^{-1/2}\lambda^{1/4} + C_{N,B_j})\|f\|_{L^2(\gamma)}$$

We will use the TT^* method. We have

$$\|T^*g\|^2_{L^2(M)} = \int_M T^*g\overline{T^*g}\,dx = \int_\gamma \big(TT^*g\big)\overline{g}\,ds \le \|TT^*g\|_{L^2(\gamma)}\|g\|_{L^2(\gamma)}$$

So by duality, it suffices to prove that

$$\|TT^*g\|_{L^2(\gamma)} \le (CN^{-1}\lambda^{1/2} + C_{N,B_j})\|g\|_{L^2(\gamma)} \tag{2.60}$$

Let $w(\tau) = (\chi(\tau))^2$. Then the kernel of TT^* is $K(\gamma(s), \gamma(s'))$ where $K(x, y)$ is the kernel of the operator $B_j \circ w(N(P - \lambda)) \circ B_j^*$. Also $\hat{w}$ is supported in $[-1, 1]$, since $\hat{w} = \hat{\chi} * \hat{\chi}$. Now

$$B_j \circ w(N(P - \lambda)) \circ B_j^* = N^{-1} \int \hat{w}(t/N) e^{-it\lambda} \Big(B_j \circ e^{itP} \circ B_j^* \Big) \, dt$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ be supported on $[-1, 1]$ with $\varphi = 1$ on $[-1/2, 1/2]$. Now, by the smoothing properties of the operators V_j and $V_j - U_j$, the difference between $B_j \circ w(N(P - \lambda)) \circ B_j^*$ and

$$N^{-1} \int \varphi(t/5L) \hat{w}(t/N) e^{-it\lambda} \Big(B_j \circ e^{itP_0} \circ B_j^* \Big) \, dt \tag{2.61}$$

has a kernel which is $\mathcal{O}(\lambda^{-m})$ for all m, so it remains to control the kernel of the operator (2.61). If $5L$ is less than the injectivity radius of M_0, then the Hadamard parametrix can be used here. Then by stationary phase arguments, it follows that the kernel of the operator (2.61) satisfies

$$|K(x, y)| \le CN^{-1}\lambda^{1/2}(d_g(x, y))^{-1/2} + C_{B_j}$$

This yields (2.60), completing the proof of Proposition 2.1.5.

Chapter 3

Lower bounds for nodal sets

3.1 Preliminaries

In this chapter, we assume (M, g) is a compact smooth Riemannian manifold with boundary. We fix $\lambda \geq 1$ and let φ be an eigenfunction of $-\Delta$, i.e. a non-zero smooth real-valued function on M with

$$-\Delta\varphi = \lambda^2\varphi$$

over the interior of M. We will assume that φ is a Dirichlet eigenfunction, meaning

$$\varphi\Big|_{\partial M} = 0$$

or a Neumann eigenfunction, meaning

$$\partial_\nu\varphi\Big|_{\partial M} = 0$$

where ν is the outward unit normal vector on ∂M and ∂_ν is the corresponding directional derivative. We normalize φ so that

$$\|\varphi\|_{L^2(M)} = 1$$

Define the nodal set

$$Z = \left\{ x \in M : \varphi(x) = 0, x \notin \partial M \right\}$$

Let n be the dimension of M and let $\mathcal{H}$ be the $(n-1)$-dimensional Hausdorff measure on M. The purpose of this chapter is to prove Theorems 1.2.5 and 1.2.6.

Define

$$P = \left\{ x \in M : \varphi(x) > 0, x \notin \partial M \right\}$$

and

$$N = \left\{ x \in M : \varphi(x) < 0, x \notin \partial M \right\}$$

We can write M as a disjoint union

$$M = P \cup N \cup \partial M \cup Z$$

Define

$$\Omega = \left\{ x \in M : \varphi(x) = 0 \right\}$$

and

$$\Sigma = \left\{ x \in \Omega : \nabla\varphi(x) = 0 \right\}$$

Lemma 3.1.1. *If φ is a Dirichlet or Neumann eigenfunction, then $\mathcal{H}(\Omega) < \infty$, and the Hausdorff dimension of Σ is at most $n-2$. If φ is a Neumann eigenfunction, then the Hausdorff dimension of $\Omega \cap \partial M$ is at most $n-2$.*

For the proof we will use the following lemma, which was proven by Hardt and Simon

[14].

Lemma 3.1.2. *Let W be an open subset of $\mathbb{R}^n$ containing the origin. Assume u is a function in $C^1(W) \cap H^2_{loc}(W)$ and satisfies*

$$\left(\sum_{i,j=1}^{n} a_{ij} D_i D_j u\right) + \left(\sum_{j=1}^{n} b_j D_j u\right) + cu = 0$$

Assume the functions a_{ij} are Lipschitz continuous and satisfy $a_{ij}(0) = \delta_{ij}$. Also assume the functions b_j and c are bounded. Assume u has finite order of vanishing at the origin. Then there is a neighborhood $U \subseteq W$ of the origin such that

$$\left\{x \in U : u(x) = 0\right\}$$

has finite $(n-1)$-dimensional Hausdorff measure, and

$$\left\{x \in U : u(x) = 0, \nabla u(x) = 0\right\}$$

has Hausdorff dimension at most $n-2$.

Proof of Lemma 3.1.1. Fix a point p in M. To prove the first statement, it suffices to find a neighborhood U of p in M such that $\mathcal{H}(\Omega \cap U) < \infty$ and $\Sigma \cap U$ has Hausdorff dimension at most $n-2$. If $\varphi(p) \neq 0$, then finding such a neighborhood U is trivial. So we assume $\varphi(p) = 0$. By Hörmander, [18], the eigenfunction φ only has finite order of vanishing at p. In fact, Donnelly and Fefferman [9] showed that the order of vanishing is at most $c\lambda$ for some positive constant c. If p is in the interior of M, we use geodesic normal coordinates about p. Then by Lemma 3.1.2, we can obtain U.

If p is on the boundary ∂M, then we use boundary normal coordinates $(x_1, \ldots, x_n)$ about p. These are defined by first letting $(x_1, \ldots x_{n-1})$ be geodesic normal coordinates on ∂M about p, with respect to the metric on ∂M induced by g. Then for fixed $x_1, \ldots, x_{n-1}$, the curves $x_n \to (x_1, \dots, x_n)$, for $x_n \geq 0$, are geodesics in M which intersect ∂M normally. These coordinates are well-defined near p and allow us to identify some neighborhood of p with

$$B_+ = \Big\{ x \in \mathbb{R}^n : |x| < \varepsilon, x_n \geq 0 \Big\}$$

for some small $\varepsilon > 0$. Here the point p is being identified with the origin in $\mathbb{R}^n$. Let g_{ij} be the Riemannian metric on B_+. Let

$$B = \Big\{ x \in \mathbb{R}^n : |x| < \varepsilon \Big\}$$

We extend the metric g_{ij} to B so that it is even in the x_n-variable. Let g^{ij} be the cometric, defined so that the matrix $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. Define

$$J = \Big(\det[g_{ij}] \Big)^{1/2}$$

The functions g_{ij}, g^{ij}, and J are Lipschitz continuous and bounded on B. If φ is a Dirichlet eigenfunction, extend φ to B so that it is odd in the x_n-variable. If φ is a Neumann eigenfunction, extend φ to B so that it is even in the x_n-variable. Then the extended function φ is in $C^1(B) \cap H^2(B)$. Let ψ be a smooth function on $\mathbb{R}^2$ with compact support contained strictly inside B. By Green's identity,

$$\sum_{i,j=1}^{n} \int_B (D_j \varphi)(D_i \psi) J g^{ij} \, dx = \int_B \lambda^2 \varphi \psi J \, dx$$

That is,

$$\left(\sum_{i,j=1}^{n} D_i J g^{ij} D_j \varphi \right) + \lambda^2 J \varphi = 0$$

We can write this equation as

$$\left(\sum_{i,j=1}^{n} Jg^{ij} D_i D_j \varphi + (D_i J g^{ij}) D_j \varphi\right) + \lambda^2 J\varphi = 0$$

Now by Lemma 3.1.2, we can obtain U.

It remains to prove the second statement. Fix a point p in $(\Omega \setminus \Sigma) \cap \partial M$. It suffices to show that there is a neighborhood V of p in ∂M such that the Hausdorff dimension of $(\Omega \setminus \Sigma) \cap V$ is at most $n-2$. The set $\Omega \setminus \Sigma$ is a hypersurface with normal vector $\nabla\varphi(p)$ at p. Since φ is a Neumann eigenfunction and $\nabla\varphi(p) \neq 0$, the sets $\Omega \setminus \Sigma$ and ∂M intersect transversally, which yields V. □

In particular, it follows that ∂P is smooth almost everywhere, with respect to $\mathcal{H}$, so the divergence theorem and Green's identities hold on P. See, e.g., Evans and Gariepy [11]. Let η be the outward unit normal on ∂P, defined at these smooth points, and let ∂_η be the corresponding directional derivative. On $Z \setminus \Sigma$, we have

$$\eta = -\frac{\nabla\varphi}{|\nabla\varphi|}$$

At any point on $\partial M \cap \partial P$ where η is defined, we have

$$\eta = \nu$$

3.2 Proofs of Theorems 1.2.5 and 1.2.6

Proof of Theorem 1.2.6. By Green's identity,

$$\begin{aligned}
\int_P \Big((\Delta+\lambda^2)f\Big)|\varphi|\,dV &= \int_P \Big((\Delta+\lambda^2)f\Big)\varphi\,dV \\
&= \int_P f(\Delta+\lambda^2)\varphi\,dV - \int_{\partial P} f\partial_\eta\varphi\,dS + \int_{\partial P}\varphi\partial_\eta f\,dS \\
&= -\int_{\partial P\cap\partial M} f\partial_\eta\varphi\,dS - \int_Z f\partial_\eta\varphi\,dS + \int_{\partial P\cap\partial M}\varphi\partial_\eta f\,dS \\
&= \int_{\partial P\cap\partial M} f|\partial_\nu\varphi|\,dS + \int_Z f|\nabla\varphi|\,dS + \int_{\partial P\cap\partial M}|\varphi|\partial_\nu f\,dS
\end{aligned}$$

The last equality holds because $-\partial_\eta\varphi = |\partial_\nu\varphi|$ over $\partial P\cap\partial M$ and $-\partial_\eta\varphi = |\nabla\varphi|$ over $\partial P\cap Z$. We can similarly obtain

$$\int_N \Big((\Delta+\lambda^2)f\Big)|\varphi|\,dV = \int_{\partial N\cap\partial M} f|\partial_\nu\varphi|\,dS + \int_Z f|\nabla\varphi|\,dS + \int_{\partial N\cap\partial M}|\varphi|\partial_\nu f\,dS$$

Now

$$\begin{aligned}
\int_M \Big((\Delta+\lambda^2)f\Big)|\varphi|\,dV &= \int_P \Big((\Delta+\lambda^2)f\Big)|\varphi|\,dV + \int_N \Big((\Delta+\lambda^2)f\Big)|\varphi|\,dV \\
&= \int_{\partial M} f|\partial_\nu\varphi|\,dS + \int_{\partial M}|\varphi|\partial_\nu f\,dS + 2\int_Z f|\nabla\varphi|\,dS
\end{aligned}$$

□

The following lemma is an analogue of (1.2).

Lemma 3.2.1. *If φ is a Dirichlet or a Neumann eigenfunction, then*

$$\lambda^{\frac{1-n}{3}} \lesssim \|\varphi\|_{L^1(M)}$$

If the boundary is strictly geodesically concave and φ is a Dirichlet eigenfunction, then

$$\lambda^{\frac{1-n}{4}} \lesssim \|\varphi\|_{L^1(M)}$$

Proof. Fix p satisfying $2 < p < \frac{2(n+1)}{n-1}$. Then, by Smith [23],

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(n-1)(p-2)}{3p}} \tag{3.1}$$

If the boundary is strictly geodesically concave and φ is a Dirichlet eigenfunction, then by Grieser [12] and Smith and Sogge [25],

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(n-1)(p-2)}{4p}}$$

Let $\theta = \frac{p-2}{2(p-1)}$. By Hölder's inequality,

$$1 = \|\varphi\|_{L^2(M)} \leq \|\varphi\|_{L^1(M)}^{\theta} \|\varphi\|_{L^p(M)}^{1-\theta}$$

The estimates now follow. $\square$

Remark. On the flat unit disc $\{|x| \leq 1\}$ in $\mathbb{R}^2$, there are whispering gallery modes, which are concentrated near the boundary. It follows from Grieser [12] that Lemma 3.2.1 is sharp for these eigenfunctions. However, for $n \geq 3$, Smith and Sogge [26] conjectured that (3.1) can be strengthened to

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(3n-2)(p-2)}{12p}} \tag{3.2}$$

Applying Hölder's inequality as above would then yield

$$\lambda^{\frac{2-3n}{12}} \lesssim \|\varphi\|_{L^1(M)}$$

The following lemma is an analogue of (1.3).

Lemma 3.2.2. *If φ is a Dirichlet or Neumann eigenfunction, then*

$$\int_Z |\nabla\varphi|^2 \, dS \lesssim \lambda^3$$

Proof. This will follow from the identity

$$-\int_M \operatorname{sgn}(\varphi) \operatorname{div}\Big(|\nabla\varphi|\nabla\varphi\Big) \, dV = \int_{\partial M} |\partial_\nu\varphi|^2 \, dS + 2\int_Z |\nabla\varphi|^2 \, dS$$

We first prove this identity. Note that $-\partial_\eta\varphi = |\nabla\varphi|$ over $Z \setminus \Sigma$. If φ is a Dirichlet eigenfunction, then we also have $|\nabla\varphi| = -\partial_\eta\varphi = |\partial_\nu\varphi|$ at any point on $\partial P \cap \partial M$ where η is defined. By the divergence theorem,

$$\begin{aligned} -\int_P \operatorname{div}\Big(|\nabla\varphi|\nabla\varphi\Big) \, dV &= -\int_{\partial P} |\nabla\varphi|\partial_\eta\varphi \, dS \\ &= \int_{\partial P \cap \partial M} |\partial_\nu\varphi|^2 \, dS + \int_Z |\nabla\varphi|^2 \, dS \end{aligned}$$

Similarly,

$$\int_N \operatorname{div}\Big(|\nabla\varphi|\nabla\varphi\Big) \, dV = \int_{\partial N \cap \partial M} |\partial_\nu\varphi|^2 \, dS + \int_Z |\nabla\varphi|^2 \, dS$$

Adding these equations establishes the identity. Now we have

$$\begin{aligned} \int_Z |\nabla\varphi|^2 \, dS &\le \int_M \Big|\operatorname{div}\Big(|\nabla\varphi|\nabla\varphi\Big)\Big| \, dV \\ &\lesssim \|\varphi\|_{H^2(M)}\|\varphi\|_{H^1(M)} \\ &\lesssim \lambda^3 \end{aligned}$$

□

For a Dirichlet eigenfunction, we also need the following lemma.

Lemma 3.2.3. *If φ is a Dirichlet eigenfunction, then*

$$\left(\int_{\partial M} |\partial_\nu \varphi|^2 \, dS\right)^{1/2} \lesssim \lambda$$

This lemma follows from a much more general result obtained by Tataru [33]. There is also the following short proof.

Proof. Let X be a smooth first-order differential operator on M with $X = \partial_\nu$ over ∂M. Then, by Green's identity,

$$\begin{aligned}
\int_M u[X,\Delta]u \, dV &= -\lambda \int_M uXu \, dV - \int_M u\Delta Xu \, dV \\
&= \int_M (\Delta u)(Xu) \, dV - \int_M u\Delta Xu \, dV \\
&= \int_{\partial M} (\partial_\nu u)(Xu) \, dS \\
&= \int_{\partial M} |\partial_\nu u|^2 \, dS
\end{aligned}$$

Since $[X,\Delta]$ is a second-order differential operator, this yields

$$\begin{aligned}
\int_{\partial M} |\partial_\nu u|^2 \, dS &= \int_M u[X,\Delta]u \, dV \\
&\lesssim \|u\|_{L^2(M)} \|u\|_{H^2(M)} \\
&\lesssim \lambda^2
\end{aligned}$$

□

We can now prove Theorem 1.2.5.

Proof of Theorem 1.2.5. First assume φ is a Neumann eigenfunction. By Theorem 1.2.6, Lemma 3.2.2, and Cauchy-Schwarz,

$$\lambda^2 \int_M |\varphi| \, dV = 2\int_Z |\nabla \varphi| \, dS \lesssim \mathcal{H}(Z)^{1/2} \lambda^{3/2}$$

We can rewrite this as

$$\lambda\left(\int_M |\varphi|\, dV\right)^2 \lesssim \mathcal{H}(Z)$$

So by Lemma 3.2.1,

$$\lambda^{\frac{5-2n}{3}} \lesssim \mathcal{H}(Z)$$

Now assume φ is a Dirichlet eigenfunction. By Theorem 1.2.6, Lemma 3.2.2, Lemma 3.2.3, and Cauchy-Schwarz,

$$\lambda^2 \int_M |\varphi|\, dV = \int_{\partial M} |\partial_\nu \varphi|\, dS + 2\int_Z |\nabla\varphi|\, dS \lesssim \lambda + \mathcal{H}(Z)^{1/2}\lambda^{3/2}$$

We can rewrite this as

$$\lambda\left(\int_M |\varphi|\, dV\right)^2 \lesssim \mathcal{H}(Z) + \lambda^{-1}$$

Now applying Lemma 3.2.1 yields the desired estimates. □

Remark. If (3.2) is true, then we would have a better lower bound for the L^1 norm of φ. If φ is a Neumann eigenfunction, this would yield

$$\lambda^{\frac{8-3n}{6}} \lesssim \mathcal{H}(Z)$$

The same bound would hold if φ is a Dirichlet eigenfunction and $n \leq 4$.

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Vitae

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