Singular Behavior of Minimal Surfaces and Mean Curvature Flow

by

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Abstract

This document records three distinct theorems that the author proved, along with his collaborators, while a graduate student at The Johns Hopkins University. The author, with M. Calle and J. Kramer, generalized a sharp estimate of Tobias Colding and William Minicozzi for the extinction time of of convex hyper-surfaces in euclidean space moving my their mean curvature vector to a much broader class of evolutions studied by Ben Andrews in (1). Also, the author gave an alternate proof, first given by D. Hoffman and B. White in (17), of very poor limiting behavior for sequences of minimal surfaces in euclidean three space. Finally, the author, together with N. Moller in (23) constructed a new family o asymptotically f conical ends that satisfy the mean curvature self shrinking equation in euclidean three space of all dimensions.

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To my family

Because I love them so...

Chapter 1

Introduction

My research has been in the fields of minimal surfaces and mean curvature flow. I have three distinct results in these fields, recorded in (8), (24), and (23). In (24), I studied laminations in $\mathbb{R}^3$ by minimal surfaces, advancing the understanding of the famous structure theorem for minimal laminations of Colding and Minicozzi given in (12). In (8), with M. Calle and J. Kramer, I generalized a well known extinction estimate of Colding and Minicozzi for convex hypersurfaces moving by mean curvature to a very general class of geometric evolutions. Very recently, with N. Moller, I constructed a new family of rotationally symmetric surfaces that shrink homothetically under mean curvature flow, recorded in (23).

1.1 Overview of Results

In (24), I constructed a sequence of smooth minimal surfaces, embedded in a ball in $\mathbb{R}^3$, with very poor limiting behavior along a line segment. Generally one can consider a sequence of minimal surfaces Σ_k contained in balls B_k with $B_k \subset B_{k+1}$ in a manifold M. Setting $B = \cup_k B_k$, and defining $K \subset B$ to be the subset of B where the curvature of Σ_k blows up on a subsequence, one sees that (after passing to a subsequence) Σ_k converges to a smooth minimal lamination of $B \setminus K$. It is natural then to try to understand the structure of the set K and to understand when the lamination extends smoothly across K. In several cases, this is well understood.

In the case that $M = \mathbb{R}^3$ and $B = \mathbb{R}^3$, Colding and Minicozzi proved that if K is non empty then it is a Lipshitz curve and the lamination is a foliation by parallel planes. In particular, the lamination extends smoothly across K.

In the case that $M = \mathbb{R}^3$ and the sets B_k are balls centered at the origin with uniformly bounded radii, the limit lamination need not be so well behaved. In particular, Colding and Minicozzi constructed a sequence of properly embedded minimal disks in a fixed ball in $\mathbb{R}^3$ that converge to a non-proper limit lamination of $B \setminus \{x_3\text{-axis}\}$ that extends smoothly at every point away from the origin. Similar results were obtained by B. Dean and S. Khan in (14) and (22), respectively. D. Hoffman and B. White in (17) gave the first example with Cantor set singularities and of singular sets with non-integer Hausdorff dimension. Inspired by this result, I proved the same, but with different methods:

Theorem 1.1.1. *Let M be a compact subset of $\{x_1 = x_2 = 0, |x_3| < 1/2\}$ and let $C = \{x_1^2+x_2^2 = 1, |x_3| < 1/2\}$. Then there is a sequence of compact embedded minimal disks $\Sigma_k \subset C$ with $\partial\Sigma_k \subset \partial C$ and containing the vertical segment $\{(0,0,t)||t| < 1/2\}$ so:*

(A) $lim_{k\to\infty}|A_{\Sigma_k}|^2(p) = \infty$ for all $p \in M$.

(B) For any $\delta > 0$ it holds $sup_k sup_{\Sigma_k \setminus M_\delta}|A_{\Sigma_k}|^2 < \infty$ where $M_\delta = \cup_{p\in M} B_\delta(p)$.

(C) $\Sigma_k \setminus \{x_3 - axis\} = \Sigma_{1,k} \cup \Sigma_{2,k}$ for multi-vauled graphs $\Sigma_{1,k}$, $\Sigma_{2,k}$.

(D) For each interval $I = (t_1, t_2)$ in the compliment of M in the x_3-axis, $\Sigma_k \cap \{t_1 < x_3 < t_2\}$ converges to an imbedded minimal disk Σ_I with $\bar{\Sigma_I} \setminus \Sigma_I = C \cap \{x_3 = t_1, t_2\}$. Moreover, $\Sigma_I \setminus \{x_3 - axis\} = \Sigma_{1,I} \cup \Sigma_{2,I}$, for multi-valued graphs $\Sigma_{1,I}$ and $\Sigma_{2,I}$ each of which spirals in to the planes $\{x_3 = t_1\}$ from above and $\{x_3 = t_2\}$ from below.

Hoffman and White had used a variational approach in their paper, which has the advantage of being extremely flexible. However I followed (13) and (22) and constructed the sequence with explicit choices of Weierstrass data for each immersion which, while more complicated in the technicalities, is advantageous for the explicitness of the construction.

In a slightly different direction, Meeks and Weber found sequences of minimal surfaces converging to a minimal lamination with singular set equal to any C^1 curve (indeed in the local case, the singular set must lie on a C^1 curve). Nonetheless, it remains an open question as to whether Cantor set like singularities can occur along curves that are not line segments. Moreover, we have the following conjecture that generalizes Theorem 1.1.1:

Conjecture 1.1.2. *Any compact subset of a geodesic segment in a three manifold is the singular limit of a sequence of embedded minimal disks, if the geodesic segment is short enough*

In (23), N. Moller and I proved the following:

Theorem 1.1.3. *In $\mathbb{R}^3$ there exists a 1-parameter family of rotationally symmetric, embedded, self-shrinking, asymptotically conical ends Σ^2 with positive mean curvature, each being smooth with boundary $\partial\Sigma$ isometric to a scaled S^1 in the x_2x_3-plane centered at the origin.*

In fact for each symmetric cone $\mathcal{C}$ in $\{x_1 \geq 0\} \subseteq \mathbb{R}^3$ with tip at the origin, there is a unique such a self-shrinker, lying outside of $\mathcal{C}$, which is asymptotic to $\mathcal{C}$ as $x_1 \to \infty$.

Most solutions for mean curvature flow $\{\Sigma_t\}$ become singular in finite time, and thus cease to exist in a classical sense. In (19), Huisken showed that as long as the principal curvatures remain bounded, the flow remains smooth, hence the existence of a singularity implies curvature blowup for the family $\{\Sigma_t\}$. Also in (18), it was shown that if the maximum curvature blows up no faster than $c(T-t)^{-1/2}$, where T denotes the singular time for the flow, then the asymptotic shape of the solution will be that of a self-shrinking shrinker. As shown in (6), the study of rotationally symmetric self-shrinking surfaces reduces to the study of the following system in the

$x - r$ plane:

$$\begin{aligned}\dot{x} &= \cos\theta \\ \dot{r} &= \sin\theta \\ \dot{\theta} &= \frac{x}{2}\sin\theta + \left(\frac{n-1}{r} - \frac{r}{2}\right)\cos\theta\end{aligned}$$

where θ is the angle the tangent vector makes with the x-axis. When rotated about the x_1-axis in $\mathbb{R}^n$ (identified with the x-axis of the $x-r$-plane), the resulting surface of revolution shrinks by homotheties under mean curvature flow. For solutions graphical over the x-axis this becomes

$$u'' - \frac{x}{2}(1+u'^2)u' + \frac{1}{2}(1+u'^2)u = (n-1)\frac{1+u'^2}{u} \tag{1.1}$$

using elementary ODE theory we derive the following identity for any such solution u, graphical over an interval $[a, b]$

$$\begin{aligned}u(x) &= (n-1)\int_x^b \frac{1}{t^2}\int_t^b \frac{1+u'^2(s)}{u(s)} e^{-\int_t^s \frac{1}{2}(1+u'^2(z)dz} ds dt &(1.2)\\ &+ \frac{u(b)}{b}x + (u(b) - u'(b)b)x\int_x^b \frac{e^{-\int_t^b \frac{1}{2}(1+u'^2(z))dz}}{t^2}dt &(1.3)\end{aligned}$$

Without much difficulty, we then show that the quantity $(u(b) - u'(b)b)$ is always positive (which is equivalent to positivity of the mean curvature). This in turn is equivalent to the monotonicity of the ratios $\frac{u(b)}{b}$. Thus, for solutions u on an interval of the form $[a, \infty]$, we derive the expression

$$u(x) = (n-1)\int_x^\infty \frac{1}{t^2}\int_t^\infty \frac{1+u'^2(s)}{u(s)} e^{-\int_t^s \frac{1}{2}(1+u'^2(z)dz} ds + \sigma x \tag{1.4}$$

Existence follows from uniform gradient estimates derived from the formula. Though not eloquent to state, we characterize all graphs over half lines $[a, \infty]$, and in the process recover a theorem of Huisken:

Corollary 1 (Huisken's Theorem from (18))**.** *Let Σ^n be a smooth self-shrinking hypersurface of revolution, which is generated by rotating an entire graph around the x_1-axis. Then Σ^n is the cylinder $\mathbb{R} \times S^{n-1}$ of radius $\sqrt{2(n-1)}$ in $\mathbb{R}^{n+1}$.*

Several questions remain open. For example, we believe our methods are sufficient to classify all complete rotationally symmetric embedded self-shrinking surfaces, of which only the sphere and cylinder should be examples. Also, our result motivates study of the Dirichlet problem for the self shrinker equation: Given a curve $\gamma \subset \mathbb{R}^3$, when is there a surfaces Σ with $\partial\Sigma = \gamma$ satisfying the self-shrinker equation.

As an application of our theorem (and in fact the primary motivation for our study of this problem in the first place) we hope to construct self-shrinking solutions in $\mathbb{R}^3$ with high genus via gluing constructions developed by N. Kapoleas for de-singularizing intersections of minimal and CMC surfaces. In (26), X. Nguyen constructed an infinite genus surface that translates at constant velocity under mean curvature flow using techniques similar to those to Kapouleas. The analogous construction for self-shrinkers was begun in the articles (27), (28). The construction was never finished, in part because, compared with that in (26), the construction Nguyen attempted in the self-shrinking case lacks a free parameter. The one-parameter family of surfaces that N. Moller and I have discovered give Nguyen's construction an extra parameter.

The general philosophy of a gluing construction is to build an "initial surface" from surfaces you wish to glue together by removing a neighborhood of their intersection and smoothly stitching in a piece of a minimal surface in its place. The resulting initial surface is–depending upon the configuration of the pieces–a good approximation to the geometric problem that you wish to solve. If you wish to construct a new minimal surface in this manner for example then your initial surfaces ought to approximately solve the minimal surface equation. Let us call this initial surface X. We are interested in finding a perturbation $\tilde{X}$ of X that satisfies

$$H_{\tilde{X}}(p) = \frac{1}{2} p \cdot \nu_{\tilde{X}}(p)$$

where p is a point in $\tilde{X}$ and $H_{\tilde{X}}, \nu_{\tilde{X}}$ denote the mean curvature and normal vector to $\tilde{X}$, respectively. In co-dimension one, the problem then reduces to the search for a function $\phi : X \to \mathbb{R}$, giving the surface $\tilde{X}$ as a normal graph of ϕ over X. ϕ must solve the non-linear equation

$$\mathcal{Q}\phi = g^{ij}(D\phi(p))D_{ij}\phi(p) - p \cdot D\phi(p) + \phi(p) = 0$$

in the right function space, which turns out to be a Holder space with symmetries and decay. Unfortunately in general the linearized operator $\mathcal{L}$ for this equation is not invertible. In fact, for our initial surface, we expect the linearized operator to be a Fredholm operator of index -1 and trivial kernel. Our initial surface is asymptotically two conical ends (oriented in three space with axis on the x_1-axis) and a plane, with a piece of a singly periodic Scherck surface stitched in near the origin. The extra parameter–that is, the choice of the angle at the vertex of the cone–in our construction can be used to generate and control a function w not contained in the image of $\mathcal{L}$ by exploiting the "balancing condition" of (21). Thus, presently, N. Moller and I are working to gratefully extend the work done by X. Nguyen to prove

Conjecture 1.1.4. *For a positive integer g sufficiently large there exists a complete, properly embedded surface without boundary Σ_g that shrinks by homotheties under motion by mean curvature*

In (8), with Maria Calle and Joel Kramer, I considered a class of evolution equations of the form

$$\frac{\partial}{\partial t}\phi(x,t) = F(\lambda(x,t))\nu(x,t) \tag{1.5}$$

where $\phi : M \times [0,T) \to \mathbb{R}^{n+1}$ is a one parameter family immersions of a fixed manifold M of dimension n into $\mathbb{R}^{n+1}$ and where the velocity function F satisfies a natural set of conditions that, among other things ensure parabolicity of the evolution. In (8), we proved the following,

Theorem 1.1.5. *Let $\{M_t\}_{t\geq 0}$ be a one-parameter family of smooth closed and strictly convex hypersurfaces in R^{n+1} satisfying $\frac{d}{dt}M_t = F\nu_t$, where ν_t is the unit normal of M_t and where F satisfies conditions stated in (1). Let $W(t)$ denote the width of the hypersurfaces M_t. Then in the sense of limsup of forward difference quotients it holds:*

$$\frac{d}{dt} \leq -\frac{C_0 4\pi}{n}, \tag{1.6}$$

and

$$W(t) \leq W(0) - \frac{C_0 4\pi}{n} \tag{1.7}$$

This theorem was motivated by the celebrated result of Colding and Minicozzi in (10) concerning the extinction time of finite volume metrics with positive Ricci curvature evolving by Ricci Flow. Similar techniques were used in a follow up paper by Colding and Minicozzi that proved an analogous result for mean curvature flow. The success of their methods towards Ricci and mean curvature flow led naturally to the question of how generally they could be applied. Both Ricci flow and mean curvature flow belong to a wide class of parabolic equations called "geometric" or "curvature flows". A rather large subclass of these flows (containing mean curvature flow as an example), had been studied by B. Andrews in (1), where Huisken's famous theorem for mean curvature flow had been generalized; that is, convex surfaces moving by these evolutions collapse in a round point in a finite amount of time.

Chapter 2

Background Material

2.1 The Volume Functional, Minimal Surfaces and the Weierstrass Representation

In differential geometry, one of the most natural and often utilized notions is that of the volume of a submanifold, and one of the most natural and studied classes of submanifold are those that, locally at least, minimize volume. More precisely, a submanifold Σ of a manifold M is minimal if its volume is unchanged by infinitesimal deformations with compact support. One sees that if Σ is twice differentiable–a very natural geometric assumption–then Σ is minimal if and only if its mean curvature vector $\vec{H}_\Sigma$ is identically zero. For a parametric surface $\Phi : M \to N$, the volume of M is computed by integrating the pullback of the volume form of N by Φ over M. In local coordinates x_i, the induced volume form has the expression $dV^M = \det g_{ij} dx^i \wedge dx^j$, where $g_{ij} = h(d\Phi(\frac{\partial}{\partial x_i}), d\Phi(\frac{\partial}{\partial x_j}))$, and h is the metric on N. Let $\mathcal{X}$ denote the space of paremetric surfaces of a fixed dimension (say k) in N. Then we aim to compute the first variation of the functional $\mathrm{Vol}^k : \mathcal{X} \to \mathbb{R}$ given by $\mathrm{Vol}^k(\{\Phi : M \to N\}) = \int_M \Phi^*(dV^N)$, where dV^N is the volume form on N. Thus, considering a smooth local perturbation of Φ (that is, a one parameter family of immersions $\Phi_t : M \to \Sigma_t \subset N$ such that the vector field $d\Phi_t(\frac{\partial}{\partial t})$ has compact support) and setting $V(t) = \mathrm{Vol}^k(\{\Phi_t : M \to N\})$, we see that the infinitesimal change in volume is equal to the time derivative of the volume form $\Phi_t^*(dV^N)$ integrated over M. In local coordinates, we get $\frac{d}{dt}V(t) = \int_\Omega \frac{d}{dt}\det\, g_{ij}dx$, where Ω is a coordinate patch. We can assume that at $p \in \Omega$ the metric is the identity matrix at time zero (geodesic normal coordinates at p). The cofactor expansion of the determinant gives that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\det\, g_{ij} &= \frac{d}{dt}\sum_{j=1}^{k}(-1)^j g_{1j}\det\ \{c_{1j}(g)\} \\ &= g_{11,t}\det\ \{c_{11}(g)\} + g_{11}\det\ \{c_{11}(g)\}_t \\ &= g_{11,t} + \det\ \{c_{11}(g)\}_t \end{aligned}$$

Here, $c_{ij}(g)$ denotes the $(i,j)-$th cofactor matrix obtained from g. Proceeding inductively, we see that, at p (and at $t = 0$), we get $\frac{d}{dt}\det\, g_{ij} = \text{trace}\ g_{ij,t}$. Recall that at p, we have normalized the determinant at one; it is easy to check that in general we have $\frac{d}{dt}\det\, g_{ij} = \det\, g_{ij}\text{trace}\ g_{ij,t}$. We then compute

$$\begin{aligned} \frac{d}{dt}g_{ij}\Big|_{t=0} &= \frac{d}{dt}h\left\{d\Phi(\frac{\partial}{\partial x_i}), d\Phi(\frac{\partial}{\partial x_j})\right\} \\ &= h\left\{\nabla_{d\Phi(\frac{\partial}{\partial t})}d\phi(\frac{\partial}{\partial x_i}), d\Phi(\frac{\partial}{\partial x_j})\right\} + h\left\{\nabla_{d\Phi(\frac{\partial}{\partial t})}d\Phi(\frac{\partial}{\partial x_j}), d\Phi(\frac{\partial}{\partial x_i})\right\} \\ &= -2h\left\{d\Phi(\frac{\partial}{\partial t}), \nabla_{d\phi(\frac{\partial}{\partial x_i})}d\Phi(\frac{\partial}{\partial x_j})\right\} \end{aligned}$$

Combining this with our previous computations, we see that

$$\frac{d}{dt}\det\, g_{ij} = -2h\left(d\Phi(\frac{\partial}{\partial t}), \sum_i \nabla_{d\phi(\frac{\partial}{\partial x_i})}d\Phi(\frac{\partial}{\partial x_i})\right)\det\, g_{ij} \tag{2.1}$$

where we have used that the coordinate vector fields $\left\{\frac{\partial}{\partial x_i}\right\}$ are orthonormal at p at time zero, the compatibility of the connection ∇ with the metric h, and the fact the coordinate vector fields are torsion free. The quantity $\sum_i \nabla_{d\phi(\frac{\partial}{\partial x_i})}d\Phi(\frac{\partial}{\partial x_i})$ is the mean curvature vector of the immersion at p, and is denoted by $\vec{H}$. Thus, we see that an immersion $\{\Phi : M \to N\}$ is stationary under normal perturbations for the volume functional Vol^k if and only if the mean curvature vector vanishes at every point, in which case we have $\frac{d}{dt}V(t) = 0$ for all compactly supported smooth normal perturbations. The mean curvature vector field for any twice differentiable surface is defined at all points on the surface, and is normal to the surface at each point. In an intuitive sense, it measures the net bending of the surfaces in all directions. For example, the mean curvature vector for a sphere is everywhere inward pointing with length equal to the dimension; as expected the mean curvature vector of a plane is everywhere zero, and there exist many non constant saddle-type surfaces that also have everywhere vanishing mean curvature vector, due to competing curvatures pulling in opposite directions. Hence the study of twice differentiable mimimal submanifolds reduces to the the study of the equation $\vec{H} = 0$. In the case that Σ is a hypersurface, this equation is equivalent to a system of degenerate elliptic PDEs for the coordinate functions of the embedding. For a C^2 immersion $\Phi : M \to N$ in which M is an orientable hypersurface, the unit normal vector field $\nu \in \Gamma(T^{\perp}M)$ is well-defined (up to a sign) and differentiable. In this case, $H = \vec{H} \cdot \nu$ is called the "mean curvature" of the immersion. Also, the eigenvalues of the second fundamental form of the immersion are called the "principal curvatures" and are denoted by $\lambda_1 \cdots \lambda_n$, where n is the dimension of M.

Given a domain $\Omega \subset \mathbb{C}$, a meromorphic function g on Ω and a holomorphic one-form ϕ on Ω, one obtains a (branched) conformal minimal immersion $F : \Omega \to \mathbb{R}^3$, given by (c.f. (29))

$$F(z) = Re\left\{\int_{\zeta\in\gamma_{z_0,z}} \left(\frac{1}{2}\left(g^{-1}(\zeta) - g(\zeta)\right), \frac{i}{2}\left(g^{-1}(\zeta) + g(\zeta)\right), 1\right)\phi(\zeta)\right\} \tag{2.2}$$

the so-called Weierstrass representation associated to Ω, g, ϕ. The triple $(\Omega,\ g,\ \phi)$ is referred to as the Weierstrass data of the immersion F. Here, $\gamma_{z_0,z}$ is a path in Ω connecting z_0 and z. By requiring that the domain Ω be simply connected, and that g be a non-vanishing holomorphic function, we can ensure that $F(z)$ does not depend on the choice of path from z_0 to z, and that $dF \neq 0$. Changing the base point z_0 has the effect of translating the immersion by a fixed vector in $\mathbb{R}^3$.

The unit normal $\mathbf{n}$ and the guass curvature of the resulting surface are then (see sections 8, 9 in (29))

$$\mathbf{n} = \left(2\mathrm{Re}\, g, \mathrm{Im}\, g, |g|^2 - 1\right) / \left(|g|^2 + 1\right), \tag{2.3}$$

$$K = -\left[\frac{4|\partial_z g||g|}{|\phi|(1+|g|^2)^2}\right]. \tag{2.4}$$

Since the pullback $F^*(dx_3)$ is $\mathrm{Re}\,\phi$, ϕ is usually called the *height differential.* The two standard examples are

$$g(z) = z, \phi(z) = dz/z, \Omega = \mathbb{C} \setminus \{0\} \tag{2.5}$$

giving a catenoid, and

$$g(z) = e^{iz}, \phi(z) = dz, \Omega = \mathbb{C} \tag{2.6}$$

giving a heliciod.

We will always write our non-vanishing holomorphic function g in the form $g = e^{ih}$, for a potentially vanishing holomorphic function h, and we will always take $\phi = dz$. For such Weierstrass data, the differential dF may be expressed as

$$\partial_x F = (\sinh v \cos u, \sinh v \sin u, 1), \tag{2.7}$$

$$\partial_y F = (\cosh v \sin u, -\cosh v \cos u, 0). \tag{2.8}$$

2.2 Mean Curvature Flow

The parabolic analogue of the minimal surface equation, often written schematically $\frac{d}{dt}x = \vec{H}(x)$, is known as mean curvature flow, justified by the fact that solutions to this equation are one-parameter families of immersed submanifolds $\{\Sigma_t\}$ that evolve in the direction of their mean curvature vector. For our purposes we will assume that the submanifolds are actually hypersurfaces. More explicitly, a family of immersions $\{\Phi_t : M \to N\}$ of a manifold M in to a manifold N (of dimension n) is a mean curvature flow if $\frac{\partial}{\partial t}\Phi_t(p)^{\perp} = \vec{H}_M(p)$; that is, the tangential velocity of a point does not play a role. From a geometric point of view, this is a consequence of the fact that the sets $\Sigma_t := \Phi_t(M)$ are determined solely by the normal component of vector field $V(t,p) := \frac{\partial}{\partial t}\Phi_t(p)$. An easy way to see this is as follows: Take a point (t,p) where $V(t,p)^{\perp}$ doesn't vanish. At such a point, the differential $d\Phi$ is non-singular, and hence there is an open set $\mathcal{U} = (-\epsilon + t, \epsilon + t) \times U$ $(U \subset M)$ such that $\Phi|_{\mathcal{U}}$ is a diffeomorphism onto its image $\phi(\mathcal{U}) := \Lambda$. We can also assume that $\mathcal{U}$ is small enough so that its image under Φ is contained in a single coordinate system in N so that we can regard Λ as a subset of euclidean space. We may then define a function $F : \Lambda \to \mathbb{R}$ by the condition $F(q) = \tau$ if and only if q is in the hypersurface Σ_τ. By construction, we have, for p in M, that $\Phi_t(p)$ is in Σ_t, and thus we get $F \circ \Phi_t(p) = t$. Differentiating this equation in t gives that $\nabla F \cdot \frac{\partial}{\partial t}\Phi_t(p) = 1$. Since, by construction, the sets $\Sigma_t \cap \Lambda$ are level sets for the function F, we have that $\nabla F : \mathcal{U} \to \mathbb{R}^n$ is non vanishing normal vector field at every point; in particular, a quick computation gives that $\nabla F = \frac{1}{|\vec{H}|^2}\vec{H}$. Moreover, if $T : \mathcal{U} \to \mathbb{R}^n$ is any tangential vector field (that is, $T(t,p)$ is in the tangent space $T_{\Phi_t(p)}\Sigma_t$ for all t and p), we can evolve points on our initial surface $\Sigma_t \cap \Lambda$ by the vector field $V_T(q) = \frac{\partial}{\partial t}\Phi_t(p) + T(t,p)$ (where the image of the point (t,p) under Φ is q); that is, for $q \in \Sigma_t \cap \Lambda$, we can consider the gradient flows $q(\tau)$, and $q_T(\tau)$ for the vector fields V and V_T respectively. The above computation gives that both $q(\tau)$ and $q_T(\tau)$ are in the level set $\Sigma_{t+\tau}$ (at least, for small τ, so that the gradient flow is defined). Now at a point interior to the closed set $\{(t,p)|V^{\perp}(t,p) = 0\}$, the surface Σ_t is stationary on an open set under the flow, and hence here we have $\Sigma_t \cap \Lambda = \Sigma_{\tau+t}$ for small τ. Thus, the evolutions given by the two vector fields V and V_T agree up to level sets and up to a set of measure zero. Continuity then says that the level sets agree wherever they are defined.

Mean curvature flow has the natural informal interpretation as the gradient flow for the area functional on submanifolds, i.e., moving a surface in the direction of its mean curvature vector decreases its volume (in the compact case) most rapidly. An easy example of a mean curvature flow that illustrates general phenomena is that of homothetically shrinking concentric spheres, which shrink to a point after a finite amount of time. In general, solutions to mean curvature flow become singular, and it has been a big challenge of the field to understand the nature of the singularities. The regularity theory for mean curvature flow is a vital, intensely researched field that has seen dramatic advances of late. In particular, there has been much recent success in the construction and understanding of various special solutions, so called translating and self-shrinking solutions. Translating and self-shrinking solutions arise as blowup solutions for mean curvature flow. That is, the asymptotic shape of a singularity in mean curvature flow is that of a self shrinking or translating solution. Thus, these special solutions are vital to the understanding the structure of singularities for mean curvature flow.

2.3 Singularities of Mean Curvature Flow and the Equation for Self Shrinkers

As mentioned, most solutions for mean curvature flow $\{\Sigma_t\}$ become singular in finite time, and thus cease to exist in a classical sense. In (19), Huisken showed that as long as the principal curvatures remain bounded, the flow remains smooth, hence the existence of a singularity implies curvature blowup for the family $\{\Sigma_t\}$. Also in (18), it was shown that if the maximum curvature blows up no faster than $c(T-t)^{-1/2}$, where T denotes the singular time for the flow, then the asymptotic shape of the solution will be that of a self-shrinking solution; that is, a sequence of rescaled solutions will converge to hyper-surface satisfying the equation

$$H = c\nu \cdot X \tag{2.9}$$

where H denotes the mean curvature vector, ν denotes the unit normal, X denotes the position vector in $\mathbb{R}^{n+1}$ with respect to an appropriate origin, and c is a positive constant.

2.4 Sweepouts and the notion of Width.

In this section we recall some notions of (9). In (9), Colding and Minicozzi gave a sharp estimate for the extinction time of a convex hypersurface of $\mathbb{R}^n$ moving by mean curvature in terms of an invariant called the "width" which we now define. The following is adapted from (9):

Let $\Phi_t : M \to M_t \subset \mathbb{R}^{n+1}$ be a continuous one parameter family of hypersurfaces. Take $\mathcal{P} = B^{n-1}$, and define Ω_t to be the set of continuous maps $\sigma : \mathbb{S}^1 \times \mathcal{P} \to M_t$ such that for each $s \in \mathcal{P}$ the map $\sigma(\cdot, s)$ is in $W^{1,2}$, such that the map $s \to \sigma(\cdot, s)$ is continuous from $\mathcal{P}$ into $W^{1,2}$, and finally such that $\sigma(\cdot, s)$ is a constant map for all $s \in \partial\mathcal{P}$. We will refer to elements of the set Ω_t as "sweepouts" of the manifold M_t. Given a sweepout $\hat{\sigma} \in \Omega_t$ representing a non-trivial homotopy class in $\pi_n M$, the homotopy class $\Omega_t^{\hat{\sigma}}$ is defined to be the set of all maps σ that are homotopic to $\hat{\sigma}$ through Ω_t. We then define the width, $W(t)$, as follows: Fix a sweepout $\beta \in \Omega_t$ representing a non-trivial homotopy class in M and let $\beta_t \in \Omega_t$ denote the corresponding sweepout of M_t. We then take

$$W(t) = W(t, \beta) = \inf_{\sigma \in \Omega_t^{\beta_t}} \max_{s \in \mathcal{P}} \text{Energy } (\sigma(\cdot, s)) \tag{2.10}$$

Here, Energy $(\sigma(\cdot, s))$ is the $W^{1,2}$-semi norm of the function $\sigma(\cdot, s)$ given by Energy $(\sigma(\cdot, s)) = \int_{\mathbb{S}^1} |\frac{d}{d\zeta}\sigma(\zeta, s)|^2 d\zeta$. We note that the width is always positive for a manifold with non-trivial fundamental group.

In the proof of the main theorem of (8), we will rely heavily on the following theorem of Colding and Minicozzi from (9):

Lemma 2. *For each t, there exists a family of sweepouts $\{\gamma^j\} \in \Omega_t$ of M_t that satisfy the following:*

1. *The maximum energy of the slices $\gamma^j(\cdot, s)$ converges to $W(t)$.*
2. *For each $s \in \mathcal{P}$, the maps $\gamma^j(\cdot, s)$ have Lipschitz bound L independent of both j and s.*
3. *Almost maximal slices are almost geodesics: That is, given $\epsilon > 0$, there exists $\delta > 0$ such that if $j > 1/\delta$ and Energy$(\gamma^j(\cdot, s)) > W(t) - \delta$ for some s, then there exists a non-constant closed geodesic η in M_t such that dist$(\eta, \gamma^j(\cdot, s)) < \epsilon$.*

The existence of such a family of sweepouts is established in (9), Theorem 1.9, and we will not include the proof here. The distance in the third statement of the previous Lemma is given by the $W^{1,2}$ norm on the space of maps from $\mathbf{S}^1$ to M.

Chapter 3

Width and Mean Curvature Flow

It is well known that a convex closed hypersurface in $\mathbf{R}^{n+1}$ evolving by its mean curvature remains convex and vanishes in finite time. In (1), Ben Andrews showed that the same holds for much broader class of evolutions. T. H. Colding and W. P. Minicozzi in (9) give a bound on extinction time for mean curvature flow in terms of an invariant of the initial hypersurface that they call the width. In this paper, we generalize this estimate to the class evolutions considered by Andrews:

Theorem 3. *Let $\{M_t\}_{t\geq 0}$ be a one-parameter family of smooth compact and strictly convex hypersurfaces in $\mathbf{R}^{n+1}$ satisfying $\frac{d}{dt}M_t = F\nu_t$, where ν_t is the unit normal of M_t and where F satisfies conditions 3.1 in (1). Let $W(t)$ denote the width of the hypersurface M_t. Then in the sense of limsup of forward difference quotients it holds:*

$$\frac{d}{dt}W \leq -\frac{C_0 4\pi}{n}, \tag{3.1}$$

and

$$W(t) \leq W(0) - \frac{C_0 4\pi}{n}t. \tag{3.2}$$

All functions F in this class are assumed to be either concave or convex. In the concave case, we require an a priori pinching condition on the initial hypersurface M_0. In the case of convexity a pinching condition is not needed, and we can take $C_0 = 1$ above. This class is of interest since it contains many classical flows as particular examples, such as the n-th root of the Gauss curvature, mean curvature, and hyperbolic mean curvature flows. The key to arriving at our estimate on extinction time is a uniform estimate on the rate of change of the width, for which preservation of convexity of the evolving hypersurface is fundamental.

We also study flows which are similar to those considered by Andrews in (1), except that we allow for higher degrees of homogeneity. The motivation for this consideration is that the degree 1 homogeneity condition on the flows given in Andrews' paper excludes some naturally arising evolutions, such as powers of mean curvature, for which preservation convexity and extinction time estimates are known (see (30)). An analogous result as (3.1) holds for the time zero derivative of width of an initially convex hypersurface for these kinds of flows. However in the general case it is, to our knowledge, unknown as to whether convexity is preserved, or even if there is a well-defined extinction time. Consequently, nothing can be said about the long term behavior of the width of a hypersurface evolving under these flows.

We conclude by taking the 2-width to be, loosely speaking, the area of the smallest homotopy 2-sphere required to pull over the surface, and we show that an analogous estimate on the derivative of the 2-width holds, and that in fact in this case the convexity assumption can be loosened to 2-convexity.

3.1 Contraction of Convex Hypersurfaces in $\mathbf{R}^{n+1}$

Let M^n be a smooth, closed manifold of dimension $n \geq 2$. Given a smooth immersion $\phi_0 : M^n \to \mathbf{R}^{n+1}$, we consider a smooth family of immersions $\phi : M^n \times [0, T) \to \mathbf{R}^{n+1}$ satisfying an equation of the following form:

$$\begin{aligned} \frac{\partial}{\partial t}\phi(x,t) &= F(\lambda(x,t))\nu(x,t) \\ \phi(x,0) &= \phi_0(x) \end{aligned} \tag{3.3}$$

for all $x \in M^n$ and $t \in [0, T)$. In this equation $\nu(x,t) = H_{M_t}/|H_{M_t}|$ is the inward pointing unit normal to the hypersurface $M_t := \phi(M^n, t) = \phi_t(M^n)$ at the point $\phi_t(x)$, and $\lambda(x,t) = (\lambda_1(x,t), ..., \lambda_n(x,t))$ are the principal curvatures of M_t at $\phi_t(x)$, that is, the eigenvalues of the Weingarten map of the immersion at that point.

We consider velocity functions F satisfying the following hypotheses, taken from (1):

Conditions 1. *1. $F : \Gamma_+ \to \mathbf{R}$ is defined on the positive cone $\Gamma_+ = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{R}^n : \lambda_i > 0, i = 1 \dots n\}$.*

2. F is a smooth symmetric function.

3. F is strictly increasing in each argument: $\frac{\partial F}{\partial \lambda_i} > 0$ for $i = 1 \dots n$ at every point in Γ_+.

4. F is homogeneous of degree one: $F(c\lambda) = cF(\lambda)$ for any positive $c \in \mathbf{R}$.

5. F is strictly positive on Γ_+, and $F(1, \dots, 1) = 1$.

6. one of the following holds:

(a) F is convex; or

(b) F is concave and either

i. $n = 2$;

ii. f approaches zero on the boundary of Γ_+; or

iii. $\sup_{x \in M} \frac{H}{F} < \liminf_{\lambda \in \partial\Gamma_+} \frac{\Sigma\lambda_i}{F(\lambda)}$

Given an initially convex smooth immersion $\phi_0 : M^n \to \mathbf{R}^{n+1}$ and a function F satisfying conditions 1 , B. Andrews proved (Corollary 3.6, Theorem 4.1 and Theorem 6.2 in (1)) that there exists a unique family of smooth immersions $\phi(x,t) : M^n \times [0,T) \to \mathbf{R}^{n+1}$ satisfying equations (3.3) for all $x \in M^n$. Moreover, he proved that the embeddings M_t remain convex and converge uniformly to a point as $t \to T$.

3.2 Width and the existence of Good Sweepouts

In this section we recall some notions of (9).

Convexity of the immersion $\phi_t : M \to M_t \subset \mathbf{R}^{n+1}$ implies that M is diffeomorphic to the n-sphere $\mathbf{S}^n$, via the Gauss map. $\mathbf{S}^n$ is then equivalent to $\mathbf{S}^1 \times B^{n-1}/\sim$, where

$\sim$ is the equivalence relation $(\theta_1, y) \sim (\theta_2, y)$ where $\theta_1, \theta_2 \in \mathbf{S}^1$ and $y \in \partial B^{n-1}$. Here B^{n-1} is the unit ball in $\mathbf{R}^{n-1}$. We use this decomposition of $M = \mathbf{S}^n$ to define the width $W(t)$ of the immersion ϕ_t.

Take $\mathcal{P} = B^{n-1}$, and define Ω_t to be the set of continuous maps $\sigma : \mathbf{S}^1 \times \mathcal{P} \to M_t$ such that for each $s \in \mathcal{P}$ the map $\sigma(\cdot, s)$ is in $W^{1,2}$, such that the map $s \to \sigma(\cdot, s)$ is continuous from $\mathcal{P}$ into $W^{1,2}$, and finally such that $\sigma(\cdot, s)$ is a constant map for all $s \in \partial\mathcal{P}$. We will refer to elements of the set Ω_t as "sweepouts" of the manifold M_t. Given a sweepout $\hat{\sigma} \in \Omega_t$ representing a non-trivial homotopy class in $\pi_n M$, the homotopy class $\Omega_t^{\hat{\sigma}}$ is defined to be the set of all maps σ that are homotopic to $\hat{\sigma}$ through Ω_t. We then define the width, $W(t)$, as follows: Fix a sweepout $\beta \in \Omega_t$ representing a non-trivial homotopy class in M and let $\beta_t \in \Omega_t$ denote the corresponding sweepout of M_t. We then take

$$W(t) = W(t, \beta) = \inf_{\sigma \in \Omega_t^{\beta_t}} \max_{s \in \mathcal{P}} Energy(\sigma(\cdot, s)) \tag{3.4}$$

We note that the width is always positive until extinction time.

In the proof of our main theorem, we will rely heavily on the following

Lemma 4. *For each t, there exists a family of sweepouts $\{\gamma^j\} \in \Omega_t$ of M_t that satisfy the following:*

1. *The maximum energy of the slices $\gamma^j(\cdot, s)$ converges to $W(t)$.*

2. *For each $s \in \mathcal{P}$, the maps $\gamma^j(\cdot, s)$ have Lipschitz bound L independent of both j and s.*

3. *Almost maximal slices are almost geodesics: That is, given $\epsilon > 0$, there exists $\delta > 0$ such that if $j > 1/\delta$ and $Energy(\gamma^j(\cdot, s)) > W(t) - \delta$ for some s, then there exists a non-constant closed geodesic η in M_t such that $dist(\eta, \gamma^j(\cdot, s)) < \epsilon$.*

The existence of such a family of sweepouts is established in (9), Theorem 1.9, and we will not include the proof here. The distance in the third statement of the previous Lemma is given by the $W^{1,2}$ norm on the space of maps from $\mathbf{S}^1$ to M.

3.3 F convex

We assume the velocity function F satisfies conditions (1) through (5) above, in addition to condition $(6a)$.

We then have the following:

Lemma 5. *For all* $(\lambda_1, ..., \lambda_n) \in \Gamma_+$*, it holds*

$$F(\lambda_1, ..., \lambda_n) \geq \frac{\sum_1^n \lambda_i}{n}.$$

Proof. Let $\lambda = (\lambda_1, ..., \lambda_n)$. Let S_n the group of permutations of $(1, ..., n)$, and let $\lambda_\sigma = (\lambda_{\sigma(1)}, ..., \lambda_{\sigma(n)})$ for each $\sigma \in S_n$. Then, since F is symmetric, $F(\lambda_\sigma) = F(\lambda)$ for all $\sigma \in S_n$.

Let $h = \frac{\sum_1^n \lambda_i}{n}$, let $\tilde{h} = (h, ..., h) \in \Gamma_+$. The we have the following:

$$\begin{aligned} \sum_{\sigma \in S_n} \lambda_\sigma &= (\sum_{\sigma \in S_n} \lambda_{\sigma(1)}, ..., \sum_{\sigma \in S_n} \lambda_{\sigma(n)}) \\ &= |S_{n-1}|(\sum_1^n \lambda_i, ..., \sum_1^n \lambda_i) = n|S_{n-1}|\tilde{h} = |S_n|\tilde{h}, \end{aligned} \tag{3.5}$$

where the last inequality follows from $|S_n| = n! = n(n-1)! = n|S_{n-1}|$. Then, since $\frac{1}{|S_n|} < 1$, by convexity we have:

$$F(\tilde{h}) \leq \frac{1}{|S_n|} \sum_{\sigma \in S_n} F(\lambda_\sigma) = \frac{|S_n|}{|S_n|} F(\lambda) = F(\lambda). \tag{3.6}$$

But then, by the forth and fifth property of F we have that $h = hF(1, ..., 1) = F(\tilde{h})$, and therefore we obtain the result. $\square$

We can then prove the following estimate (corollary 2.9 in (9)):

Lemma 6. *Let* $\Sigma \subset M_0$ *be a closed geodesic and* $\Sigma_t \subset M_t$ *the corresponding evolving closed curve. Let* E_t *be the energy of* Σ_t*. Then:*

$$\frac{d}{dt}_{t=0} E_t \leq -\frac{4\pi}{n}. \tag{3.7}$$

Proof. Observe that, since M_0 is convex and Σ is minimal, H_Σ points in the same direction as ν. By convexity, we have that $|H_\Sigma| \leq |H_{M_0}|$. Let V_t be the length of the closed curve Σ_t. Then, by the first variation formula for volume (see 9.3 and 7.5' in (31)) we have the following:

$$\begin{aligned} \frac{d}{dt}_{t=0} V_t &= -\int_\Sigma \langle H_\Sigma, F(x,t)\nu(x,t)\rangle \\ &\leq -\frac{1}{n}\int_\Sigma |H_\Sigma||H_{M_0}| \leq -\frac{1}{n}\int_\Sigma |H_\Sigma|^2. \end{aligned} \tag{3.8}$$

Here the first inequality follows from Lemma 5, and the second inequality follows from $0 \leq |H_\Sigma| \leq |H_{M_0}|$. Then we can compute the variation of energy as follows:

$$\pi \frac{d}{dt}_{t=0} E_t = V_0 \frac{d}{dt}_{t=0} V_t \leq -\frac{V_0}{n}\int_\Sigma |H_\Sigma|^2 \leq -\frac{1}{n}\left(\int_\Sigma |H_\Sigma|\right)^2 \leq -\frac{4\pi^2}{n}. \tag{3.9}$$

Here the first inequality follows from (3.8), the second from the Cauchy-Schwarz inequality, and the last inequality follows since by Borsuk-Fenchel's theorem every closed curve in $\mathbf{R}^{n+1}$ has total curvature at least 2π (see (7), (16)). $\square$

3.4 F concave

We assume the velocity function F satisfies conditions (1) through (5) above, in addition to condition $(6b)$. We also require that the initial convex immersion $\phi_0 : M \to \mathbf{R}^{n+1}$ satisfies an a priori pinching condition

$$max\{\lambda_1(x,0), ..., \lambda_n(x,0)\} \leq C_0 min\{\lambda_1(x,0), ..., \lambda_n(x,0)\} \tag{3.10}$$

for all $x \in M$. Here we have chosen an orientation on M so that the sign of the principle curvatures is positive. We then have

$$\sup_{x\in M_0} \frac{|H_{M_0}|}{nF} = C_0. \tag{3.11}$$

We then have the following:

Lemma 7. *For all $t \geq 0$,*

$$\sup_{x\in M_t} \frac{|H_{M_t}|}{nF} \leq \sup_{x\in M_0} \frac{|H_{M_0}|}{nF} =: C_0.$$

Proof. By the parabolic maximum principle. See proof of Theorem 4.1 in (1). □

We can then prove the following estimate:

Lemma 8. *Let $\Sigma \subset M_0$ be a closed geodesic and $\Sigma_t \subset M_t$ the corresponding evolving closed curve. Let E_t be the energy of Σ_t. Then:*

$$\frac{d}{dt}_{t=0} E_t \leq -\frac{4\pi}{nC_0}. \tag{3.12}$$

Proof. The proof follows exactly as in the proof of Lemma 6, except that here Lemma 7 gives the inequality

$$\frac{d}{dt}_{t=0} V_t \leq -\frac{1}{nC_0} \int_\Sigma |H_\Sigma|^2 \tag{3.13}$$

We then have as before

$$\pi \frac{d}{dt}_{t=0} E_t = V_0 \frac{d}{dt}_{t=0} V_t \leq -\frac{V_0}{nC_0} \int_\Sigma |H_\Sigma|^2 \leq -\frac{1}{nC_0} \left(\int_\Sigma |H_\Sigma| \right)^2 \leq -\frac{4\pi^2}{nC_0}. \tag{3.14}$$

3.5 Extinction Time

□

We can now prove our main theorem:

Theorem 9. *Let $\{M_t\}_{t\geq 0}$ be a one-parameter family of smooth compact and strictly convex hypersurfaces in $\mathbf{R}^{n+1}$ flowing by equation (3.3). Let $C_0 = 1$ if F is convex, and C_0 as in Lemma 7 if F is concave. Then in the sense of limsup of forward difference quotients it holds:*

$$\frac{d}{dt} W \leq -\frac{4\pi}{nC_0}, \tag{3.15}$$

and

$$W(t) \leq W(0) - \frac{4\pi}{nC_0} t. \tag{3.16}$$

As a consequence, we get the following bound on the extinction time:

Corollary 10. *Let $\{M_t\}_{t\geq 0}$ be as above, then it becomes extinct after time at most*

$$\frac{nC_0 W(0)}{4\pi}.$$

Observe that in the concave case, the constant C_0 depends on the pinching of the initial hypersurface M_0, whereas in the convex case the bound on the extinction time is independent of the evolving hypersurface.

We will need the following consequence of the first variation formula for energy in the proof of our main theorem: If σ_t, η_t are two families of curves evolving by a C^2 vector field $\mathbf{V}$, then

$$|\frac{d}{dt}Energy(\eta_t) - \frac{d}{dt}Energy(\sigma_t)| \leq C||\mathbf{V}||_{C^2}||\sigma_t - \eta_t||_{W^{1,2}}(1 + \sup|\sigma_t'|^2) \qquad (3.17)$$

Proof of Theorem 9. Here we follow the outline of Theorem 2.2 in (9)

Fix a time τ. Throughout the proof, C will denote a constant depending only on M_τ, but will be allowed to change from inequality to inequality. Let γ^j be a sequence of sweepouts in M_τ given by Lemma 4. For $t \geq \tau$, let $\sigma_s^j(t)$ denote the curve in M_t corresponding to $\gamma_s^j = \gamma^j(\cdot, s)$, and set $e_{s,j}(t) = Energy(\sigma_s^j(t))$. We will use the following claim to establish an upper bound for the width: Given $\epsilon > 0$, there exist $\delta > 0$ and $h_0 > 0$ so that if $j > 1/\delta$ and $0 < h < h_0$, then for all $s \in \mathcal{P}$

$$e_{s,j}(\tau + h) - \max_{s_0} e_{s_0,j}(\tau) \leq \left[\frac{-4\pi}{nC_0} + C\epsilon\right] h + Ch^2. \qquad (3.18)$$

The result follows from (3.18) as follows: take the limit as $j \to \infty$ in (3.18) to get

$$\frac{W(\tau + h) - W(\tau)}{h} \leq \frac{-4\pi}{nC_0} + C\epsilon + Ch. \qquad (3.19)$$

Taking $\epsilon \to 0$ in (3.19) gives (3.15).

It remains to establish (3.18). First, let $\delta > 0$ be given by Lemma 4. Since β is homotopically non-trivial in M_τ, $W(\tau)$ is positive and we can assume that $\epsilon^2 < W(\tau)/3$ and $\delta < W(\tau)/3$. If $j > 1/\delta$, and $e_{s,j}(\tau) > W(\tau) - \delta$, then Lemma 4 gives a non-constant closed geodesic η in M_τ with $dist(\eta, \gamma_s^j) < \epsilon$. Letting η_t denote the image of η in M_t, we have, using (3.17) with $\mathbf{V} = F\nu$ and the uniform Lipschitz bound L for the sweepouts at time τ, that

$$\frac{d}{dt}_{t=\tau} e_{s,j}(t) \leq \frac{d}{dt}_{t=\tau} Energy(\eta_t) + C\epsilon||F\nu||_{C^2}(1 + L^2) \leq \frac{-4\pi}{nC_0} + C\epsilon \qquad (3.20)$$

Since $\sigma_s^j(t)$ is the compositon of γ_s^j with the smooth flow and γ_s^j has Lipschitz bound L independent of j and s, the energy function $e_{s,j}(\tau + h)$ is a smooth function of h

with a uniform C^2 bound independent of both j and s near $h = 0$. In particular, the Taylor expansion for $e_{s,j}(\tau + h)$ gives

$$e_{s,j}(\tau + h) - e_{s,j}(\tau) = \frac{d}{dh} e_{s,j}(\tau) h + R_1(e_{s,j}(\tau + h)) \tag{3.21}$$

where $R_1(e_{s,j}(\tau + h))$ denotes the first order remainder. The uniform C^2 bounds on $e_{s,j}(t)$ give uniform bounds on the remainder terms $R_1(e_{s,j}(\tau + h))$, and using (3.20), we get

$$e_{s,j}(\tau + h) - e_{s,j}(\tau) \leq \left\{ \frac{-4\pi}{nC_0} + C\epsilon \right\} h + Ch^2 \tag{3.22}$$

from which the claim follows. In the case $e_{s,j}(\tau) \leq W(\tau) - \delta$, the claim (3.18) automatically holds after possibly shrinking $h_0 > 0$:

$$e_{s,j}(\tau + h) - \max_{s_0} e_{s_0,j}(\tau) \leq e_{s,j}(\tau + h) - e_{s,j}(\tau) - \delta \tag{3.23}$$

Taking h sufficiently small, so that $-\delta \leq \frac{-4\pi}{nC_0} h$, and using the differentiability of $e_{s,j}(t)$, we get (3.18). To get (3.16), note that for any $\epsilon > 0$, the set $\{t | W(t) \leq W(0) - (\frac{4\pi}{nC_0} - \epsilon)t\}$ contains 0, is closed, and (3.15) implies that is also open: Take $F(t) = W(t) - W(0) + (\frac{4\pi}{nC_0} - \epsilon)t$, and note that $\frac{d}{dt} F(t) < 0$ in the sense of limsup of forward difference quotients. We therefore have $W(t) \leq W(0) - (\frac{4\pi}{nC_0} - \epsilon)t$ for all t up to extinction time. Taking $\epsilon \to 0$ gives (3.16).

$\square$

3.6 F homogeneous of degree k

We assume the function F has the same properties as in section 3.3, except that it is homogeneous of degree k for some integer $k > 0$, i.e., $F(c\lambda_1, ..., c\lambda_n) = c^k F(\lambda_1, ..., \lambda_n)$. Then we can prove the following lemma:

Lemma 11. *For all* $(\lambda_1, ..., \lambda_n) \in \Gamma_+$, *it holds*

$$F(\lambda_1, ..., \lambda_n) \geq \left(\frac{\sum_1^n \lambda_i}{n} \right)^k.$$

Proof. Analogous to the proof of Lemma 5. $\square$

We can then prove the following estimate (analogous to lemma 6):

Lemma 12. *Let $\Sigma \subset M_0$ be a closed geodesic and $\Sigma_t \subset M_t$ the corresponding evolving closed curve. Let E_t be the energy of Σ_t. Then:*

$$\frac{d}{dt}_{t=0} E_t^{\frac{k+1}{2}} \leq -\frac{k+1}{n^k}(2\pi)^{\frac{k+1}{2}}. \tag{3.24}$$

Proof. Observe that, since M_0 is convex and Σ is minimal, the mean curvature vector of Σ in $\mathbf{R}^{n+1}$ points in the same direction as ν. We have that $|H_\Sigma| \leq |H_{M_0}|$ by convexity. Let V_t be the length of the close curves Σ_t, we can use the first variation formula for volume (see 9.3 and 7.5' in (31)) to obtain the following:

$$\begin{aligned} \frac{1}{k+1}\frac{d}{dt}_{t=0} V_t^{k+1} &= -V_0^k \int_\Sigma \langle H_\Sigma, F(x,t)\nu(x,t)\rangle \leq -\frac{1}{n^k} V_0^k \int_\Sigma |H_\Sigma||H_{M_0}|^k \qquad (3.25)\\ &\leq -\frac{1}{n^k} V_0^k \int_\Sigma |H_\Sigma|^{k+1} \leq -\frac{1}{n^k}\left(\int_\Sigma |H_\Sigma|\right)^{k+1} \leq -\frac{1}{n^k}(2\pi)^{k+1}. \end{aligned}$$

Here the first inequality follows from Lemma 5, the second inequality follows from $0 \leq |H_\Sigma| \leq |H_{M_0}|$, the third one follows from Hölder's inequality and the last inequality follows as above from Borsuk-Fenchel's theorem. Then we can compute the variation of energy as follows:

$$\frac{d}{dt}_{t=0} E_t^{\frac{k+1}{2}} = \frac{1}{(2\pi)^{\frac{k+1}{2}}}\frac{d}{dt}_{t=0} V_t^{k+1} \leq -\frac{k+1}{n^k}(2\pi)^{\frac{k+1}{2}}. \tag{3.26}$$

□

Finally, we could use this estimate to give a bound on the decrease rate of the width as in Theorem 9, to obtain the following:

Theorem 13. *Let $\{M_t\}_{t\geq 0}$ be a one-parameter family of smooth compact and strictly convex hypersurfaces in $\mathbf{R}^{n+1}$ flowing by equation (3.3), with F as in this section. Then in the sense of limsup of forward difference quotients it holds:*

$$\frac{d}{dt} W^{k+1} \leq -\frac{k+1}{n^k}(2\pi)^{\frac{k+1}{2}}, \tag{3.27}$$

and

$$W(t) \leq W(0) - \frac{k+1}{n^k}(2\pi)^{\frac{k+1}{2}} t \tag{3.28}$$

for as long as the evolving manifold remains smooth and strictly convex.

3.7 2-Width and 2-Convex Manifolds

Until now, we have assumed that the evolving manifold is strictly convex. Also, we have defined the width using sweepouts by curves. We can generalize the result for manifolds which are "2-convex", that is, such that the sum of any two principal curvatures is positive, by using a different width, defined by sweepouts of 2-spheres. The fundamental point is that in the case of 2-width, as earlier, one can rely on the existence of a sequence of "good sweepouts" as comparisons in order to estimate the derivative of the width. For higher dimensional sweepouts the analogous result is not known, and so we cannot argue as before in trying to prove an extinction time estimate. A motivation for dealing with general higher dimensional widths, as opposed to 1-width, is that it allows for a relaxation of the convexity condition.

In defining the width above, we chose to decompose $\mathbf{S}^n$ topologically into the space $\mathbf{S}^1 \times B^{n-1}$, with each set $\{(t,s) : t \in \mathbf{S}^1\}$ collapsed to a point, for each $s \in \partial B^{n-1}$. The purpose of this decomposition is that, since M is a closed convex hypersurface, it is topologically $\mathbf{S}^n$, and the decomposition ensures that the sweepouts induce non-trivial maps on $\pi_n(\mathbf{S}^n)$ - i.e. they are not homotopically trivial - and hence that the width is positive until the hypersurface becomes extinct. If we require that the initial immersion $\phi_0 : M \to \mathbf{R}^{n+1}$ is only 2-convex, we lose information about the global topology of M, and so it makes sense to use more general parameter spaces than those that ensure the sweepouts are homotopy $\mathbf{S}^n$'s.

Let $\mathcal{P}$ be a compact finite dimensional topological space, and let Ω_t be the set of continuous maps $\sigma : \mathbf{S}^2 \times \mathcal{P} \to M$ so that for each $s \in \mathcal{P}$ the map $\sigma(\cdot, s)$ is in $C^0 \cap W^{1,2}(\mathbf{S}^2, M)$, the map $s \to \sigma(\cdot, s)$ is continuous from $\mathcal{P}$ to $C^0 \cap W^{1,2}(\mathbf{S}^2, M)$, and finally σ maps $\partial\mathcal{P}$ to point maps. Given a map $\hat{\sigma} \in \Omega_t$, the homotopy class $\Omega_t^{\hat{\sigma}} \subset \Omega_t$ is defined to be the set of maps $\sigma \in \Omega_t$ that are homotopic to $\hat{\sigma}$ through maps in Ω_t. To define the 2-width, fix a homotopy class $\beta \in \Omega_0$ representing a non-trivial homotopy class in M_0 and let $\beta_t \in \Omega_t$ represent the corresponding sweepout in M_t. Then, for each t, we define

$$W_2(t) = W_2(t, \hat{\sigma}) = \inf_{\sigma \in \Omega_t^{\hat{\sigma}}} \max_{s \in \mathcal{P}} Energy(\sigma(\cdot, s)), \tag{3.29}$$

where the energy is given by

$$Energy(\sigma(\cdot, s)) = \frac{1}{2}\int_{\mathbf{S}^2} |\nabla_x \sigma(x,s)|^2 dx. \tag{3.30}$$

It was shown in (10) and (11) that we could also define the energy using area, and we would obtain the same quantity:

$$W_2^A = \inf_{\sigma\in\Omega_t^{\hat{\sigma}}} \max_{s\in\mathcal{P}} Area(\sigma(\cdot, s)) = W_2. \tag{3.31}$$

As in the case of 1-width defined above, we will use the existence of a sequence of "good sweepouts" that approximate the width. Namely, the following theorem was proven in (10), Theorem 1.14:

Theorem 14. *Given a map $\beta \in \Omega_t$ representing a non-trivial class in $\pi_n(M_t)$, there exists a sequence of sweepouts $\gamma^j \in \Omega_t^\beta$ with $\max_{s\in\mathcal{P}} Energy(\gamma^j(\cdot, s)) \to W_2(\beta)$, and so that given $\epsilon > 0$, there exist $\bar{j}$ and $\delta > 0$ so that if $j > \bar{j}$ and*

$$Area(\gamma^j(\cdot, s)) > W_2(\beta) - \delta, \tag{3.32}$$

then there are finitely many harmonic maps $u_i : \mathbf{S}^2 \to M_t$ with

$$d_V(\gamma^j(\cdot, s), \cup_i\{u_i\}) < \epsilon. \tag{3.33}$$

Here d_V denotes varifold distance as defined in (10).

We consider now a family $M_t^n \in \mathbf{R}^{n+1}$ evolving by the evolution equation (3.3). However, now the manifolds M_t are not necessarily convex, but they are *2-convex*: we may choose an orientation on the ambient space $\mathbf{R}^{n+1}$ and on M so that the sum of any two principal curvatures is always positive. Equivalently, if the principal curvatures are $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$, then $|\lambda_1| \leq \lambda_2$. Observe that this implies that M_t is mean convex, that is, the mean curvature is positive in this orientation. We assume also that $n > 3$.

The velocity function F now is defined in all $\mathbf{R}^n$, and satisfies the rest of conditions in Section 3.1. Observe that the third condition on F ensures that the evolution equation is parabolic, and therefore a smooth solution exists at least on a short time interval (see Theorem 3.1 in (20)). Also, Lemma 5 or 7 still hold in this case. Then we can generalize Lemmas 6 and 8 as follows:

Lemma 15. *Let $\Sigma \subset M_0$ be a closed minimal surface and Σ_t the corresponding surface in M_t. Let $V_t = Area(\Sigma_t)$. Then:*

$$\frac{d}{dt}_{t=0} V_t \leq -\frac{16\pi}{nC_0}, \tag{3.34}$$

where C_0 is 1 if F is convex, and it is as in Lemma 7 if F is concave.

Proof. Observe that, since Σ is minimal in M_0, the mean curvature vector of Σ is perpendicular to M_0. Let k_1 and k_2 be the principal curvatures of Σ in M_0, that is, $H_\Sigma = k_1 + k_2$. Then by 2-convexity (and because $n > 3$), we can chose an orientation on $\mathbf{R}^{n+1}$ and M so that:

$$0 \leq k_1 + k_2 \leq \lambda_{n-1} + \lambda_n \leq (\lambda_1 + \lambda_2) + \lambda_{n-1} + \lambda_n \leq \sum_1^n \lambda_i. \tag{3.35}$$

This shows that H_Σ and H_{M_0} in fact point in the same direction, with $0 \leq |H_\Sigma| \leq |H_{M_0}|$. As before, we can then compute the first variation of volume as:

$$\begin{aligned} \frac{d}{dt}_{t=0} V_t &= -\int_\Sigma \langle H_\Sigma, F(x,t)\nu(x,t)\rangle \leq -\frac{1}{n}\int_\Sigma |H_\Sigma||H_{M_0}| \\ &\leq -\frac{1}{n}\int_\Sigma |H_\Sigma|^2 \leq -\frac{16\pi}{n}, \end{aligned} \tag{3.36}$$

where the last inequality is from, e.g. (32). ☐

Then we can prove the following theorem:

Theorem 16. *Let $\{M_t\}_{t\geq 0}$ be a one-parameter family of smooth compact 2-convex hypersurfaces in $\mathbf{R}^{n+1}$ (with $n > 3$) flowing as represented in equation (3.3). Then in the sense of limsup of forward difference quotients it holds:*

$$\frac{d}{dt} W_2 \leq -\frac{16\pi}{n} \tag{3.37}$$

and

$$W_2(t) \leq W_2(0) - \frac{16\pi}{n} t \tag{3.38}$$

for as long as the solution remains smooth and 2-convex.

This result gives a bound for the time of the first singularity. Unlike in the convex case, we don't know that the submanifold contracts into a point by the first singularity. The proof is analogous to the proof of Theorem 9 (see also (10) and (11)).

Chapter 4

A Singular Lamination

In this chapter we record the alternate proof, recorded in (24), of the main theorem of D. Hoffman and B. White in (17). The theorem generalizes an example of Colding and Minicozzi in (12) that was the first known example of a sequence of properly embedded minimal surfaces converging to a non-proper limit:

Theorem 17. *Let M be a compact subset of $\{x_1 = x_2 = 0, |x_3| < 1/2\}$ and let $C = \{x_1^2 + x_2^2 = 1, |x_3| < 1/2\}$. Then there is a sequence of compact embedded minimal disks $\Sigma_k \subset C$ with $\partial\Sigma_k \subset \partial C$ and containing the vertical segment $\{(0,0,t) | |t| < 1/2\}$ so that:*

(A) $lim_{k\to\infty} |A_{\Sigma_k}|^2(p) = \infty$ *for all* $p \in M$.

(B) For any $\delta > 0$ *it holds* $sup_k sup_{\Sigma_k \setminus M_\delta} |A_{\Sigma_k}|^2 < \infty$ *where* $M_\delta = \cup_{p\in M} B_\delta(p)$.

(C) $\Sigma_k \setminus \{x_3 - axis\} = \Sigma_{1,k} \cup \Sigma_{2,k}$ *for multi-vauled graphs* $\Sigma_{1,k}$, $\Sigma_{2,k}$.

(D) For each interval $I = (t_1, t_2)$ *in the compliment of* M *in the* x_3*-axis,* $\Sigma_k \cap \{t_1 < x_3 < t_2\}$ *converges to an imbedded minimal disk* Σ_I *with* $\bar{\Sigma_I} \setminus \Sigma_I = C \cap \{x_3 = t_1, t_2\}$*. Moreover,* $\Sigma_I \setminus \{x_3 - axis\} = \Sigma_{1,I} \cup \Sigma_{2,I}$*, for multi-valued graphs* $\Sigma_{1,I}$ *and* $\Sigma_{2,I}$ *each of which spirals in to the planes* $\{x_3 = t_1\}$ *from above and* $\{x_3 = t_2\}$ *from below.*

It follows from (D) that the $\Sigma_k \setminus M$ converge to a limit lamination of $C \setminus M$. The leaves of this lamination are given by the multi-valued graphs Σ_I given in (D), indexed by intervals I of the compliment of M, taken together with the planes $\{x_3 = t\} \cap C$ for $t \in M$. This lamination does not extend to a lamination of C, however, as every boundary point of M constitutes a non-removable singularity. Theorem 17 is inspired by the result of Hoffman and White in (17).

Throughout this chapter, we will use coordinates (x_1, x_2, x_3) for vectors in $\mathbb{R}^3$, and $z = x + iy$ on $\mathbb{C}$. For $p \in \mathbb{R}^3$, and $s > 0$, the ball in $\mathbb{R}^3$ is $B_s(p)$. We denote the sectional curvature of a smooth surface Σ by K_Σ. When Σ is immersed in $\mathbb{R}^3$, A_Σ will be its second fundamental form. In particular, for Σ minimal we have that $|A_\Sigma|^2 = -2K_\Sigma$. Also, we will identify the set $M \subset \{x_3\text{-axis}\}$ with the corresponding subset of $\mathbb{R} \subset \mathbb{C}$; that is, the notation will not reflect the distinction, but will be clear from context. Our example will rely heavily on the Weierstrass Representation, which we introduce here.

4.1 The Structure of the Proof

We will be dealing with a family of immersions $F_{k,a} : \Omega_{k,a} \to \mathbb{R}^3$ that depend on a parameter $0 < a < 1/2$ given by Weierstrass data of the form $\Omega_{k,a}, G_{k,a} = e^{iH_{k,a}}, \phi = dz$, and a sequence $M_k \subset M$ that converge to a dense subset of M. Each function $H_{k,a}$ will be real valued when restricted to the real line in $\mathbb{C}$. That is, writing $H_{k,a} = U_{k,a} + iV_{k,a}$ for real valued functions $U_{k,a}, V_{k,a} : \Omega_k \to \mathbb{R}^3$, we have that $H_{k,a}(x, 0) = U_{k,a}(x, 0)$. Moreover, we will show that $V_{k,a}(x, y) > 0$ when $y \neq 0$. A look at the expression for the unit normal given above in (2.3) then shows that all of the surfaces $\Sigma_{k,a}$ will be multi-valued graphs away from the x_3-axis (since $|g(x, y)| = 1$ is equivalent to $y = 0$). The dependence on the parameter $0 < a < 1/2$ will be such that $\lim_{a\to 0}|A_{\Sigma_{k,a}}|^2(p) = \infty$ for all $p \in M_k$, and such that $|A_{\Sigma_{k,a}}|^2$ remains uniformly bounded in k and a away from M. We will then choose a suitable sequence $a_k \to 0$, and set $F_k = F_{k,a_k}$, $\Omega_k = \Omega_{k,a_k}$, $G_k = G_{k,a_k}$, and $H_k = H_{k,a_k}$. Immediately, (A), (B) and (C) of Theorem 17 are satisfied by the diagonal subsequence. In fact, we will assume throughout that we are working with any diagonal sequence determined by a sequence $a_k \to 0$ satisfying a_k The bulk of the work will go towards establishing (D). To this end, we will show that

Lemma 18. *(a) The horizontal slice $\{x_3 = t\} \cap F_k(\Omega_k)$ is the image of the vertical segment $\{x = t\}$ in the plane, i.e., $x_3(F_k(t, y)) = t$.*

(b) The image of $F_k(\{x = t\})$ is a graph over a line segment in the plane $\{x_3 = t\}$(the line segment will depend on t)

(c) The boundary of the graph in (b) is outside the ball $B_{r_0}(F_k(t, 0))$ for some $r_0 > 0$.

This gives that the immersions $F_k : \Omega_k \to \mathbb{R}^3$ are actually embeddings, and that the surfaces Σ_k given by $F_k(\Omega_k)$ are all embedded in a fixed cylinder $C_{r_0} = \{x_1^2 + x_2^2 = r_0^2, |x_3| < 1/2\}$ about the x_3-axis in $\mathbb{R}^3$. This will then imply that the surfaces Σ_k converge smoothly on compact subsets $C_{r_0} \setminus M$ to a limit lamination of C_{r_0}. The claimed structure of the limit lamination(that is, that on each interval of the compliment it consists of two multi-valued graphs that spiral into planes from above and below) will be established at the end.

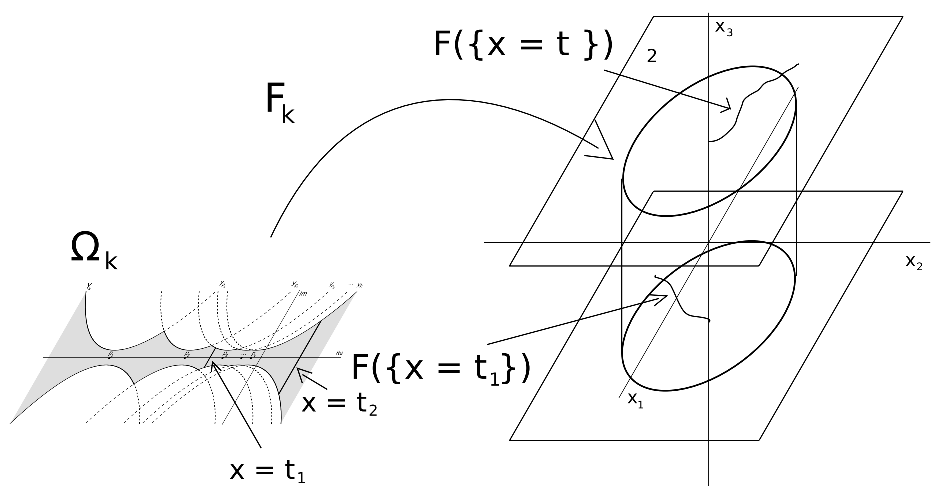

Figure 4.1 The functions F_k

Throughout the paper, all computations will be carried out and recorded only on the upper half plane in $\mathbb{C}$, as the corresponding computations on the lower half plane are completely analogous. By scaling it suffices to prove Theorem 17 (D) for some C_{r_0}, not C_1 in particular.

4.2 definitions

Let $M \subset [0,1]$ be a closed set. Fix $\gamma > 1$, and take M_{-1} to be the empty set. Then for k a non-negative integer, we inductively define two families of sets m_k, and M_k as follows: Assuming m_{k-1} and M_{k-1} are already defined, take m_k to be any maximal subset of M with the property that, for $p, q \in m_k$, $r \in M_{k-1}$, it holds that $|p-q|, |p-r| \geq \gamma^{-k}$. Then define $M_k = M_{k-1} \cup m_k$ and $M_\infty = \cup_k M_k$. Take

$$h(z) = \int_0^z \frac{dz}{\left(z^2+a^2\right)^2} = u(z) + iv(z) \tag{4.1}$$

and

$$y_0(x) = \epsilon \left(x^2+a^2\right)^{5/4}$$

for ϵ to be determined. For $p \in \mathbb{R}$ we define

$$h_p(z) = h(z-p) = u_p(z) + iv_p(z)$$

and

$$y_p(x) = y_0(x-p)$$

We then take

$$h_l(z) = \sum_{p \in m_l} h_p(z) = u_l(z) + iv_l(z) \tag{4.2}$$

and

$$y_l(x) = min_{p \in m_l} y_p(x)$$

We take

$$H_k(z) = \sum_{l=0}^{k} \mu^{-l} h_l(z) = U_k(z) + iV_k(z) \tag{4.3}$$

for a parameter $\mu > \gamma$ to be determined. We take

$$Y_k(x) = \min_{l \leq k} y_l(x)$$

Note that the dependence on the parameter a is ommited from the notation. We take

$$\omega = \{x+iy||y| \leq y_0(x)\}, \omega_p = \{x+iy||y| \leq y_p(x)\}$$

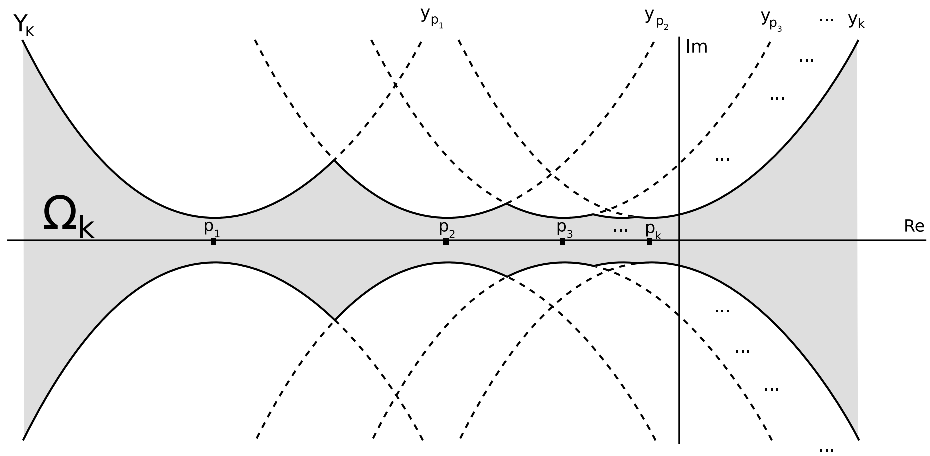

Figure 4.2 A schematic rendering of the domain Ω_k in the case of $M = \{p_l = -2^{-l} | l \in \mathbb{N}\}$

and

$$\omega_l = \{x + iy | |y| \leq y_l(x)\}, \Omega_k = \{x + iy | |y| \leq Y_k(x)\}$$

and lastly set $\Omega_\infty = \cap_k \Omega_k$.

4.3 Preliminary Results

We record some elementary properties of the sets M_k and m_k defined above which will be needed later.

Lemma 19. $|m_k| \leq \gamma^k + 1$

Proof. Let $p_1 < \ldots < p_n$ be n distinct elements of m_k, ordered least to greatest. By construction we have that $p_{k+1} - p_k \geq \gamma^{-k}$. Also, since $p_1, p_n \in M$ we get

$$1 \geq p_n - p_1 = \sum_{k=1}^{n-1} p_{k+1} - p_k \geq (n-1)\gamma^{-k}$$

□

Lemma 20. *For all p in M, there is a q in M_k such that $|p - q| < \gamma^{-k}$*

Proof. If not, m_k is not maximal. □

Lemma 21. *The union $\cup_{k=0}^{\infty} m_k = \cup_{k=0}^{\infty} M_k \equiv M^{\infty}$ is a dense subset of M*

Proof. Suppose not. Then there is a $q \in M$, and a positive integer k such that $|p-q| > \gamma^{-k}$, $\forall p \in M^{\infty}$. In particular, this implies that m_k is not maximal. $\square$

In order to avoid disrupting the narrative, the proofs of the remaining results in this section will be recorded later in the Appendix at the end. The proofs are somewhat tedious, though easily verified.

Lemma 22. *For ϵ sufficiently small, $h_p(z)$ is holomorphic on ω_p, h_l is holomorphic on ω_l, and H_k is holomorphic on Ω_k.*

We will also need the following estimates:

Lemma 23. *On the domain ω_p it holds that:*

$$\left|\frac{\partial}{\partial y} u_p(x,y)\right| \leq \frac{c_1|x-p||y|}{((x-p)^2+a^2)^3}$$

and

$$\frac{\partial}{\partial y} v_p(x,y) > \frac{c_2}{((x-p)^2+a^2)^2}$$

Integrating the above estimates from 0 to the upper boundary of ω_p gives

$$u_p(x, y_p(x)) - u_p(x,0) \leq \epsilon^2 c_1$$

and

$$\min_{[y_p(x)/2, y_p(x)]} v(x,y) > \frac{\epsilon c_2}{((x-p)^2+a^2)^{3/4}}$$

These estimates immediately give

Corollary 24.

$$|U_k(x, Y_k(x)) - U_k(x,0)| \leq \epsilon^2 c_1 \sum_{l=0}^{k} (\gamma/\mu)^l + \mu^{-l} \leq \epsilon^2 c_1' \tag{4.4}$$

$$V_k(x, Y_k(x)/2) > \frac{\epsilon c_2}{2} \sum_{l=0}^{k} \mu^{-l} \sum_{p \in m_l} \frac{Y_k(x)}{y_p(x)} ((x-p)^2+a^2)^{-3/4} \equiv \frac{1}{2} c_2 q_k(x) \tag{4.5}$$

Here, the quantity q_k is defined by the last equality. Lastly we record

Lemma 25.

$$q_k(x) \geq \frac{Y_k(x)}{Y_{k-1}(x)} q_{k-1}(x) \tag{4.6}$$

4.4 Proof of lemma 18

We will first concern ourselves with establishing lemma 18. (a) follows from (2.2) and the choice of $z_0 = 0$ Choosing $\epsilon < \epsilon_0 < c_1'^{-1/2}$, where c_1' is the constant in (4.4), and using (2.8) we get

$$\langle \partial_y F_k(x,y), \partial_y F_k(x,0)\rangle = \cosh V_k(x,y)\cos(U_k(x,y_0(x) - U_k(x,0)) > \cosh V_k(x,y)/2 \tag{4.7}$$

Here we have used that $\cos(1) > 1/2$. This gives that all of the maps $F_k : \Omega_k \to \mathbb{R}^3$ are indeed embeddings (for all values of a) and proves (b) of lemma 18.

Now, integrating (4.7) from $Y_k(x)/2$ to $Y_k(x)$ gives

$$\langle F_k(x, Y_k(x)) - F_k(x,0), \partial_y F_k(x,0)\rangle > \frac{Y_k(x)}{2} e^{\min_{[Y_k/2, Y_k]} V_k} \tag{4.8}$$

Using the bound for V_k recorded in (4.5), we get that

$$\langle F_k(x, Y_k(x)) - F_k(x,0), \partial_y F_k(x,0)\rangle > \frac{Y_k(x)}{2} e^{\frac{1}{2}c_2 q_k(x)} \tag{4.9}$$

Take $r_k(x) \equiv \frac{1}{2} Y_k(x) e^{\frac{1}{2}c_2 q_k(x)}$. We will show that, for an appropriate sequence $a_k \to 0$, $r_k(x) \geq c_k r_{k-1}(x)$ for a sequence c_k with $0 < c_k < 1$ such that $\prod_{l=0}^{\infty} c_l > c_0$; this establishes (c) of lemma 18.

There are two cases to consider when comparing $r_k(x)$ with $r_{k-1}(x)$. If $Y_{k-1}(x) = Y_k(x)$, there is nothing to do. If $Y_k(x) < Y_{k-1}(x)$ then we may assume that $Y_k(x) = y_p(x)$ for some $p \in m_k$. This is the case we are concerned with.

Consider the immersion given by data $g(z) = e^{i\mu^{-k} h_p(z)}$, $\phi = dz$, with domain ω_p. Arguing as above, we can estimate radius of the cylinder in which it is embedded by

$$r_{p,\mu^{-k}}(x) = \frac{1}{2}\epsilon((x-p)^2 + a^2)^{5/4} e^{\epsilon c_2 \mu^{-k} \frac{1}{2}((x-p)^2 + a^2)^{-3/4}}$$

which we rewrite as

$$r_{p,\mu^{-k}}(s(x)) = \frac{\epsilon}{2} s^{5/3} e^{\frac{1}{2}\epsilon c_2 \mu^{-k} s^{-1}} \tag{4.10}$$

with s $= ((x-p)^2 + a^2)^{3/4}$. The right hand side of (4.10) has a minimum at $s = c\mu^{-k}$ with value $c\mu^{-5/3k}$ (here c is used to denote potentially different positive constants and should be thought of as denoting "on the order of"). as k goes to infinity, the embedded surfaces $\Sigma_{p,\mu^{-k}}$ "pinch off" at the singular point, and the limiting immersion is not embedded in any neighborhood of our line segment. On the other hand, to complete our construction, we are forced "scale" the data by uniformly summable factors, so that the curvature remains bounded away from the singular set.

Lemmas 26 and 27 show that we can choose a σ so that as long as $|x-p| < c\mu^{-2/3(1+\sigma)k}$(with a small enough, $\sigma > 0$), then $r_{p,\mu^{-k}}(x) > 1$. In the following we set $\Phi(s) = \frac{\epsilon}{2}s^{5/3}e^{\frac{\epsilon}{2}c_2 s^{-1}}$.

Lemma 26. *$\forall \alpha > 0, \exists \delta = \delta(\alpha)$ s.t.*

$$\Phi(s) \geq s^{-\alpha} \tag{4.11}$$

for $0 < s < \delta$

Proof.

$$lim_{s\to 0}\frac{\epsilon}{2}s^{5/3+\alpha}e^{\frac{1}{2}\epsilon c_2 s^{-1}} = \infty$$

for all α. □

The point now is to choose μ and σ so that $\mu^{2/3} < \mu^{(1+\sigma)2/3} < \gamma < \mu$. We must also choose α so that $\alpha\sigma - 5/3 \geq 0$, as will be seen in the following. In fact, for later applications, we demand $\alpha\sigma - 5/3 \geq 1$.

Lemma 27. *For*

$$|x-p|, a \leq \mu^{-2/3(1+\sigma)k}\left(\frac{\delta(\alpha)^{2/3}}{\sqrt{2}}\right)$$

We have that

$$r_{p,\mu^{-k}}(x) > 1$$

Proof. The assumptions immediately give that

$$s = ((x-p)^2 + a^2)^{3/4} < \mu^{-(1+\sigma)k}\delta < \delta$$

which we re-write as

$$\mu^k s \leq \mu^{-\sigma k}\delta$$

applying (4.11), we find that

$$\Phi(\mu^k s) > \left(\mu^{-\sigma k}\delta\right)^{-\alpha}$$

Equivalently,

$$r_{p,\mu^{-k}}(x) = \frac{\epsilon}{2} s^{5/3} e^{\frac{\epsilon}{2}c_2\mu^{-k}s-1} > \mu^{(\alpha\sigma-5/3)k}\delta^{-\alpha} > 1$$

since we have choosen $\alpha\sigma - 5/3 \geq 1$, and we may assume $\delta < 1$. □

We are ready to prove:

Lemma 28. *set* $\tau = \frac{\mu^{(1+\sigma)2/3}}{\gamma} < 1$ *and* $c_0 = \delta^{2/3}/\sqrt{2}$. *Then either*

$$r_k(x) \geq \left[\frac{1}{1+\gamma c_0^{-1}\tau^k}\right]^{5/2} \left[\frac{\epsilon c_0^{5/2}\mu^{-5/3k}}{2}\right]^{c\tau^k}$$

or

$$r_k(x) > \left[\frac{1}{1+\gamma c_0^{-1}\tau^k}\right]^{5/2} \left[\frac{\epsilon c_0^{5/2}\mu^{-5/3k}}{2}\right]^{c\tau^k} r_{k-1}(x)$$

where

Proof. We may assume $Y_k(x) = y_p(x)$ for some $p \in m_k$. If $|x-p|, a < \mu^{-2/3(1+\sigma)k}\delta^{2/3}/\sqrt{2}$, then

$$r_k(x) > r_{p,\mu^{-k}}(x) > 1$$

by lemma 27. So we assume that $|x-p| > c_0\mu^{-2/3(1+\sigma)k}$ with $c_0 = \delta^{2/3}/\sqrt{2}$. By the construction of the sets m_k, M_k, there is a $q \in M_{k-1}$ such that $|p-q| < \gamma^{-k+1}$. We may assume that q is the closest point to x in M_{k-1} so that $Y_{k-1}(x) = y_q(x)$.

Then we may estimate

$$\left[\frac{Y_k(x)}{Y_{k-1}(x)}\right]^{4/5} = \left[\frac{y_p(x)}{y_q(x)}\right]^{4/5} > \frac{|x-p|^2 + a^2}{(|x-p|+|p-q|)^2 + a^2}$$

or

$$\left[\frac{Y_k(x)}{Y_{k-1}(x)}\right]^{4/5} > \frac{1 + a^2/|x-p|^2}{(1+|p-q|/|x-p|)^2 + a^2/|x-p|^2} > \frac{1}{(1+|p-q|/|x-p|)} \quad (4.12)$$

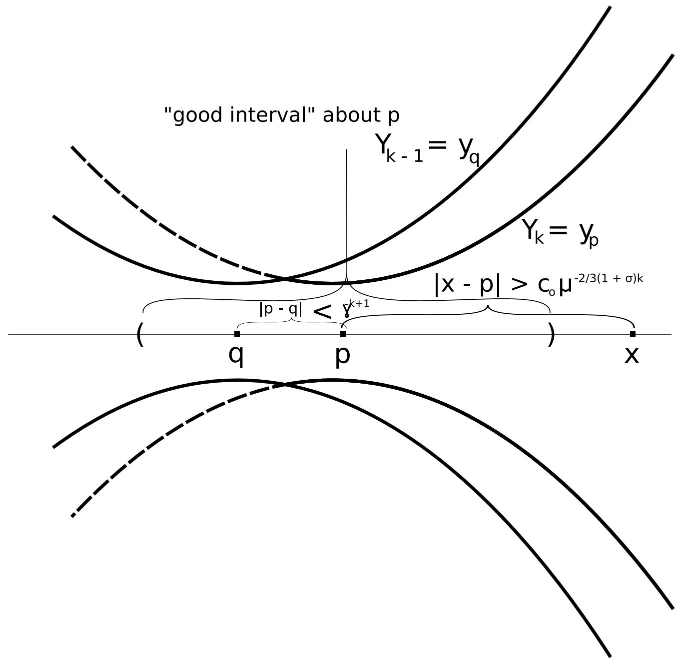

Figure 4.3 The "good interval about p"

The last inequality in (4.12) follows since

$$\frac{d}{dx}\left(\frac{1+x}{c+x}\right) > 0$$

for $c > 1$ and $x > 0$. Recalling that $|p-q| < \gamma^{-k+1}$, $|x-p| > c_0\mu^{-2/3(1+\sigma)k}$ and setting $\tau = \mu^{(1+\sigma)2/3}/\gamma$ (note that $\tau < 1$), we get that

$$\frac{Y_k(x)}{Y_{k-1}(x)} > \left[\frac{1}{1+\gamma c_0^{-1}\tau^k}\right]^{5/2} \tag{4.13}$$

Recalling (4.6) and (4.13) above, we obtain

$$\begin{aligned}
r_k(x) &= \frac{1}{2}Y_k(x)e^{\frac{1}{2}c_2 q_k(x)} > \frac{1}{2}Y_k(x)e^{\frac{1}{2}c_2\frac{Y_k}{Y_{k-1}}q_{k-1}(x)} \\
&> \frac{Y_k(x)}{Y_{k-1}(x)}\left[\frac{1}{2}Y_{k-1}(x)\right]^{1-\frac{Y_k}{Y_{k-1}}}\left[\frac{1}{2}Y_{k-1}(x)e^{\frac{1}{2}c_2 q_{k-1}}\right]^{\frac{Y_k}{Y_{k-1}}} \\
&> \left[\frac{1}{1+\gamma c_0^{-1}\tau^k}\right]^{5/2}\left[\frac{1}{2}Y_{k-1}(x)\right]^{1-\frac{Y_k}{Y_{k-1}}}\left[r_{k-1}(x)\right]^{\frac{Y_k}{Y_{k-1}}}
\end{aligned}$$

Now, we have that

$$1 - \frac{Y_k}{Y_{k-1}} < 1 - \left[\frac{1}{1+\gamma c_0^{-1}\tau^k}\right]^{5/2} < c\tau^k$$

for some positive constant c. Also, we assumed that $|x-p| > c_0\mu^{-2/3(1+\sigma)}$, so that $|x-q| > |x-p| > c_0\mu^{-2/3(1+\sigma)}$ so that $Y_{k-1}(x) = y_q(x) > \epsilon c_0^{5/2}\mu^{-5/3(1+\sigma)k}$. Combining these, we get

$$r_k(x) > \left[\frac{1}{1+\gamma c_0^{-1}\tau^k}\right]^{5/2} \left[\frac{\epsilon}{2}c_0^{5/2}\mu^{-5/3(1+\sigma)k}\right]^{c\tau^k} [r_{k-1}(x)]^{\frac{Y_k}{Y_{k-1}}}$$

and the conclusion follows by considering the separate cases $r_{k-1}(x) \geq 1$ and $r_{k-1}(x) < 1$. $\square$

The immediate corollary is

Corollary 29. *Either*

$$r_k(x) > \prod_0^\infty c_l \tag{4.14}$$

or

$$r_k(x) > \left(\prod_0^\infty c_l\right) r_0(x) \tag{4.15}$$

for $c_l = \left[\frac{1}{1+\gamma c_0^{-1}\tau^l}\right]^{5/2} \left[\frac{\epsilon}{2}c_0^{5/2}\mu^{-5/3(1+\sigma)l}\right]^{c\tau^l}$

Since $\prod_0^\infty c_l > 0$, we get (c) of lemma 18

4.5 Proof of Theorem 17(A), (B) and (C)

Note that (2.4) and our choice of Weierstrass data gives that

$$K_{\Sigma_k}(z) = \frac{-|\partial_z H_k|^2}{\cosh^4 V_k}. \tag{4.16}$$

For $p \in m_l$, it is clear that $F_k(p) = (0,0,p)$ for all k. Thus, for $k > l$ we can then estimate

$$|\partial_z H_k(p)| > \frac{\mu^{-l}}{a_k^4} \tag{4.17}$$

since $V_k(x,0) = 0$ for all $x \in \mathbb{R}$ and hence $|A_{\Sigma_k}(p)|^2 \to \infty$. For $p \in M \setminus M_\infty$, there is a sequence of points $p_{l_j} \in m_{l_j}$ such that $|p - p_{l_j}| < \gamma^{-l_j}$. We then get

$$|\partial_z H_{l_j}(p)| > \frac{\mu^{-l_j}}{((p - p_{l_j}) + a_{l_j}^2)^2} > \frac{\mu^{-l_j}}{(\gamma^{-2l_j} + a_{l_j}^2)^2} \tag{4.18}$$

Taking $j \to \infty$ and $a_k < \gamma^{-k}$ gives that $|A_{\Sigma_{l_j}}(p)|^2 \to \infty$ and proves (A) of Theorem 17.

Since $V_k(x,y) > 0$ for $y \neq 0$, we see that $x_3(\mathbf{n}(x,y)) \neq 0$, and hence Σ_k is graphical away from the x_3- axis, which proves (C) of Theorem 17.

Now, for $\delta > 0$ set $S_\delta = \{z | \mathrm{dist}(\mathrm{Re}\, z, M) > \delta\}$. From (2.4), it is immediate that

$$\sup_k \sup_{\Omega_k \setminus S_\delta} |A_{\Sigma_k}(z)|^2 < \infty \tag{4.19}$$

for any $\delta > 0$. This combined with Heinz's curvature estimate for minimal graphs gives (B)

4.6 Proof of Theorem 17 (D) and The Structure Of The Limit Lamination

Lemma 30. *A subsequence of the embeddings $F_k : \Omega_k \to \mathbb{R}^3$ converges to a minimal lamination of the cylinder C*

Proof. Let K be a compact subset of Ω_∞. Then for $p \in K$, we have that $\sup_k |\frac{d}{dz} H_k(z)| < \infty$. Montel's theorem then gives a subsequence converging smoothly to a holomorphic function on K. By continuity of integration this gives that the embeddings $F_k : K \to \mathbb{R}^3$ converge smoothly to a limiting embedding. Thus the surfaces Σ_k converge to a limit lamination of B_{r_0} that is smooth away from the x_3- axis. □

Let $I = (t_1, t_2) \subset \mathbb{R}$ be an interval comprising the compliment of the M in $\mathbb{R}$ and consider $\Omega_I = \Omega_\infty \cap \{\mathrm{Re}\, z \in I\}$. Then Ω_I is topologically a disk, and by lemma 30, the surfaces $\Sigma_{k,I} \equiv F_k(\Omega_I)$ are contained in $\{t_1 < x_3 < t_2\} \subset \mathbb{R}^3$ and converge to an embedded minimal disk Σ_I. Now, Theorem 17(C) (which we have already established), gives that Σ_I consists of two multi-valued graphs Σ_I^1, Σ_I^2 away from the x_3- axis. We will show that each graph Σ_I^j is $\infty-$ valued and spirals into the $\{x_3 = t_1\}$ and $\{x_3 = t_2\}$ planes as claimed.

Note that by (2.8) and (C), the level sets $\{x_3 = t\} \cap \Sigma_I^j$ for $t_1 < t < t_2$ are graphs over lines in the direction

$$\lim_{k\to\infty} (\cos U_k(t,0), -\text{sine}\, U_k(t,0), 0) \tag{4.20}$$

First, suppose $t_1 \in m_l$ for some l. Then we get that, for any $k > l$ and any $t < \frac{t_2-t_1}{2}$.

$$U_k(t_1+2t,0) - U_k(t_1+t,0) = \int_{t_1+t}^{t_1+2t} \partial_s U_k(s,0)ds > c_2\mu^{-l} \int_{t_1+t}^{t_1+2t} \frac{ds}{((s-t_1)^2+a_k^2)^2} \tag{4.21}$$

(by the Cauchy-Reimann equations $U_{k,x} = V_{k,y}$). Then, since $a_k \to 0$ as $k \to \infty$, we get that

$$\lim_{k\to\infty} U_k(t_1+2t,0) - U_k(t_1+t,0) > c_2\mu^{-l} \int_{t_1+t}^{t_1+2t} \frac{ds}{(s-t_1)^4} > \frac{c_2\mu^{-l}}{64t^3} \tag{4.22}$$

and hence $\{t_1 + t < |x_3| < t_1 + 2t\}$ contains an embedded N_t-valued graph, where

$$N_t > \frac{c_2\mu^{-l}}{64t^3}. \tag{4.23}$$

Note that $N_t \to \infty$ as $t \to 0$ from above and hence Σ_I spirals into the plane $\{x_3 = t_1\}$.

Now, suppose that $t_1 \notin M_\infty$. Then there exists a sequence of non-negative integers l_j such that $t_1 - p_{l_j} < \gamma^{-l_j}$, with $p_{l_j} \in m_{l_j}$. Then set $t^j = t_1 + \gamma^{-l_j}$ and consider the intervals

$$I_j = [t^{j+1}, t^j]. \tag{4.24}$$

Note that for j large $I_j \subset I$. Then, for $k > l_j$ and $s \in I_j$ we may estimate

$$\partial_s U_k(s,0) > \frac{c_2\mu^{-l_j}}{((s-p_{l_j})^2+a_k^2)^2} > \frac{c_2\mu^{-l_j}}{(4\gamma^{-2l_j}+a_k^2)^2} \tag{4.25}$$

since $s - p_{l_j} < 2\gamma^{-l_j}$. We then get

$$U_k(t^j,0) - U_k(t^{j+1},0) > |I_j| \frac{c_2\mu^{-l_j}}{(4\gamma^{-2l_j}+a_k^2)^2} \geq \frac{c_2\mu^{-l_j}(1-\gamma^{-1})\gamma^{-l_j}}{(4\gamma^{-2l_j}+a_k^2)^2} \tag{4.26}$$

Taking limits, we get

$$\lim_{k\to\infty} U_k(t^j,0) - U_k(t^{j+1},0) > \frac{c_2(1-\gamma^{-1})}{16}\left(\frac{\gamma^3}{\mu}\right)^{l_j} \tag{4.27}$$

Thus we see that $\{t^{j+1} < x_3 < t^j\} \cap \Sigma_I^j$ contains an embedded N_j-valued graph, where

$$N_j \approx \frac{c_2(1-\gamma^{-1})}{32\pi}\left(\frac{\gamma^3}{\mu}\right)^{l_j} \tag{4.28}$$

This again shows that Σ_I^j spirals into the plane $\{x_3 = t_1\}$ since as $j \to \infty$, $t^j \to t_1$, and $N_j \to \infty$. Now for $t \in \text{int}(M)$, every singly graphical component of F_{l_j} contained in the slab $\{t - \gamma^{-l_j}, t < x_3 < \gamma^{-l_j}\}$ (by (4.28) there are many) is graphical over $\{x_3 = 0\} \cap B_{r_{l_j}}(0)$ where lemma 27 gives. $r_{l_j} \to \infty$, which implies each component comverges to the plane $\{x_3 = t\}$. This proves Theorem 17 (D).

4.7 Appendix

Here we provide the computations that were omitted from section 4.3

proof of lemma 22. It suffices to show that $h = u + iv$ is holomorphic on ω. Recall that

$$h(z) = \int_0^z \frac{dz}{(z^2+a^2)^2} \tag{4.29}$$

It is clear that the points $\pm ia$ lie outside of ω. Moreover, as ω is the subgraph of a function, it is obviously simply connected, so that $\int_0^z \frac{dz}{(z^2+a^2)^2}$ gives a well-defined holomorphic function on ω. ☐

proof of lemma 23. We compute the real and imaginary components of $(z^2 + a^2)^2$:

$$z^2 + a^2 = x^2 - y^2 + a^2 + 2ixy,$$

$$\left(z^2+a^2\right)^2 = \left(x^2-y^2+a^2\right)^2 - 4x^2y^2 + 4ixy\left(x^2-y^2+a^2\right).$$

Set

$$a = Re\left\{\left(z^2+a^2\right)^2\right\} = \left(x^2-y^2+a^2\right)^2 - 4x^2y^2,$$

$$b = Im\left\{\left(z^2+a^2\right)^2\right\} = 4xy\left(x^2-y^2+a^2\right),$$

and

$$c^2 = \left|\left(z^2+a^2\right)^2\right|^2 = a^2 + b^2 = \left\{\left(x^2-y^2+a^2\right)^2 - 4x^2y^2\right\}^2 + 16x^2y^2\left(x^2-y^2+a^2\right)^2.$$

Now on ω (that is, on the set where $|y| \le y_0(x)$), we get the bounds

$$a \ge (1-\epsilon^2)^2 \left(x^2+a^2\right)^2 - 4\epsilon^2(x^2+a^2)^2 = \left\{(1-\epsilon^2)^2 - 4\epsilon^2\right\}(x^2+a^2)^2,$$

$$a \le (x^2+a^2)^2,$$

$$b \le 4\epsilon(x^2+a^2)^{11/4} \le 4\epsilon(x^2+a^2)^2$$

since by assumption $|x|, a < \frac{1}{2}$. Using that $c^2 = a^2+b^2$,

$$\left\{(1-\epsilon^2)^2 - 4\epsilon^2\right\}^2 (x^2+a^2)^4 \le c^2 \le \left\{1+16\epsilon^2\right\}\left(x^2+a^2\right)^4.$$

Recalling that

$$\frac{\partial}{\partial y}u(x,y) = Im\left\{\frac{1}{\left(z^2+a^2\right)^2}\right\} = \frac{-b}{c^2}, \frac{\partial}{\partial y}v(x,y) = Re\left\{\frac{1}{\left(z^2+a^2\right)^2}\right\} = \frac{a}{c^2}$$

We get

$$\left|\frac{\partial}{\partial y}u_p(x,y)\right| \le \frac{4}{\{(1-\epsilon^2)^2-4\epsilon^2\}^2}\frac{|x-p||y|}{\left((x-p)^2+a^2\right)^3}$$

and

$$\frac{\partial}{\partial y}v_p(x,y) \ge \frac{\{(1-\epsilon^2)^2-4\epsilon^2\}}{1+16\epsilon^2}\frac{1}{\left((x-p)^2+a^2\right)^2}$$

If we restrict $\epsilon < \epsilon_0$ for ϵ_0 sufficiently small we get that

$$\frac{4}{\{(1-\epsilon^2)^2-4\epsilon^2\}^2} < c_1,$$

and

$$\frac{\{(1-\epsilon^2)^2-4\epsilon^2\}}{1+16\epsilon^2} > c_2.$$

for constants c_1 and c_2. which immediately gives the lemma. □

proof of corollary 24. Recalling definitions (4.2) and (4.3), we get

$$
\begin{aligned}
|U_k(x,Y_k(x))-U_k(x,0)| &\le \sum_{l=0}^{k}\mu^{-l}\int_0^{Y_k(x)}\left|\frac{\partial}{\partial y}u_l(x,y)\right| \\
&\le \sum_{l=0}^{k}\mu^{-l}\sum_{p\in m_l}\int_0^{Y_k(x)}\left|\frac{\partial}{\partial y}u_p(x,y)\right| \\
&\le \sum_{l=0}^{k}\mu^{-l}\sum_{p\in m_l}\int_0^{y_p(x)}\left|\frac{\partial}{\partial y}u_p(x,y)\right| \\
&\le c_1\epsilon^2\sum_{l=0}^{k}\mu^{-l}(\gamma^l+1)
\end{aligned} \tag{4.30}
$$

and

$$
\begin{aligned}
min_{[Y_k(x)/2,Y_k(x)]}V_k(x,y) &\ge \sum_{l=0}^{k}\int_0^{Y_k(x)/2}\frac{\partial}{\partial y}v_l(x,y) \\
&> \sum_{l=0}^{k}\mu^{-l}\sum_{p\in m_l}\int_0^{Y_k(x)/2}\frac{\partial}{\partial y}v_p(x,y) \\
&= \frac{\epsilon c_2}{2}\sum_{l=0}^{k}\mu^{-l}\sum_{p\in m_l}\frac{Y_k(x)}{y_p(x)}((x-p)^2+a^2)^{-3/4} \\
&= \frac{c_2}{2}q_k(x)
\end{aligned} \tag{4.31}
$$

□

proof of lemma 25.

$$
\begin{aligned}
q_k(x) &= \epsilon\sum_{l=0}^{k}\mu^{-l}\sum_{p\in m_l}\frac{Y_k(x)}{y_p(x)}((x-p)^2+a^2)^{-3/4} \\
&\ge \epsilon\sum_{l=0}^{k-1}\mu^{-l}\sum_{p\in m_l}\frac{Y_k(x)}{y_p(x)}((x-p)^2+a^2)^{-3/4} \\
&= \epsilon\frac{Y_k(x)}{Y_{k-1}(x)}\sum_{l=0}^{k-1}\mu^{-l}\sum_{p\in m_l}\frac{Y_{k-1}(x)}{y_p(x)}((x-p)^2+a^2)^{-3/4} \\
&= \frac{Y_k(x)}{Y_{k-1}(x)}q_{k-1}(x)
\end{aligned} \tag{4.32}
$$

□

Chapter 5

Self-Shrinkers with a symmetry

5.1 Non-compact, rotationally symmetric, embedded, self-shrinking, asymptotically conical ends with positive mean curvature

We consider n-dimensional hypersurfaces $\Sigma^n \subseteq \mathbb{R}^{n+1}$ with boundary $\partial\Sigma \neq \emptyset$, satisfying the self-shrinker equation for mean curvature flow

$$H = \frac{\langle \vec{x}, \vec{\nu} \rangle}{2}, \tag{5.1}$$

away from $\partial\Sigma$, and normalize to extinction time $T = 1$.

In the paper (18) (or is $n > 2$ another paper?), Huisken showed that the only positive mean curvature $H > 0$ rotationally symmetric surface Σ^n, defined by revolution of an entire graph over the x_1-axis is the cylinder. However, we will prove the following theorem.

Theorem 31. *In $\mathbb{R}^{n+1}$ there exists a 1-parameter family of rotationally symmetric, embedded, self-shrinking, asymptotically conical ends Σ^n with $H > 0$. Each is smooth with boundary $\partial\Sigma = S^{n-1}$ a scaled S^{n-1} in the $\mathbb{R}^n$-hyperplane, centred at the origin and with diameter $d < \sqrt{2(n-1)}$.*

In fact for each symmetric cone $\mathscr{C}$ in $\{x_1 \geq 0\} \subseteq \mathbb{R}^{n+1}$ with tip at the origin, there is such a self-shrinker, lying outside of $\mathscr{C}$, which is asymptotic to $\mathscr{C}$ as $x_1 \to \infty$.

Remarks 1.

(1) *Theorem 31 shows in particular that not all solutions will eventually stop being imbedded (becoming immersed only), for increasing (positive) x-values. This fact is an important and necessary part of understanding the question of uniqueness of embedded genus 0 self-shrinkers. Such uniqueness is thus a quite global property.*

(2) *It is interesting to note that such self-shrinking ends could potentially have applications in gluing constructions for self-shrinkers.*

5.2 Initial conditions "at infinity" for a non-linear ODE with cubic gradient term

The theorem we will prove is more precisely stated as follows:

Theorem 32 (:= Theorem 1'). *For each fixed ray from the origin,*

$$r^\sigma(x) = \sigma x, \quad r^\sigma : (0,\infty) \to \mathbb{R}_+,$$

there exists a unique smooth graphical solution $u^\sigma : (0,\infty) \to \mathbb{R}_+$, of the rotational self-shrinker ODE

$$u''(x) = \Big[\frac{x}{2}u'(x) + \frac{n-1}{u(x)} - \frac{u(x)}{2}\Big]\big(1 + (u'(x))^2\big) \tag{5.2}$$

satisfying $u^\sigma \geq r^\sigma$, and which is $O(\frac{1}{x})$-asymptotic to r^σ as $x \to +\infty$. Moreover any solution $u : (0,\infty) \to \mathbb{R}^+$ to 5.2 belongs to this family.

We will need the following lemma, which observes that, generally speaking, solutions become non graphical.

Lemma 33. *Consider a fixed ray from the origin $r^\sigma(x) = \sigma x$, where $\sigma > 0$. If $a \in (\frac{\sqrt{2}}{\sigma}, \infty)$, and (a, x_a^∞) is a maximally extended graphical solution to the initial* *value problem*

$$\begin{cases} u'' = \left[\frac{x}{2}u' + \frac{1}{u} - \frac{u}{2}\right]\left(1 + (u')^2\right), \\ u(a) = \sigma a, \\ u'(a) \geq \sigma, \end{cases} \tag{5.3}$$

then $x_a^\infty < \frac{3a}{2}$*.*

Proof of Lemma 33. If we define for u the quantity

$$\Phi(x) := \frac{x}{2}u' + \frac{1}{u} - \frac{u}{2},$$

then by Equation (5.2) we have $\Phi(a) = \frac{1}{\sigma a}$. We claim that in fact $\Phi(a) \geq \frac{1}{\sigma a}$ always. Namely assuming this holds up to x we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{x}{2}u' + \frac{1}{u} - \frac{u}{2}\right) = \frac{x}{2}u'' - \frac{u'}{u^2} &= \frac{x}{2}\left[\frac{x}{2}u' + \frac{1}{u} - \frac{u}{2}\right]\left(1 + (u')^2\right) - \frac{u'}{u^2} \\ &\geq \frac{x}{2}\frac{1}{\sigma a}\left(1 + (u')^2\right) - \frac{u'}{u^2} \geq 0, \end{aligned}$$

under the assumption that $a \geq \frac{\sqrt{2}}{\sigma}$. Hence the condition $\Phi(a) \geq \frac{1}{\sigma a}$ is preserved by the ODE (as we move to the right, i.e. increasing x), and implies in particular $u' \geq \sigma$ always.

Using this, we can estimate: $u'' \geq \frac{1}{\sigma a}\left(1 + (u')^2\right)$. Integrating this inequality gives

$$u'(x) \geq \tan\left[2\frac{x-a}{\sigma a} + \arctan\sigma\right],$$

which finally leads to

$$x_\infty < \frac{\sigma a}{2}\left(\frac{\pi}{2} - \arctan\sigma\right) \leq \frac{3a}{2}.$$

$\square$

Lemma 34 (A useful bootstrapping identity)**.** *Any solution* $u : (0, \infty) \to \mathbb{R}$ *to (5.13) satisfies the identity*

$$u(x) = (n-1)x\int_x^\infty \frac{1}{t^2}\left\{\int_t^\infty \frac{(1 + u'(s)^2)}{u(s)} e^{-1/2\int_t^s z\left(1+u'(z)^2\right)dz} dt\right\} ds, \tag{5.4}$$

Proof. Suppose first that we are given a solution $u : (0,a) \to \infty$ over an interval $(0,a)$. We can regard the solution u as solving the inhomogeneous linear equation determined by freezing the coefficients of (**??**) at u:

$$u'' - \big(1+\varphi'^2\big)\frac{x}{2}u' + \big(1+\varphi'^2\big)\frac{u}{2} = (n-1)\frac{\big(1+(\varphi'+\sigma)^2\big)}{\varphi} \tag{5.5}$$

where we have set $u = \phi$ above. We simply solve the resulting linear equation with variable coefficients, for $x \in (0,a)$, by making the observation that a pair of spanning solutions of the homogeneous linear equation are

$$u_1(x) = x, \quad \text{and} \quad u_2(x) = x\int_x^a \frac{e^{\int_s^a z(1+\varphi'^2)dz}}{s^2}ds, \tag{5.6}$$

Then computing the Wronskian $W(s) = e^{F_a(s)}$, and matching the initial conditions in (5.13) gives

$$\begin{aligned} u(x) &= \frac{u(a)}{a}x + (u(a) - u'(a)a)x\int_x^a \frac{e^{\int_s^a z(1+\varphi'^2)dz}}{s^2}ds & (5.7)\\ &+ +(n-1)x\int_x^a \frac{1}{t^2}\left\{\int_t^a \frac{(1+u'(s)^2)}{u(s)}e^{-1/2\int_t^s z\left(1+u'(z)^2\right)dz}dt\right\}ds, & (5.8)\end{aligned}$$

The remainder of the proof will be concerned with asymptotic behavior of the terms in(5.7) as $a \to \infty$. That is, we will show, using elementary arguments, that

$$\frac{u(a)}{a} \to \sigma$$

for some limit σ, and

$$(u(a) - u'(a)a)x\int_x^a \frac{e^{\int_s^a z(1+\varphi'^2)dz}}{s^2}ds \to 0.$$

Observe that for any solution $u : (0,\infty) \to \mathbb{R}$ the quantity $\Phi(x) = xu'(x) - u(x)$ is always negative. For suppose $\Phi(x_1) \geq 0$ for some point x_1. Then

$$\Phi'(x_1) = x_1 u''(x_1) \tag{5.9}$$

But

$$u''(x_1) > \frac{1}{2}\Phi(x_1) + \frac{1}{u(x_1)} > 0 \tag{5.10}$$

so that once Φ is non-negative, it is strictly increasing. Moreover, one must also have $u'(x_1) > 0$, since the positivity of Φ implies the positivity of u'. Thus, we will eventually have $u > \sqrt{2}$, $\Phi > 0$, $u' > c$ on some interval $[d, \infty)$ at which point we can apply Lemma 33 to show that u has to become non graphical. This then implies that ratio $\frac{u(a)}{a}$ is monotonically decreasing in a, and hence converges to some limit $\sigma \geq 0$. The positivity of Φ also implies that

$$u(x) > (n-1)x \int_x^a \frac{1}{t^2} \left\{ \int_t^a \frac{(1+u'(s)^2)}{u(s)} e^{-1/2 \int_t^s z\left(1+u'(z)^2\right)dz} dt \right\} ds$$

This then gives that there exists an sequence a_k increasing to infinity such that $u(a_k) \geq \sqrt{2}$: Otherwise, one would have that $u(x) < \sqrt{2}$ for large enough x. Then appealing to the inequality (5.10) gives the contraction

$$u(x) > (n-1)x \int_x^a \frac{1}{t^2} \left\{ \int_t^a \frac{(1+u'(s)^2)}{u(s)} e^{-1/2 \int_t^s z\left(1+u'(z)^2\right)dz} dt \right\} ds > \frac{2}{\sqrt{2}} = \sqrt{2}(n-1).$$

Moreover, we can choose such a sequence a_k so that $u''(a_k) \geq 0$: If u'' were positive only when $u < \sqrt{2}$, a glance at equation (5.2) gives that u would eventually be below the line $y = \sqrt{2}$. For such a sequence a_k, one has,

$$0 < u(a_k) - u'(a_k)a_k < \sqrt{2}.$$

and

$$\begin{aligned} \int_x^{a_k} \frac{e^{-1/2 \int_s^{a_k} z(1+u(z)'^2)dz}}{s^2} ds &< \int_x^{a_k/2} \frac{e^{\frac{1}{2}(s^2 - {a_k}^2)}}{s^2} + \int_{a_k/2}^{a_k} \frac{e^{\frac{1}{2}s^2 - {a_k}^2}}{s^2} dt \\ &< e^{\frac{1}{2}(1/4-1)a_k^2} \left[\frac{1}{x} - \frac{2}{a_k} \right] + \left[\frac{2}{a_k} - \frac{1}{a_k} \right] \end{aligned} \tag{5.11}$$

Thus, taking this sequence $a_k \to \infty$ in (5.7), we derive the limiting expression for u as

$$u(x) = (n-1)x \int_x^\infty \frac{1}{t^2} \int_t^\infty \frac{s(1+u'(s)^2)}{u(s)} e^{-1/2 \int_t^s (1+u'(z)^2)dz} ds dt + \sigma x \tag{5.12}$$

□

Hereafter, we will denote the quantities $1/2x(1+u'^2(x))$ and $\frac{1+u'^2(x)}{u}$ by $p(x,u)$ and $g(x,u)$, respectively. When there is no chance of confusion, we will simply use

$p(x)$ and $g(x)$, respectively. With this notation, we may write 5.2 as

$$u'' - p(x)u' + \frac{p(x)}{x}u = g(x)$$

and the identity 34 becomes

$$u(x) = 2(n-1)x \int_x^\infty \frac{1}{t^2} \int_t^\infty \frac{1}{u(s)} p(s) e^{-\int_t^s p} ds dt$$

As an immediate consequence, we see that $u(x) \geq \sigma x$. This then gives

$$|u(x) - \sigma x| < \frac{2(n-1)}{\sigma} \int_x^\infty \frac{1}{t^2} \int_t^\infty p(x) e^{-\int_t^s p} ds dt = \frac{2(n-1)}{x}$$

and by similar reasoning

$$|u'(x) - \sigma| < \frac{2(n-1)}{\sigma x^2}$$

In particular, we see that any solution $u : (0, \infty) \to \mathbb{R}^+$ is monotonic over some interval (d, ∞) for $d > 0$. We next show that u is actually monotonic over all of $(0, \infty)$.

To finish the proof, we find it illustrative to construct each solution u_σ as a limit of approximating solutions. More specifically, fixing a $\sigma \geq 0$, we solve the initial value problem

$$\begin{cases} u'' = \left[\frac{x}{2}u' + \frac{n-1}{u} - \frac{u}{2}\right]\left(1 + (u')^2\right), \\ u(a) = a\sigma, \\ u'(a) = \sigma. \end{cases} \tag{5.13}$$

for a positive. On any interval where the solution exists, one derives the analogous identity

$$u(x) = (n-1)x \int_x^a \frac{1}{t^2} \left\{ \int_t^a \frac{(1 + u'(s)^2)}{u(s)} e^{-1/2 \int_t^s z\left(1+u'(z)^2\right) dz} dt \right\} ds, \tag{5.14}$$

for $x < a$. The lack of terms in this expression corresponding to the homogeneous equation is a special property initial conditions. One derives analogous uniform estimates for solutions

$$|u(x) - \sigma x| < \frac{2(n-1)}{x}$$

and

$$|u'(x) - \sigma| < \frac{2(n-1)}{\sigma x^2}$$

for any $x < a$. This then gives that each solution $u_{\sigma,a}$ extends to $(0, a)$, and that the family $\{u_a\}_{a>0}$ converges to a limiting solution u_σ on $(0, \infty)$. Note however, that each approximate solution is really only approximate: lemma 33 implies that they do not remain graphical for values of x much larger than a.

Proposition 1. *For each σ, the map Φ given by*

$$\Phi(u) = (n-1)x \int_x^\infty \frac{1}{t^2} \left\{ \int_t^\infty \frac{(1 + u'(s)^2)}{u(s)} e^{-1/2 \int_t^s z\left(1+u'(z)^2\right)dz} dt \right\} ds$$

is a contraction

Proof. for two functions u_1, u_2 we may write

$$\begin{aligned} \Phi(u_2) - \Phi(u_1) &= 2x \int_x^\infty \frac{1}{t^2} \int_t^\infty \left(\frac{1}{u_2} - \frac{1}{u_1} \right) p_2 e^{-\int_t^s p_2} \\ &+ 2x \int_x^\infty \frac{1}{t^2} \int_t^\infty \frac{p_2}{u_1} \left(e^{-\int_t^s p_2} - e^{-\int_t^s p_1} \right) \\ &+ 2x \int_x^\infty \frac{1}{t^2} \int_t^\infty \frac{1}{u_1} (p_2 - p_1) e^{-\int_t^s p_1} \\ &= a + b + c \end{aligned} \tag{5.15}$$

We estimate term a by

$$a < 2\frac{||u_2 - u_1||}{\sigma^2 x^2}$$

Term c may be estimated by

$$c < ||u_2' - u_1'||||u_2' + u_1'||\frac{1}{\sigma} \int_x^\infty \frac{1}{t^2} \int_t^\infty s e^{1/4(t^2 - s^2)} ds dt = ||u_2' - u_1'||||u_2' + u_1'||\frac{2}{\sigma x}$$

To estimate term b, we note that, for real numbers $x, y < c$, one has $|e^y - e^x| < e^c|y - x|$ so that we may estimate term b as follows:

$$\begin{aligned} b &< ||u_2' + u_1'||||u_2' - u_1'||||1 + u_2'^2||\frac{x}{\sigma} \int_x^\infty \frac{1}{t^2} \int_t^\infty 1/4(s^2 - t^2) e^{1/4(t^2 - s^2)} ds dt \\ &< ||u_2' + u_1'||||u_2' - u_1'||||1 + u_2'^2||\frac{2x}{\sigma} \int_x^\infty \frac{1}{t^3} \int_t^\infty 1/2(s^2 - t^2) e^{1/8(t^2 - s^2)} s ds dt \\ &= ||u_2' + u_1'||||u_2' - u_1'||||1 + u_2'^2||\frac{2x}{\sigma} \int_x^\infty \frac{1}{t^3} \int_0^\infty \tau e^{-\tau} d\tau dt \\ &< ||u_2' + u_1'||||u_2' - u_1'||||1 + u_2'^2||\frac{2}{\sigma x} \end{aligned}$$

Here, all norms considered are sup norms over the interval (x, ∞)

$\square$

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