The Geometry of Renormalization

Susama Agarwala
John Hopkins University

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Abstract

This thesis constructs a geometric object (called a renormalization bundle), on which the β-function of a renormalizable scalar field theory over a general compact Riemannian background space-time manifold is expressed as a connection. This is a generalization of work by Connes and Marcolli, arXiv:hep-th/0411114, who originally created a renormalization bundle for a scalar quantum field theory on a flat background. This connection also defines a "generalized β-function" for non-conformal theories as a Laurent expansion on the regularization parameter.

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Contents

1 Introduction

Renormalization theory, as developed by Feynman, Schwinger, and Tomonaga in the late 1940's and 1950's, predicts with great precision experimental results of many subtle and difficult to understand phenomena across subatomic physics. However, it doesn't make much mathematical sense. A landmark series of papers, starting with A. Connes and D. Kreimer's work in 2000 [5] and 2001 [6], and culminating in 2006, in a paper by A. Connes and M. Marcolli [8], outlines a program for rationalizing the renormalization methods of perturbative Quantum Field Theory (QFT) in geometric terms.

Perturbative QFT needs to be renormalized because the Lagrangian of a particular QFT predict probability amplitudes that are infinitely different from the quantities measured in a lab. There are several algorithms which yield a finite correct solution from an infinite incorrect solution. Dimensional regularization is a popular one because it preserves many of the symmetries of the QFT in question. It involves rewriting physically interesting integrals over space-time as formal, but conceptually meaningless, integrals, in which the dimension of space-time becomes a complex number D. The integrals can then be written in terms of Laurent series in a complex parameter, $z = D - d$, with a pole at $z = 0$. The point $z = 0$ corresponds to the original dimension of the problem, d. Finite values for divergent integrals are then extracted as residues taken around paths avoiding the singular points, by an application of Cauchy's theorem. This process of extraction, called minimal subtraction, is the renormalization method studied in this paper. However, naive dimensional regularization does not account for the fact that a complicated interaction may have additional sub-interactions which are also divergent. In the 1950's and 1960's Bogoliubov, Parasiuk, Hepp, and Zimmermann developed, corrected and proved the BPHZ algorithm for iteratively subtracting off divergent sub-interactions. This algorithm applies to dimensional regularization and other regularization schemes.

However, the question remained as to why such infinities can be "swept under the rug" using Cauchy's rule, or any other algorithm. A problem from the study of classical fields motivates this type of subtraction [7]. Consider the classical situation of an object floating in a fluid. One can apply a force F to the object, measure its acceleration, and naively calculate its inertial mass, m_i, using

$$F = m_i a \, .$$

The inertial mass, however, will be much greater than the bare mass, m, which is the mass of the object measured outside any fluid. This is because the object interacts with the surrounding field of fluid. Its inertial mass is

$$m_i = m + \alpha M \, ,$$

where α is a constant determined by the viscosity of the fluid and M is the mass of the displaced fluid (Archimedes' principle). In this scenario, the inertial mass is the renormalized mass, the bare mass, m, is the unrenormalized mass, and the M is the interaction mass, or the counterterm. This terminology carries over from the classical to the quantum context. In QFT, the particle is itself a field which can interact with itself. Therefore, one needs to subtract off its infinite self-interaction mass, the counterterm, in order for theory to match experiment. Connes and Kreimer reformulate earlier work by Kreimer and others on combinatorially-defined Hopf algebras of Feynman graphs in the language of loop groups. They then apply this new language to dimensional regularization to extract finite values from divergent integrals. Finally, they express the BPHZ renormalization process as the process of Birkhoff decomposition of loops into a Lie group defined by the Hopf algebra [5].

Connes and Marcolli [7], [8] formulate dimensional regularization and BPHZ renormalization in terms of a connection on a principal bundle over a complex two-manifold B of complex renormalization parameters (corresponding to mass and space-time dimension). This bundle along with the corresponding connection seems to be a new object in both mathematics and physics. Similar bundles can be constructed for many QFTs that satisfy certain regularization conditions and are renormalized by minimal subtraction. This thesis extends their construction to ζ-function regularization over a general curved space-time background, M. This requires treating the mass parameter as a conformal density, and interpreting B as the fiber of a bundle over M. Connes and Marcolli's construction of the β-function extends to this context. As a test case I consider the case of the conformally invariant $\phi^{\frac{2n}{n-2}}$ model in dimensions three, four and six.

All of the work in [5], [6], [7], [8] and this paper is done for a fictional scalar field living in six dimensional space-time given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(|d\phi|^2 - m^2\phi^2) + g\phi^3 \; . \tag{1}$$

I choose thi Lagrangian because working with this fictional scalar field keeps the calculations simple. The six dimensional space-time is that it is the simplest renormalizable theory, as shown in section 2.1.1. The work can be generalized to physical field theories, although many of the calculations in doing so become more difficult. Some of this generalization has been done in [22].

There are many classic textbooks from which I draw my physics background. I cite Itzykson and Zuber [18], Peskin and Schroeder [32], Ryder [37], and Ticciati [40] at various points in the paper. The AMS has published a two volume series recording the lectures from the 1996-1997 Special Year at the Institute for Advanced Studies. The first volume is a thorough overview of QFT. Several chapters of this volume are cited throughout the paper. Finally, techniques useful in understanding ζ-function regularization can be found in [13]. For understanding the algebraic aspects of the Hopf algebra of Feynman graphs, Kreimer, Ebrahimi-Fard, Guo and Manchon have done extensive work exploring the structure of the graphs and their algebra homomorphisms [9], [10], [11], [12], and [25]. Along with the work of Connes and Kreimer [4], [5], [6] which established this field, the four above mentioned persons and their co-authors have produced an extensive body of literature on this topic.

1.1 Organization of this paper

The key tool to renormalization, Feynman diagrams (also referred to as Feynman graphs), is introduced in section two of this paper. These are graphical representations of the possible particle interactions in a QFT. They can be represented by the Feynman rules as distributions on the space of test functions describing the momenta of the particles involved in an interaction. Let n be the number of particles involved in the interaction represented by a Feynman diagram and d be the space-time dimension of the theory. Each particle is represented as a test function in momentum space, $\mathbb{R}^d$. Let $E = C_c^\infty(\mathbb{R}^d)$ be the space of test functions. The distribution associated to a diagram by the Feynman rules acts on an n-fold symmetric product of these test functions, $S^n(E)$. As such, the distributions can be considered element in the restricted dual space $S^n(E)^\vee$. These distributions, written as integrals, are called Feynman integrals. The Feynman rules translate between the diagrams and the distributions.

$$\text{Feynman rules}: \text{ Feynman diagrams} \leftrightarrow \text{Feynman Integrals } \subset S^n(E)^\vee \; .$$

Feynman integrals frequently yield divergent results. The Feynman diagrams capture much of the information about the nature of these divergences, which can then be extracted by knowing the details of the particular QFT and the Feynman rules translating between the distributions and the diagrams.

Kreimer's realization that the Feynman diagrams formed a Hopf algebra was an important step to rigorously codifying the renormalization process. Section three develops the Hopf algebra generated by the Feynman diagrams of a specific Lagrangian, $\mathcal{H}_{\mathcal{L}}$, as a bigraded algebra, defines the associated Lie group, $G_{\mathcal{L}}$, and establishes the associated Lie algebra, $\mathfrak{g}_{\mathcal{L}}$, through the Milnor-Moore theorem. The Hopf algebra and its associated Lie algebra and Lie group are the algebraic geometric cornerstones of this method of renormalization. The structures of and the relationships between these three objects and their dual objects are key to the construction of Connes and Marcolli's renormalization bundle.

The final step in codifying renormalization was Connes and Kreimer's realization that the BPHZ renormalization algorithm was exactly a problem of Birkhoff decomposition on the abstract space of complex space-time dimension. The regularization process rewrites the Feynman integrals that have divergent evaluations on E as Laurent polynomials with distribution valued coefficients, which are convergent when the regularization parameter z is away from 0:

$$\text{Regularization} \;:\; \text{Feynman integrals} \to \mathrm{Hom}_{\mathrm{vect}}(S^n(E), \mathbb{C}\{\{z\}\}) \;.$$

Evaluating a regularized Feynman integral on a test function gives a Laurent series with a non-zero radius of convergence, $\mathbb{C}\{\{z\}\}$. Connes and Kreimer express this as the set of algebra homomorphisms from $\mathcal{H}_{\mathcal{L}}$ to the ring $\mathbb{C}\{\{z\}\}$, called $G_{\mathcal{L}}(\mathbb{C}\{\{z\}\})$. These homomorphisms are in correspondence with maps from closed loops in the infinitesimal complex space of the regularization parameter to the complex Lie group $G_{\mathcal{L}}$. This fact allows the problem of renormalization to be cast in the language of loop groups.

Section four introduces the Birkhoff decomposition theorem, and discusses how it solves the problem of BPHZ renormalization. Presley and Segal [33] discuss the Birkhoff decomposition theorem in chapter 8 of their book "Loop Groups". Work done by Ebrahimi-Fard, Guo and Kreimer [10] shows that since $\mathbb{C}\{\{z\}\}$ can be given the structure of a Rota-Baxter algebra, renormalization can be studied in the context of algebra homomorphisms from $\mathcal{H}_{\mathcal{L}}$ to a Rota-Baxter algebra. Then the renormalized homomorphisms can be written as a sub-Lie group of the Lie group, G. This substructure is necessary for later analysis. However, as the Rota-Baxter algebra is only tangential to the construction of the bundle, the Rota Baxter structure relating to the algebra of Feynman diagrams is dealt with in Appendix A. Appendix B is a summary of some algebraic notation.

In the process of dimensional regularization, one introduces a mass parameter to balance the fact that one is "changing the dimensions" of space-time. This gives rise to a family of effective QFTs, parametrized by the renormalization mass. The family of effective theories can be expressed as a one parameter family of automorphisms (or a $\mathbb{C}^\times$ action) on the group of algebra homomorphisms on the space of Feynman integrals, called the renormalization group. The renormalization group flow describes the effect of this automorphism on the renormalized Feynman integrals. The renormalization group flow generator, β, called the beta-function, is key to understanding this flow. Section five discusses the effects of these $\mathbb{C}^\times$ actions, and develops corresponding renormalization groups, flows and generators.

Section six constructs the renormalization bundle as sketched below. I construct a trivial connection of the bundle associated to each section of the bundle. This defines a global connection on the renormalization bundle. A class of these sections satisfy the equisingularity condition outlined by Connes and Marcolli in [8], and the corresponding connections are uniquely defined by β-functions. These sections may represent different Lagrangians having the same Feynman graphs as the Lagrangian listed above, or to different regularization schemes that have the same divergence structure as the Hopf algebra studied in [8] and this paper. Furthermore, the global connection is defined on sections that do not satisfy the equisingularity condition. However, this may provide a means of relating physical and non-physical renormalization schemes.

Section seven constructs a similar bundle for a scalar field theory on a Riemannian manifold, under ζ-function renormalization. Dimensional regularization and ζ-function regularization can be written as sections of the same bundle on a flat background. Instead of changing the dimension of the QFT, ζ- function regularization replaces the "propagators" or Green's functions of the Laplacian on a manifold, written $-\Delta^{-1}$ by operators raised to complex powers $(-\Delta)^{-1+r}$. On a general manifold, ζ-function regularization depends on the metric, and thus the position over the manifold. Thus the bundle for ζ-function regularization is built

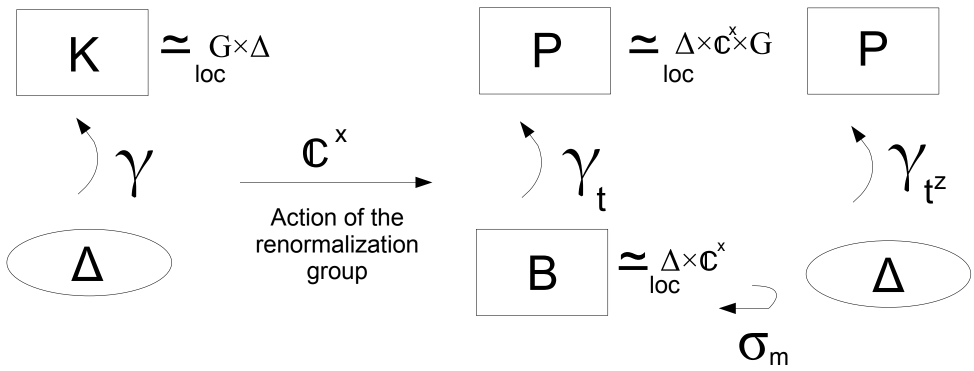


Figure 1: Schematic representation of the renormalization bundle and before and after the action of the renormalization group.

over the manifold as a base. The β-function of this theory is uniquely determined by the counterterms of this section, and is a function of the curvature of the manifold.

1.2 Overview of the renormalization bundle

Figure 1 sketches the Connes-Marcolli renormalization bundle. The regularized QFTs are geometrized as sections of the $K \to \Delta$ bundle on the left. K can be written as a trivial principal $G_{\mathcal{L}}$ fiber bundle over Δ where $G_{\mathcal{L}} = \text{Spec } \mathcal{H}_{\mathcal{L}}$. $G_{\mathcal{L}}$ is an affine pro-unipotent group with an underlying Lie group structure.

The infinitesimal disk, $\Delta \simeq \text{Spec } \mathbb{C}\{z\}$, is the space defined by the regularization parameter of a QFT. The regularized integrals may have a singularity at $z = 0$, which corresponds to the unregularized theory.

B is a trivial $\mathbb{C}^\times$ principal fiber bundle over Δ. The $\mathbb{C}^\times$ comes from a mass term that must be incorporated to perform dimensional regularization. While the physical mass scale is real, the underlying symmetry extends to $\mathbb{C}$ as explained in section 5.2.

The $P \to B$ bundle incorporates the action of the renormalization group. It is a trivial $G_{\mathcal{L}}$ principal bundle over B and equivariant under a $\mathbb{C}^\times$ action. The $P \to \Delta$ bundle is the pullback of the $P \to B$ bundle along a specific action of the renormalization group. Sections of this bundle are geometric representations of the fixing of the energy scale for an effective Lagrangian corresponding to a section of $K \to \Delta$. The details are described in section 5.2.

Let $W \subset K$ be the fiber over 0. Then $K \setminus W \to \Delta \setminus 0$ is written $K^* \to \Delta^*$. Consider the group of sections of $K^* \to \Delta^*$:

$$\gamma : \Delta^* \to G_{\mathcal{L}} \ .$$

These γ are the relevant maps in the Birkhoff decomposition. Notice that $\Delta^* \simeq \text{Spec } (\mathcal{A})$ where $\mathcal{A} = \mathbb{C}\{\{z\}\} = \mathbb{C}\{z\}[z^{-1}]$ is the localization of the ring of functions on Δ away from 0. Therefore, the sections, γ, can be rewritten

$$\gamma : \text{Spec } \mathcal{A} \to \text{Spec } \mathcal{H} \ .$$

The group $\text{Maps}(\Delta^*, G_{\mathcal{L}})$ is isomorphic to the group $G_{\mathcal{L}}(\mathcal{A}) = \text{Hom}_{\text{alg}}(\mathcal{H}_{\mathcal{L}}, \mathcal{A})$. A regularized QFT maps polynomials of the test functions on E to elements of $G_{\mathcal{L}}(\mathcal{A})$. The singular behavior of elements of $G_{\mathcal{L}}(\mathcal{A})$ is captured in the fiber of K over $0 \in \Delta$.

Let $V \subset B$ be the fiber of B over the origin in Δ, and $B^* = B \setminus V$. In the principal $G_{\mathcal{L}}$ bundle over B^*, the singular behavior of sections is captured in $V \times G_{\mathcal{L}}$. The bundle over the punctured disk defined by $P^* = P \setminus (V \times G_{\mathcal{L}})$ is the object of interest in this paper. The Laurent series corresponding to sections of this bundle are well defined over the punctured disk.

Let σ_m be the sections of the bundle $B \to \Delta$ defined by

$$\sigma_m(z) = (z, m^z) .$$

Then the pullbacks

$$\gamma_t \circ \sigma_m : \Delta^* \to P^*$$

are interesting because they display to the effect of the renormalization group on the dimensionally regularized Lagrangian. This is the notation used in Section 5 to be consistent with dimensional regularization. In Section 6, σ is allowed to be any section of $B \to \Delta$. There is a subset of algebra homomorphisms $G^{\Phi}_{\mathcal{L}}(\mathcal{A}) \subset G_{\mathcal{L}}(\mathcal{A})$ which satisfy certain physical conditions, called locality, also satisfied by regularized Feynman integrals. This subset is defined in [25]. It is this subset that defines the flat equisingular connections of Connes and Marcolli which lets one find a β function for the QFT in question. These various parts of the renormalization bundle are explained in greater detail as they appear throughout this paper.

2 Feynman graphs

A particular QFT is defined by a Lagrangian, which can be written as a sum of a free part, $\mathcal{L}_F$, and an interaction part $\mathcal{L}_V$

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_V$$

where $\mathcal{L}_F$ is a quadratic term involving derivatives of the fields, and $\mathcal{L}_V$ is a polynomial function of the fields (with minimal degree = 3). A Lagrangian prescribes the set of possible interactions of a theory, which can be written either as integrals or diagrams; the Feynman rules translate between the two. While everything in this paper can be generalized to different dimensions and more fields, I use the following relatively simple Lagrangian with space-time dimension six as the example Lagrangian:

$$\mathcal{L} = \frac{1}{2}(|d\phi|^2 - m^2\phi^2) + g\phi^3 . \tag{2}$$

Here $\mathcal{L}_F = \frac{1}{2}(|d\phi|^2 - m^2\phi^2)$ with d an exterior derivative, and $\mathcal{L}_V = g\phi^3$ where ϕ is a real scalar field and g is called the coupling constant. For development of and calculations involving fields of this type following well established traditions in the physics literature see [40], chapter two, [41] and [19]. As there are several very good text books on QFT which develop these Lagrangians as well as the forms that the fields must take, I will not go into the details here. A classic text book is Peskin and Schroeder [41], while a less conventional but more axiomatic book is by Ticciati [40].

Definition 1. A Feynman diagram is an abstract representation of an interaction of several fields. It is drawn as a connected, not necessarily planar, graph with possibly differently labeled edges. The orientation of the graph in the plane does not matter. It is a representative element of the equivalence class of planar embeddings of connected non-planar graphs. The types of edges, vertices, and the permitted valences are determined by the Lagrangian of the theory in the following way:

1. The edges of a diagram are labeled by the different fields in the Lagrangian.
2. The degrees of the monomial summands in $\mathcal{L}_V$ correspond to permissible valences of the Feynman diagrams. The composition of these monomials determine the types of edges that may meet at a vertex.

3. Vertices of valence one are replaced by half edges. That is, the vertex is “cut off”, leaving a stub. These are different than vertices with valence one, which do not exist in this formulation of the Feynman rules.

These rules apply to the Lagrangian in equation (2) as follows:

1. The Lagrangian in equation (2) only involves one field, and therefore only has one type of edge. For more complicated Lagrangians, differently labeled edges are often portrayed as different types of lines (dashed, wavy, etc.) in drawing the Feynman diagram. For a detailed treatment of the Lagrangian involved in QED, for instance, see [22].

2. Since $\mathcal{L}_V = g\phi^3$, only valence three vertices are allowed for this Lagrangian. If $\prod_i |\phi_i|^{n_i}$ is a monomial in $\mathcal{L}_V$, then vertices of valence $\sum_i n_i$ are allowed, with n_i legs of type ϕ_i meeting at a vertex.

The definitions and terminology in this section follows [21]. The structure of the graph is defined as follows:

Definition 2. 1. An external edge is an edge that is connected to only one vertex. External edges are also called half edges as above.

2. Internal edges are edges connected to two vertices.

3. Let $\Gamma^{[0]}$ be the set of vertices of the graph Γ, and let $\Gamma^{[1]}$ be the set of edges. Let $\Gamma^{[1]}_{\mathrm{ext}}$ be the external edges of the graph and $\Gamma^{[1]}_{\mathrm{int}}$ be the internal edges.

Remark 1. For purposes of drawing Feynman diagrams, external edges are drawn primarily as a book keeping device for the valence and type of a vertex. This is more important for more complicated Lagrangians. For purposes of evaluating the corresponding Feynman integrals, the external legs indicate the number of particles involved in the interaction. In the case of multiple field types, the external legs also keep track of the types of test functions that the Feynman integral acts on.

Graphs that satisfy the above condition can be classified as follows:

Definition 3. 1. Vacuum to vacuum graphs have no external edges.

2. Particle to particle graphs have external edges.

3. A one particle irreducible graph is a connected Feynman graph such that the removal of any internal edge still results in a connected graph.

I ignore vacuum to vacuum graphs and graphs with only one external leg in the sequel, following [42]. For particle to particle diagrams, I am mainly concerned with one particle irreducible (1PI) diagrams. These are the building blocks of the set of Feynman diagrams, as any non 1PI Feynman diagram can be created by gluing together 1PI graphs along external edges [42].

2.1 Subgraphs

Given a Feynman graph Γ associated to a divergent Feynman integral, the BPHZ renormalization process iteratively subtracts off divergent subgraphs, γ, and considers the remaining contracted graph $\Gamma//\gamma$. For details, see [18], Section 8.2 and [42]. This section describes what these divergent subgraphs look like, and how to construct the contracted graphs.

2.1.1 Divergent graphs

Because of the Feynman rules translating between possibly divergent distributions and Feynman graphs, one can define a quantity $\omega(\Gamma)$ called the superficial degree of divergence for the integral associated to the 1PI graph Γ. Given information about the fields involved, the dimension and level of divergence of a particular QFT, $\omega(\Gamma)$ also gives information about possible valences of vertices and number of external legs allowed for graphs in that theory. A more complete and general exposition of this subject is given in [18] Section 8.1 for a theory in four space-time dimensions, and in [42] for certain classes of theories in various dimensions.

For the scalar field, the superficial degree of divergence is given by

$$\omega(\Gamma) = dL(\Gamma) - 2I(\Gamma) \tag{3}$$

where d is the space-time dimension of the theory, $I(\Gamma)$ is the number of internal lines and L is the loop number,

$$L(\Gamma) = I(\Gamma) - V(\Gamma) + 1 \tag{4}$$

or Euler characteristic of the graph, with $V(\Gamma)$ the number of vertices in the graph. If $\omega(\Gamma) \geq 0$, the graph is called superficially divergent. These graphs are generally associated to divergent integrals. (Graphs with $\omega(\Gamma) < 0$ are generally associated to convergent integrals.)

Plugging equation 4 into 3 gives

$$\omega(\Gamma) = (d-2)I(\Gamma) - dV(\Gamma) + d\ .$$

Let E_v be the number of external edges meeting a vertex v in Γ. Also, let $\Gamma^{[0]}$ be the set of vertices in the graph Γ. Furthermore, an external leg is attached to one vertex, while an internal edge is attached to two. Therefore,

$$2I = \sum_{v\in\Gamma^{[0]}} (n_v - E_v)$$

where n_v is the valence of the vertex v. This gives

$$\omega(\Gamma) = \sum_{v\in\Gamma^{[0]}} (\frac{d-2}{2} n_v - d) - \sum_{v\in\Gamma^{[0]}} \frac{d-2}{2} E_v + d\ . \tag{5}$$

The total number of external edges of Γ is

$$E(\Gamma) = \sum_{v\in\Gamma^{[0]}} E_v\ .$$

The contribution of each vertex to $\omega(\Gamma)$ is given by $\frac{d-2}{2} n_v - d = \omega_v$.

The dependence of the superficial degree of divergence on the number of vertices of a graph classifies QFTs into 3 classes:

1. Super-renormalizable theories have

   $$\omega_v < 0\ .$$

   The degree of divergence decreases with the number of vertices of graphs in this theory.

2. Non-renormalizable theories are such that

   $$\omega_v > 0\ .$$

   The degree of divergence increases with the number of vertices of graphs in this theory.

3. Renormalizable theories are those where

$$\omega_v(\Gamma) = 0 \; . \tag{6}$$

The degree of divergence does not increase or decrease as graphs get more complicated. All graphs contributing to divergences here have the same degree of divergence. Therefore, renormalization can be done with a finite number of parameters.

Renormalizable theories are the topic of this paper. In this case equation (6) gives

$$0 = \omega_v = \frac{(d-2)n_v}{2} - d$$

for all v. For a theory with only one type of vertex,

$$n_v = \frac{2d}{d-2} \tag{7}$$

for all vertices. Notice that since n_v is the valence of a vertex it must be an integer.

Remark 2. Write the polynomial

$$\mathcal{L}_V = \sum_{n \geq 3} g_n \phi^n$$

for a scalar theory in dimension d. If the only non-zero term of this sum is for $n = \frac{2d}{d-2}$, then by equation (7) the theory is *renormalizable*. In a similar calculation to the one above, if the only non-zero coefficients are for

$$n < \frac{2d}{d-2}$$

then the theory is *super-renormalizable*. Likewise if the only non-zero terms are for

$$n > \frac{2d}{d-2}$$

the theory is *non-renormalizable*. Notice that for *renormalizable* theories n_v is an integer only if $d \in \{3, 4, 6\}$. Specifically for $d = 6$, $n_v = 3$.

Graphs with superficial degrees of divergence also have an extra condition on the number of external legs they can have. Equation (5) gives

$$0 \leq \omega(\Gamma) = \sum_{v \in \Gamma^{[0]}} \omega_v - \frac{E(\Gamma)(d-2)}{2} + d \; .$$

That is,

$$E(\Gamma) \leq \frac{2(d + \sum_{v \in \Gamma^{[0]}} \omega_v)}{d-2} \; .$$

For super-renormalizable theories, the number of external legs in a superficially divergent graph decreases with the complexity of the graph. For non-renormalizable theories, the number of external legs increases with the complexity of the graph. Only for renormalizable theories is there a fixed bound on the number of external legs a superficially divergent graphs can have regardless of the number of vertices of Γ.

Remark 3. For a superficially divergent subgraph in a theory of dimension d, the external leg structure of Γ satisfies

$$E(\Gamma) \leq \frac{2d}{d-2} \; .$$

Therefore, if $d = 6$ superficially divergent graphs can only have 2 or 3 external legs.

2.1.2 Admissible subgraphs

The previous section defined the Feynman graphs that have divergent integrals: call these Γ. If Γ can be found as a proper subgraph of another Feynman graph Γ' then one needs to isolate the contribution of Γ to the divergence of Γ'. The contribution of Γ to the divergence of Γ' is called a subdivergence. In this case Γ is said to be a subgraph of Γ', even though Γ is a Feynman graph in its own right. For this reason, in this paper, the word subgraph refers to both the subgraph (as a collection of edges and vertices) inside a larger Feynman graph, and the Feynman diagram that collection of vertices and edges forms on its own. Bogoliubov, Parasiuk, Hepp and Zimmerman in the 1950's and 1960's developed a method for accurately subtracting off the subdvergent contribution of divergent subgraphs. This is called the BPHZ renormalization process.

Definition 4. For a 1PI Feynman diagram Γ, γ is an admissible subgraph if and only if:

1. γ is subgraph of Γ, as a collection of vertices and edges.
2. The collection of edges and vertices in γ form a superficially divergent 1PI Feynman diagram, or a disjoint union of such diagrams.

If γ is connected, it is called a connected admissible subgraph of Γ. If it is disconnected it is called a disconnected admissible subgraph of Γ. I first develop connected admissible subgraphs as these are the building blocks of all admissible subgraphs. The following terminology can be found in [21]. Let Γ be a 1PI graph.

Definition 5. For a $v \in \Gamma^{[0]}$, the set of edges meeting v is denoted $f_v = \{f \in \Gamma^{[1]} | f \cap v \neq \emptyset\}$. Then $|f_v|$ is the valence of v, and the types of lines in f_v determine the type of the vertex v.

Using this notation I am now ready to define connected admissible subgraphs.

Definition 6. A connected admissible subgraph γ of Γ consists of the following data:

1. A subset of vertices of Γ: $\gamma^{[0]} \subset \Gamma^{[0]}$.
2. A collection of interior edges meeting these vertices: $\gamma^{[1]}_{\text{int}} \subset \bigcup_{v \in \gamma^{[0]}} f_v$.
3. The subgraph γ is connected and 1PI.

To calculate the divergences that these subgraphs contribute to the overall divergence of Γ, one needs to express the subgraphs as Feynman graphs in their own right. One needs to discuss their external leg structure. In the sequel, admissible subgraph will refer both to the Feynman graph associated to an admissible subgraph and the admissible subgraph itself. Defining the Feynman graph associated to the admissible subgraph requires two more rules:

Definition 7. The internal legs of γ are those legs specified in the subset $\Gamma^{[1]}_{\text{int}}$. The external legs are not in this subset and are given by the elements of $\Gamma^{[1]}$ that intersect with γ. The Feynman diagram associated to an admissible subgraph γ of Γ is the subgraph with the following external leg structure:

1. The exterior edges of the subgraph are given by the map

$$\begin{array}{rcl} \rho : \left(\bigcup_{v \in \gamma^{[0]}} f_v \right) \setminus \gamma^{[1]}_{\text{int}} & \rightarrow & \gamma^{[1]}_{\text{ext}} \\ f & \mapsto & \begin{cases} f_1, & |f \cap \gamma^{[0]}| = 1 \\ \{f_1, f_2\}, & |f \cap \gamma^{[0]}| = 2 \,. \end{cases} \end{array}$$

That is, if an edge of Γ meets a single vertex of γ, it is represented by an external edge of γ. If it meets γ at two vertices, and is not an internal edge of γ, then it is represented by two external legs of γ.

2. The external edges of γ must correspond to a prescribed configuration of edges for a divergent subgraph.

This last condition ensures that only subgraphs contributing to subdivergences are considered, as shown in the example in the next section. If teh exterior leg structure does not satisfy the configuration for a divergernt graph, γ is not a divergent subgraph and does not need to be considered for BPHZ renormalization. In other words, it is not an admissible subgraph.

To define the more general concept of disconnected admissible subgraphs, I first define non-overlapping subgraphs.

Definition 8. 1. Let γ_1 and γ_2 be two graphs. They are non-overlapping subgraphs of Γ if, for $j = 1, 2$, there exists an insertion map i_j such that

$$i_j : \gamma_j^{[0]} \to \Gamma^{[0]} \qquad ; \qquad \bigcap_j i_j(\gamma_j) = \emptyset \, .$$

2. If $\gamma = \gamma_1 \coprod \gamma_2$ for γ_1, γ_2 non-overlapping subgraphs of Γ, then γ is a *disconnected* admissible subgraph of Γ.

This is the most general form of an admissible subgraph. In the sequel, the word subgraph will refer only to admissible subgraphs, either connected or disconnected, unless otherwise specified.

Notice that the entire graph is also an admissible subgraph. One can also define the empty graph to be an admissible (trivial) subgraph. The set of *proper subgraphs* of Γ is the set of all subgraphs of Γ connected or not, minus the entire graph and the empty subgraph.

2.1.3 Contracted graphs

Along with identifying the subgraphs and subdivergences, the BPHZ renormalization process identifies the divergences remaining in the diagram after the subtraction of a subdivergence. To do this, the connected admissible subgraph associated to the subdivergence is contracted to a vertex, and the divergences of the resulting Feynman diagram is studied. In the case of a disconnected admissible subgraph, each connected component is contracted. The remaining Feynman diagram is called a contracted graph.

Definition 9. Let γ be a disconnected admissible subgraph of Γ consisting of the connected components $\gamma_1 \ldots \gamma_n$. A contracted graph is the Feynman graph derived by replacing each connected subgraph, with a vertex v_{γ_i}. So for each $1 \leq i \leq n$,

$$\begin{array}{rcl} \Gamma & \to & \Gamma//\gamma \\ f \in \gamma_i^{[1]},\ v \in \gamma_i^{[0]} & \mapsto & v_{\gamma_i} \notin \Gamma^{[0]} \\ f' \notin \gamma^{[1]},\ v' \notin \gamma^{[0]} & \mapsto & f', v' \, . \end{array}$$

The resulting contracted graph is written $\Gamma//\gamma$.

Notice that $\Gamma//\gamma$ is always 1PI. Definition 7 part 2 ensures that the contracted graph $\Gamma//\gamma$ is a valid Feynman graph for the Lagrangian under consideration [18]. In the case of the ϕ^3 in dimension six theory, this means that each connected admissible subgraph must have either two or three external edges.

Example 1. The decomposition of the top graph in Figure 2 is not allowed for the example Lagrangian in equation (2). All vertices of Γ are included in γ, and the topmost internal leg becomes two external legs of γ, for a total of 4 external legs. This means that γ does not contribute to the superficial divergence of Γ. It leaves $\Gamma//\gamma$ with a valence 4 vertex, which is not allowed in the theory. The decomposition of the bottom graph is valid, however. The vertex in $\Gamma//\gamma$ that replaces γ is denoted by $\times$ in this figure for the reader's convenience. For purposes of calculations on the graph $\Gamma//\gamma$, the vertex denoted $\times$ is usually indistinguishable from any other vertex in the graph. The notable exceptions to this are vertices of valence two. These types of vertices exist only for the purpose of calculating subdivergences and counterterms. That is, this type of vertex is only found in contracted graphs (of the form $\Gamma//\gamma$), and not in a Feynman diagram generated from the Lagrangian.

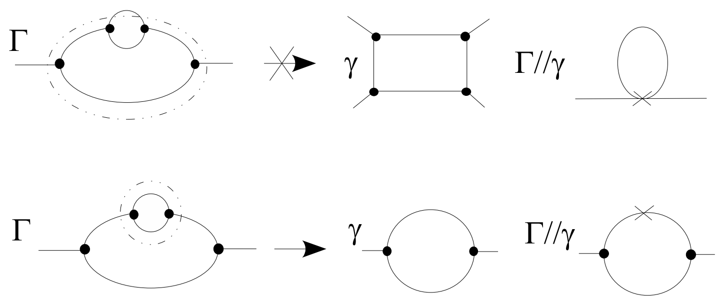


Figure 2: Inadmissible and admissible subgraphs of $\mathcal{L}$

2.2 Feynman rules

Feynman rules translate between Feynman graphs and Feynman integrals. Because of the structure of the diagrams, it is sufficient to study Feynman rules on 1PI diagrams:

$$\text{Feynman Rules} : \text{1PI Diagrams} \leftrightarrow \text{Feynman integrals} .$$

Each Feynman diagram corresponds to a distribution, which can be written as an integral on the space of test functions in momentum space I call the external leg data, E. Let $E = C_c^\infty(\mathbb{R}^6)$ be the space of test functions in momentum space for each leg. The test functions for general Feynman integrals are elements of the symmetric algebra on E, $S(E)$. Section 2.1.1 shows that for a six dimensional renormalizable scalar theory, the 1PI graphs can only have two or three external legs. The Feynman integrals for the 1PI diagrams act on test functions in $S^n(E)$ for $n \in \{2, 3\}$. General Feynman graphs are associated to integrals that are distributions on $S^n(E)$ for $n \in \mathbb{Z}_{\geq 2}$. These distributions correspond to linear maps

$$\text{Feynman Integrals} \in \text{Hom}_{\text{vect}}(S^n(E), \mathbb{C}) .$$

In practice, however, they are usually divergent. This is why QFTs need to be regularized. Regularization introduces a parameter z which captures the divergence as z goes to a predetermined limit. For the class of regularization schemes studied in this paper the divergence is captured as $z \to 0$ and regularized Feynman rules define maps:

$$\text{Feynman diagrams} \leftrightarrow \text{ Regularized Feynman integrals} \hookrightarrow \text{Hom}_{\text{vect}}(S^n(E), \mathbb{C}\{\{z\}\}) .$$

Section 4 discusses this in more detail.

To maintain consistency with the physics literature, I write down the unregularized Feynman rules, even though the resulting integrals are not well defined. In the case of six dimensional Lorentz space, for a fixed graph Γ the Feynman integrals are constructed as follows:

- To each edge, internal and external, $k \in \Gamma^{[1]}$ assign a momentum $p_k \in \mathbb{R}^6$ and a propagator $\frac{i}{p_k^2 - m^2}$.
- To each vertex $v \in \Gamma^{[0]}$ assign a factor of $-ig\delta(\sum_{j\in f_v} p_j)$, where δ is the Dirac delta function. The sum is taken over the momenta assigned to the edges meeting at v.
- Take the product of all the factors assigned to the edges and vertices and integrate over the internal momenta

$$\int_{\mathbb{R}^{6|\Gamma^{[1]}_{int}|}} \prod_k^{|\Gamma^{[1]}|} \frac{i}{p_k^2 - m^2} \prod_v^{|\Gamma^{[0]}|} -ig\delta(\sum_{j\in f_v} p_j) \prod_i^{|\Gamma^{[1]}_{\text{int}}|} d^6 p_i .$$

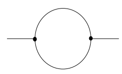
- Divide by the symmetry factor of the graph.

Notice that the second step applies conservation of momentum at each vertex, and also for the external legs of the graph. By counting the number of free variables, one sees that this is a distribution on the momentum space in $6n$ variables where n is the number of external legs. Specifically, if Γ is 1PI, $n \in \{2, 3\}$ for a renormalizable scalar theory in six dimensions. More details on Feynman rules and how they are derived can be found in [41] and [40].

Remark 4. The definition of a Feynman graph identifies two graphs if they differ only by their embedding in the plane. Dividing by the symmetry factor of the graph implements this equivalence class. The symmetry factor is the number of ways a graph can be embedded in a plane up to homeomorphism.

Example 2.

The Feynman integral for the 1PI graph



is given by

$$G(p,-p) = \frac{g^2}{2}\int_{\mathbb{R}^6} \frac{d^6k}{(k^2-m^2)((k-p)^2-m^2)} \ .$$

This is a distribution on $S^2(E)$ acting on test functions $f, g \in E$ as

$$\int_{\mathbb{R}^6} G(p,-p)f(p)g(-p)d^6p = \int_{\mathbb{R}^6} \frac{g^2}{2}\int_{\mathbb{R}^6} \frac{f(p)g(-p)d^6k}{(k^2-m^2)((k-p)^2-m^2)} \ .$$

3 The Hopf algebra of Feynman graphs

The previous section describes what the physical Feynman graphs are, what they represent and what they look like. This section moves away from the physical interpretation and looks at them as algebraic objects. Assigning variables, x_Γ, to each 1PI graph, Γ, generates a polynomial algebra which is also a commutative bigraded Hopf algebra. In general, commutative Hopf algebras can be interpreted as a ring of functions on a group. Since the spectrum of a commutative ring is an affine space, the group in question is affine group scheme, Spec $\mathcal{H}_\mathcal{L}$. This structure gives a geometric analog throughout the paper to the algebraic calculations on this Hopf algebra. This section also examines the dual of this Hopf algebra.

3.1 Constructing the Hopf algebra

Let k be a field of characteristic 0. Assign to each

$$\Gamma \in \{\text{1PI graphs of } \mathcal{L} = \frac{1}{2}(|d\phi|^2 - m^2\phi^2) + g\phi^3\}$$

the variable x_Γ where the empty graph is associated to the unit

$$1 = x_\emptyset \ .$$

The x_Γ are called the indecomposable elements of $\mathcal{H}$.

Definition 10. The vector space $k\langle x_\Gamma | \Gamma \in \{\text{1PI graphs of } \mathcal{L}\}\rangle$ is generated by the indecomposable elements of $\mathcal{H}$. The Hopf algebra associated to this Lagrangian is the polynomial algebra

$$\mathcal{H}_\mathcal{L} = k\left[x_\Gamma | \Gamma \in \{\text{1PI graphs of } \mathcal{L}\}\right] \ .$$

For ease of notation, I drop the subscript $\mathcal{L}$, as there is no confusion over the Lagrangian generating the graphs. The polynomial algebra structure of $\mathcal{H}$ allows for the study of disjoint unions of 1PI graphs (and thus disconnected admissible subgraphs for BPHZ renormalization) as shown below. This section shows that $\mathcal{H}$ satisfies the axioms of a commutative Hopf algebra. Section 3.3 defines the underlying Lie group.

The algebra structure on $\mathcal{H}$ is given by a multiplication m and a unit η. Let Γ and Γ' be 1PI graphs. Multiplication of the variables

$$\begin{array}{rccc} m: & \mathcal{H} \otimes \mathcal{H} & \to & \mathcal{H} \\ & x_\Gamma \otimes x_{\Gamma'} & \mapsto & x_\Gamma x'_\Gamma \end{array}$$

translates to a disjoint union of the 1PI graphs on the space of graphs:

$$x_\Gamma x_{\Gamma'} \leftrightarrow \Gamma \coprod \Gamma' .$$

Therefore, this product is commutative. This extends to multiplication on all of $\mathcal{H}$ by linearity.

The unit is defined as $\eta : k \to \mathcal{H}$ such that $\eta(1_k) = 1_{\mathcal{H}}$ where $1_{\mathcal{H}}$ is the empty graph, $1_{\mathcal{H}} = x_\emptyset$. It is easy to check that these operations satisfy the rules of an algebra. When the context is clear, I drop the subscript $\mathcal{H}$ and write $x_\emptyset = 1$.

Lemma 3.1. *$(\mathcal{H}, m, \eta)$ is a commutative, associative, unital algebra.*

One can also impose a coalgebra structure on this by defining a comultiplication Δ and a counit ε. I use a variation on Sweedler's notation, where $\sum_{(\Gamma)}$ indicates the sum over all proper admissible subgraphs (connected or disconnected) of Γ.

$$\begin{array}{rccc} \Delta: & \mathcal{H} & \to & \mathcal{H} \otimes \mathcal{H} \\ & x_\Gamma & \to & 1 \otimes x_\Gamma + x_\Gamma \otimes 1 + \sum_{(\Gamma)} x_\gamma \otimes x_{\Gamma//\gamma} . \end{array}$$

Sometimes, I use the shorthand

$$\sum_{(\Gamma)} x_{\Gamma'} \otimes x_{\Gamma''}$$

instead of

$$\sum_{(\Gamma)} x_\gamma \otimes x_{\Gamma//\gamma}$$

where $x_{\Gamma''} := x_{\Gamma//\Gamma'}$. The coproduct can be extended to a ring homomorphism on all of $\mathcal{H}$ by requiring

$$\Delta(x_{\gamma_1} x_{\gamma_2}) = \Delta(x_{\gamma_1})\Delta(x_{\gamma_2})$$

for all $x_{\gamma_1}, x_{\gamma_2} \in \mathcal{H}$. For a general element, y of $\mathcal{H}$, I write

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \tilde{\Delta}(y) .$$

For an indecomposable $x_\Gamma \in \mathcal{H}$, the definition of $\tilde{\Delta}$ simplifies to

$$\tilde{\Delta}(y) = \sum_{(\Gamma)} x_{\Gamma'} \otimes x_{\Gamma''} .$$

Definition 11. $x_\Gamma \in \mathcal{H}$ is primitive if $\tilde{\Delta}(x_\Gamma) = 0$, that is $\Delta(x_\Gamma) = x_\Gamma \otimes 1 + 1 \otimes x_\Gamma$.

The counit is defined on indecomposable graphs as

$$\begin{array}{rcl} \varepsilon: \quad \mathcal{H} & \to & k \\ x_\Gamma & \to & \begin{cases} x_\Gamma & \Gamma = \emptyset \\ 0, & \text{else} . \end{cases} \end{array}$$

The counit can be extended to all of $\mathcal{H}$ by multiplication. Again, it is easy to check that these operations satisfy the rules of a coalgebra.

Definition 12. Co-associativity means that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta .$$

Lemma 3.2. *$(\mathcal{H}, \Delta, \varepsilon)$ forms a non-cocommutative, co-associative, unital co-algebra.*

Proof. Proof that $(\mathcal{H}, \Delta, \varepsilon)$ forms a co-associative coalgebra can be found in [5]. □

Since Δ and ε are algebra homomorphisms,

Lemma 3.3. *$(\mathcal{H}, m, \eta, \Delta, \varepsilon)$ forms an associative, commutative, non-cocommutative unital k bialgebra.*

Recall the following definition.

Definition 13. A Hopf algebra is a bialgebra with an antipode map S satisfying

$$m(S \otimes \mathrm{id})\Delta = \varepsilon\eta = m(\mathrm{id} \otimes S)\Delta . \tag{8}$$

Given this definition, by recursively defining an antipode on $\mathcal{H}$ as follows,

$$\begin{array}{rcl} S: \mathcal{H} & \to & \mathcal{H} \\ x_\Gamma & \to & -x_\Gamma - \sum_{(\Gamma)} m(S(x_\gamma) \otimes x_{\Gamma//\gamma}) , \end{array}$$

one has the desired result.

Theorem 3.4. *$(\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ is a Hopf algebra.*

Proof. See [5] □

Given the one-to-one correspondence between the variables x_Γ and the space of 1PI graphs of $\mathcal{L}$, one can omit the x notation and view $\mathcal{H}$ as a Hopf algebra on the graphs themselves. Write

$$\mathcal{H} = k[\Gamma | \Gamma \in \{\text{1PI graphs of } \mathcal{L}\}]$$

The indecomposable elements are the 1PI graphs. Multiplication on the algebra is given by

$$m(\Gamma \otimes \Gamma') = \Gamma \coprod \Gamma'$$

on 1PI graphs. This product is written $\Gamma\Gamma'$. Comultiplication is given by

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{(\Gamma)} \gamma \otimes \Gamma//\gamma$$

on 1PI graphs, and the antipode is given by

$$S(\Gamma) = -\Gamma - \sum_{(\Gamma)} m(S(\gamma) \otimes \Gamma//\gamma) .$$

These operations are extended to the entire Hopf algebra as before. A primitive graph is one where

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma$$

i.e. $\tilde{\Delta}(\Gamma) = 0$. This is the notation for the rest of this paper.

3.2 The grading and filtration on $\mathcal{H}$

Certain properties of the Feynman graphs induce a grading and a filtration on $\mathcal{H}$.

Definition 14. A connected Hopf algebra is a graded Hopf algebra, $\mathcal{H}$, with grading bounded above or below, and $\mathcal{H}^0 \simeq k$.

Definition 15. The grading on $\mathcal{H}$ is given by the loop number $L(\Gamma)$, or rk $H_1(\Gamma)$ the rank of the first homology group of Γ. For Γ a monomial in $\mathcal{H}$, one says $\Gamma \in \mathcal{H}^l$ if and only if rk $H_1(\Gamma) = l$. $\mathcal{H}^0 = k$.

Lemma 3.5. *$\mathcal{H}^l$ is a finite dimensional vector space for all l.*

Proof. This amounts to showing that there are only finitely many connected 1PI graphs of a given loop number. Since each graph has either 2 or 3 external edges, and each vertex has valence 3, the number of possible internal legs for a 1PI graph is

$$I = \frac{3V-3}{2} \text{ or } I = \frac{3V-2}{2} \; .$$

For a fixed loop number $l = I - V + 1$,

$$l = \begin{cases} \frac{3V-3}{2} - V + 1, & \text{or} \\ \frac{3V-2}{2} - V + 1, & \end{cases}$$

(Since I must be an integer, this means that a graph with 3 external legs has an odd number of vertices and a graph is 2 external legs has an even number.) That is, the number of possible vertices a 1PI graph in $\mathcal{H}^l$ can have is

$$V = 2l + 1 \quad \text{or} \quad 2l \; .$$

The number of ways to connect $2l+1$ labeled vertices with $\frac{3V-3}{2}$ edges is $\left(\begin{array}{c} \binom{2l+1}{2} \\ \frac{3V-3}{2} \end{array} \right) < \infty$ [35]. For $2l$ labeled vertices and $\frac{3V-2}{2}$ internal edges is $\left(\begin{array}{c} \binom{2l}{2} \\ \frac{3V-2}{2} \end{array} \right) < \infty$. The Feynman diagrams in $\mathcal{H}^l$ have more restrictions on valence, have unlabeled vertices (the embedding in the plane does not matter), and must be 1PI. Therefore, there are fewer possible 1PI generators in $\mathcal{H}^l$, i.e. the number is finite for each l. Thus, the number of monomial generators of $\mathcal{H}^l$ is finite. ☐

Theorem 3.6. *The Hopf algebra of Feynman diagrams $\mathcal{H}$ is a connected, graded Hopf algebra. That is, for $\Gamma_1 \in \mathcal{H}^l$ and $\Gamma_2 \in \mathcal{H}^m$,*

$$m(\Gamma_1 \otimes \Gamma_2) \in \mathcal{H}^{l+m} \; .$$

For any monomial $\Gamma \in \mathcal{H}^l$,

$$\Delta\Gamma \in \sum_{i=0}^{l} \mathcal{H}^i \otimes \mathcal{H}^{l-i}$$

and

$$S(\Gamma) \in \mathcal{H}^l \; .$$

Proof. This Hopf algebra is connected under this grading by definition. Since (rk H_1) is additive under disjoint union, multiplication is preserved by this grading.

For comultiplication, one only needs to check that

$$\Delta(\Gamma) \in \sum_{i=0}^{l} \mathcal{H}^i \otimes \mathcal{H}^{l-i}$$

because $1 \in \mathcal{H}^0$. For $\Gamma \in \mathcal{H}^l$ 1PI,

$$\tilde{\Delta} = \sum_{(\Gamma)} \Gamma' \otimes \Gamma''$$

where $\Gamma' \in \mathcal{H}^i$ with $0 < i < l$. Let Γ' have n connected components labeled Γ'_j with $1 \leq j \leq n$. Then

$$L(\Gamma'') = (I(\Gamma) - \sum_j I(\Gamma'_j)) - (V(\Gamma) - \sum_j V(\Gamma'_j)) + n + 1 = L(\Gamma) - \sum_j L(\Gamma'_j) = l - i \ .$$

Therefore, comultiplication is preserved by the grading for all 1PI graphs. Since $\Delta(\Gamma\Gamma') = \Delta(\Gamma)\Delta(\Gamma')$, the property holds for the entire Hopf algebra because it is an algebra homomorphism. The antipode is preserved by this grading by the same argument. □

The Hopf algebra of Feynman diagrams is also a filtered Hopf algebra, where the filtration is given by the maximum number of embedded admissible subgraphs. This filtration is also discussed in [2]. Consider the operator

$$\tilde{\Delta}^n \quad : \mathcal{H} \to \mathcal{H}^{\otimes n+1}$$

defined recursively as

$$\tilde{\Delta}^1 = \tilde{\Delta}$$

$$\tilde{\Delta}^n = \left(\sum_{i=1}^{n} \otimes_1^{i-1} \mathrm{id} \otimes \tilde{\Delta} \otimes_{i+1}^{n} id \right) \circ \tilde{\Delta}^{n-1} \ .$$

I also define

$$\tilde{\Delta}^{(0)} = \mathrm{id} - \varepsilon$$

for notational convenience. For an indecomposable element of $\mathcal{H}$, $\tilde{\Delta}(\Gamma)$ is a sum of all tensor products of *proper* subgraphs and their residues. Therefore, $\tilde{\Delta}^n(\Gamma)$ is the sum of all n-tensor products of n proper subgraphs and residues. This gives a filtration on $\mathcal{H}$.

Definition 16. For $\Gamma \in \ker \varepsilon$, $\Gamma \in \mathcal{H}^{(n)}$ if $\tilde{\Delta}^n(\Gamma) = 0$. This is an increasing filtration on $\mathcal{H}$. Define $k = \mathcal{H}^{(0)}$, and the space $\mathcal{H}^{(1)}$ contains the space of primitive elements of $\mathcal{H}$ by Definition 11.

Lemma 3.7. $\tilde{\Delta}^{(n+1)} = \sum_0^n \binom{n}{i} (\tilde{\Delta}^{(i)} \otimes \tilde{\Delta}^{(n-i)})\tilde{\Delta} \quad .$

Proof. By definition, this holds for $n = 1$. For $n = 2$,

$$\tilde{\Delta}^{(2)} = (\mathrm{id} \otimes \tilde{\Delta} + \tilde{\Delta} \otimes \mathrm{id})\tilde{\Delta} \ .$$

If this holds for n, then for $n + 1$,

$$\tilde{\Delta}^{(n+1)} = (\sum_{i=0}^{n} \mathrm{id}^{\otimes i} \otimes \tilde{\Delta} \otimes \mathrm{id}^{\otimes n-i-1})(\sum_{j=0}^{n-1} \binom{n-1}{j} (\tilde{\Delta}^{(j)} \otimes \tilde{\Delta}^{(n-j-1)})\tilde{\Delta}) \ .$$

I can rewrite this as

$$\sum_{i=0}^{n-1}\sum_{j>i}^{n} \mathrm{id}^{\otimes j} \otimes \tilde{\Delta} \otimes \mathrm{id}^{\otimes n-j} \binom{n-1}{i} (\tilde{\Delta}^{(i)} \otimes \tilde{\Delta}^{(n-i-1)})\tilde{\Delta} + \\ \mathrm{id}^{\otimes n-j} \otimes \tilde{\Delta} \otimes \mathrm{id}^{\otimes j} \binom{n-1}{i} (\tilde{\Delta}^{(n-i-1)} \otimes \tilde{\Delta}^{(i)})\tilde{\Delta} \; .$$

For the first (resp. last) term of this sum the first (last) $i+1$ identity maps are applied to $\tilde{\Delta}^{(i)}$ while the sum of the last (first) $n-i$ terms applied to $\tilde{\Delta}^{(n-i-1)}$ gives $\tilde{\Delta}^{(n-i)}$. Thus

$$\begin{aligned}\tilde{\Delta}^{(n+1)} &= \sum_{i=0}^{n-1} \binom{n-1}{i} (\tilde{\Delta}^{(i)} \otimes \tilde{\Delta}^{(n-i)})\tilde{\Delta} + \\ &\qquad \binom{n-1}{i} (\tilde{\Delta}^{(n-i)} \otimes \tilde{\Delta}^{(i)})\tilde{\Delta} \\ &= \sum_{0}^{n} \binom{n}{i} (\tilde{\Delta}^{(i)} \otimes \tilde{\Delta}^{(n-i)})\tilde{\Delta} \; .\end{aligned}$$

□

A second grading on $\mathcal{H}$ is the associated grading to this filtration.

$$\mathrm{Gr}_i\mathcal{H} = \mathcal{H}^{(i)}/\mathcal{H}^{(i-1)} \; .$$

Remark 5. I show next that $\mathcal{H}$ is a graded filtered Hopf algebra. The module given by the i^{th} loop grading and the j^{th} filtration level is written $\mathcal{H}^{i,(j)}$. I write $\mathcal{H}^{i,j}$ to mean $\mathcal{H}^{i,(j)}/\mathcal{H}^{i,(j-1)}$, the i^{th} loop grading and the j^{th} filtration grading in $\mathrm{Gr}(\mathcal{H})$ is written .

Lemma 3.8. *The filtration $\mathcal{H}^{(n)}$ is preserved under multiplication and comultiplication and the antipode. That is,*

$$\begin{array}{rccl} m: & \mathcal{H}^{(p)} \otimes \mathcal{H}^{(q)} & \hookrightarrow & \mathcal{H}^{(p+q)} \\ \Delta: & \mathcal{H}^{(n)} & \hookrightarrow & \bigoplus_{p+q=n} \mathcal{H}^{(p)} \otimes \mathcal{H}^{(q)} \\ S: & \mathcal{H}^{(n)} & \hookrightarrow & \mathcal{H}^{(n)} \; . \end{array}$$

Proof. First check comultiplication on $\Gamma \in \mathcal{H}^{(n)}$.

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \tilde{\Delta}(\Gamma) \; .$$

The first two terms of this sum are of the correct form. Lemma 3.7 shows that all summands in $\tilde{\Delta}$ are also of the correct form.

Suppose a summand of $\Delta(\Gamma) \in \mathrm{Gr}_p\mathcal{H} \otimes \mathrm{Gr}_q\mathcal{H}$, with $p+q>n$. Call this term $\gamma_1 \otimes \gamma_2$. Then

$$\sum_{l+k=n,\, k,l\geq 1} \tilde{\Delta}^k(\gamma_1) \otimes \gamma_2 + \gamma_1 \otimes \tilde{\Delta}^l(\gamma_2) \neq 0 \; .$$

That is, $\tilde{\Delta}^n(\Gamma) \neq 0$.

Consider two 1PI graphs such that $\Gamma_1 \in \mathcal{H}^{(i)}$ and $\Gamma_2 \in \mathcal{H}^{(j)}$. We show preservation under multiplication inductively on the sum $n = i+j$. If $i = 0$ then $\Gamma_1 \in k$, and the result is trivial. Consider $\Gamma_1, \Gamma_2 \in \mathcal{H}^{(1)}$. Then

$$\Delta(\Gamma_1\Gamma_2) = 1 \otimes \Gamma_1\Gamma_2 + \Gamma_1\Gamma_2 \otimes 1 + \Gamma_1 \otimes \Gamma_2 + \Gamma_2 \otimes \Gamma_1 \; .$$

Then

$$\tilde{\Delta}(\Gamma_1\Gamma_2) = \Gamma_1 \otimes \Gamma_2 + \Gamma_2 \otimes \Gamma_1 \ .$$

So

$$\tilde{\Delta}^{(2)}(\Gamma_1\Gamma_2) = 0 \ .$$

Thus $\Gamma_1\Gamma_2 \in \mathcal{H}^{(2)}$.

If $i + j = n$, then

$$\tilde{\Delta}(\Gamma_1\Gamma_2) = \sum_{(\Gamma_1)}\sum_{(\Gamma_2)} \Gamma_1'\Gamma_2' \otimes \Gamma_1^{''}\Gamma_2^{''} + \sum_{(\Gamma_1)} \Gamma_1'\Gamma_2 \otimes \Gamma_1^{''} + \sum_{(\Gamma_1)} \Gamma_1' \otimes \Gamma_1^{''}\Gamma_2 + \sum_{(\Gamma_2)} \Gamma_1\Gamma_2' \otimes \Gamma_2^{''} + \sum_{(\Gamma_2)} \Gamma_2' \otimes \Gamma_1\Gamma_2^{''}$$

Since this filtration is preserved under comultiplication, if $\Gamma_1' \in \mathcal{H}^{(l)}$, with $0 < l < i$, $\Gamma_1^{''} \in \mathcal{H}^{(i-l)}$. The same is true for Γ_2. By induction, since multiplication preserves the first $n-1$ filtered levels, $\Gamma_1\Gamma_2 \in \mathcal{H}^{(n)}$.

We check the antipode by induction. Recall that for $\Gamma \in \mathcal{H}^{(1)}$

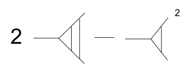
$$\tilde{\Delta}S(\Gamma) = -\tilde{\Delta}(\Gamma) = 0$$

so $S(\Gamma) = 0$. For $\Gamma \in \mathcal{H}^{(n)}$,

$$S(\Gamma) = -\Gamma - \sum_{(\Gamma)} S(\Gamma')\Gamma'' \ .$$

Since the filtration is preserved under co-multiplication, $\Gamma' \in \mathcal{H}^{(p)}$ and $\Gamma'' \in \mathcal{H}^{(n-p)}$for some $p < n$. Therefore, $S(\Gamma') \in \mathcal{H}^{(p)}$ as is $S(\Gamma')\Gamma''$. $\square$

Example 3. While both terms in the polynomial



are in $\mathrm{Gr}_2\mathcal{H}$, the polynomial itself is primitive. Before verifying this statement, for typographical ease, rewrite the above expression as

$$\Gamma = 2\gamma_1 - \gamma_2^2$$

Since γ_2 is the only proper subgraph of γ_1,

$$\Delta(\Gamma) = 2(\gamma_1 \otimes 1 + 1 \otimes \gamma_1 + \gamma_2 \otimes \gamma_2) - (\gamma_2 \otimes 1 + 1 \otimes \gamma_2)^2$$
$$= 2(\gamma_1 \otimes 1 + 1 \otimes \gamma_1 + \gamma_2 \otimes \gamma_2) - (\gamma_2\gamma_2 \otimes 1 + 2\gamma_2 \otimes \gamma_2 + 1 \otimes \gamma_2\gamma_2) = \Gamma \otimes 1 + 1 \otimes \Gamma \ .$$

3.3 The affine group scheme

Since $\mathcal{H}$ is a commutative Hopf algebra, one can define $G = \mathrm{Spec}\ \mathcal{H}$. Recall that Spec is a contravariant functor that assigns to a commutative algebra its underlying variety. Furthermore, recall that a Hopf algebra obeys the following three relations

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$$
$$(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$$
$$m(S \otimes \mathrm{id})\Delta = \varepsilon\eta$$

which covariantly define a multiplication, identity and an inverse on G. Thus G is an affine scheme that satisfies the axioms of a group, and is thus called an affine group scheme.

One can also look at G as a covariant functor associating the group of A-valued points $G(A) = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, A)$ to a unital k algebra A. The product structure on $G(A)$ is induced by the insertion product, and is the same as that on $\mathcal{H}^\vee$ (introduced below):

$$f \star g = m(f \otimes g)\Delta$$

with f, $g \in G(\mathcal{A})$. Since $f \in G(A)$ is an algebra homomorphism,

$$f(\gamma_1\gamma_2) = f(\gamma_1)f(\gamma_2) ,$$

where this product is the product on A, and not the $\star$ product.

Finally, since the grading on $\mathcal{H}$ is locally finite dimensional, one can create a series of finitely generated k algebras

$$\mathcal{H}_i = k[\Gamma_1, \ldots \Gamma_{n_i}]$$

where the set

$$\{\Gamma_1, \ldots \Gamma_{n_i}\}$$

is the set of all 1PI graphs with loop number at most i. Then we create a set of affine schemes $G_i = \mathrm{Spec}\ \mathcal{H}_i$. Next we can write

$$G = \varprojlim_i G_i \ .$$

Definition 17. G is a pro-unipotent affine group scheme.

For an explicit treatment of what these G_is look like as matrices, see [11]. For more details, see [7].

3.4 The Lie algebra structure on $\mathcal{H}$ and $\mathcal{H}^\vee$

There is a well established relationship between Hopf algebras and Lie algebras over fields of characteristic 0.

Theorem 3.9. ***Milnor-Moore*** *[28] Given a connected, graded, cocommutative, locally finite Hopf algebra, H, over $\mathbb{Q}$, there is a Hopf algebra isomorphism, $H \simeq \mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is the graded Lie algebra generated by the indecomposable elements of H and $\mathcal{U}$ is the universal enveloping algebra.*

The full dual algebra of $\mathcal{H}$ is not a Hopf algebra. However, the restricted dual of a commutative, connected, graded, locally finite Hopf algebra is a cocommutative, connected, graded, locally finite Hopf algebra written

$$\mathcal{H}^\vee = \bigoplus_l (\mathcal{H}^l)^\vee = \bigoplus_l \mathcal{H}_l \ . \tag{9}$$

Since $\mathcal{H}$ is a graded filtered Hopf algebra one expects $\mathcal{H}^\vee$ to also be a graded filtered Hopf algebra.

Remark 6. The grading on $\mathcal{H}^\vee$ is given by the loop number as above. Notice that $\mathcal{H}^\vee$ is still finite dimensional at each level. An increasing filtration on $\mathcal{H}^\vee$ is given by $\mathcal{H}^\vee_{(l)} = (\bigoplus_{i\leq l} \mathcal{H}^l)^\vee$

The Milnor-Moore theorem reduces to the following statement in this case.

Corrollary 3.10. *[26] There is an isomorphism of bigraded Hopf algebras, $\mathcal{H}^\vee \simeq \mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is the bigraded Lie algebra of the affine group scheme G.*

The structure of $\mathfrak{g}$ is defined below. There is a second (decreasing) filtration on this space corresponding to an (increasing) filter on $\mathcal{H}$ formed by $\tilde{\Delta}$ which I will use to define the topology of $\mathcal{H}^\vee$. Before introducing that structure, I will discuss the Hopf algebra properties of $\mathcal{H}^\vee$.

Definition 18. Define $\mathcal{L}(\mathcal{H}, A)$ to be the vector space of module homomorphisms from $\mathcal{H}$ to some k algebra A. Then $\mathcal{L}(\mathcal{H}, k) \supset \mathcal{H}^\vee$.

Remark 7. The group $G(A) \subset \mathcal{L}(\mathcal{H}, A)$ is the group of ungraded algebra homomorphisms from $\mathcal{H}$ to A. That is, if $f \in G(A)$, then for $x\, y \in \mathcal{H}$

$$f(xy) = f(x)f(y) .$$

Another way of stating this is that if f is an algebra homomorphism from $\mathcal{H} \to A$ then

$$\Delta(f) = f \otimes f$$

or that algebra homomorphisms from $\mathcal{H} \to A$ are group-like elements of $\mathcal{L}(\mathcal{H}, A)$.

The indecomposable elements of $\mathcal{H}^\vee$ are in one-to-one correspondence with the indecomposable elements of $\mathcal{H}$ in the usual way:

$$\Gamma \leftrightarrow \delta_\Gamma(x)$$

where Γ is an 1PI graph, x a polynomial in $\mathcal{H}$, and $\delta_\Gamma(x)$ is the Kronecker delta function.

$$\delta_\Gamma(x) = \begin{cases} 1, & x = \Gamma, \text{ a 1PI graph;} \\ 0, & \text{otherwise.} \end{cases}$$

That is, there is an isomorphism of vector spaces

$$k\langle \delta_\Gamma | \Gamma \in \{\text{1PI graphs of } \mathcal{L}\}\rangle \simeq k\langle \Gamma | \Gamma \in \{\text{1PI graphs of } \mathcal{L}\}\rangle . \tag{10}$$

Corrollary 3.11. *The indecomposable elements of $\mathcal{H}$ correspond to the primitive elements of $\mathcal{H}^\vee$.*

Proof. Follows from the Milnor Moore Theorem. □

Let δ_Γ and $\delta_{\Gamma'}$ be indecomposable elements of $\mathcal{H}^\vee$. That is, Γ and Γ' are 1PI graphs. We can define the Hopf algebra operations on the indecomposable elements of $\mathcal{H}^\vee$ in a similar manner to that of $\mathcal{H}$. Multiplication on $\mathcal{H}^\vee$ is a convolution product induced by the insertion product.

$$\begin{array}{rccl} \star : & \mathcal{H}^\vee \otimes \mathcal{H}^\vee & \to & \mathcal{H}^\vee \\ & \delta_\Gamma \star \delta_{\Gamma'} & \to & m(\delta_\Gamma \otimes \delta_{\Gamma'})\Delta . \end{array}$$

This can be extended to all of $\mathcal{H}^\vee$ by coassociativity of Δ and linearity. $\mathcal{H}^\vee$ is a non-commutative Hopf algebra. In the dual, the co-unit maps 1 to ε. For Γ 1PI, i.e. in $\ker \varepsilon$,

$$\delta_\Gamma(1) = \varepsilon(\Gamma) = 0 . \tag{11}$$

As stated in corollary (3.11), the indecomposable elements of $\mathcal{H}^\vee$ are primitive. Notice that for Γ a 1PI graph,

$$\Delta(\delta_\Gamma)(x \otimes y) = \delta_\Gamma(m(x \otimes y)) \quad x, y \in \mathcal{H}$$

since a product becomes a coproduct in the dual space. But $m(x \otimes y)$ is not an indecomposable element of $\mathcal{H}$, unless $x \in k$ and y is 1PI. Therefore,

$$\begin{array}{rccl} \Delta : & \mathcal{H}^\vee & \to & \mathcal{H}^\vee \otimes \mathcal{H}^\vee \\ & \delta_\Gamma & \mapsto & \varepsilon \otimes \delta_\Gamma + \delta_\Gamma \otimes \varepsilon . \end{array} \tag{12}$$

The primitive elements of $\mathcal{H}^\vee$, in this case, also the indecomposable elements of $\mathcal{H}^\vee$, are the generators of the Lie algebra $\mathfrak{g}$. They are called infinitesimal algebra homomorphisms. One can write

$$\mathfrak{g} = k\langle \delta_\Gamma | \Gamma \in \{\text{1PI graphs of } \mathcal{L}\}\rangle .$$

Lemma 3.12. *The grading from equation 9 is preserved under the convolution product and the coproduct. That is, for monomials $f \in \mathcal{H}_l^\vee$ and $g \in \mathcal{H}_m^\vee$,*

$$f \star g(\Gamma) \neq 0 \Rightarrow \Gamma \in \mathcal{H}^{l+m}$$

and

$$\Delta(f)(\gamma' \otimes \gamma'') \neq 0 \Rightarrow \gamma' \in \mathcal{H}^i \text{ and } \gamma'' \in \mathcal{H}^{l-i} .$$

Proof. To show that the grading is preserved under multiplication, it is sufficient to show it is preserved in the product of two primitive elements of $\mathcal{H}^\vee$. For the indecomposable elements $\gamma_1 \in \mathcal{H}^l$ and $\gamma_2 \in \mathcal{H}^m$, there are $\delta_{\gamma_1} \in \mathcal{H}_l^\vee$ and $\delta_{\gamma_2} \in \mathcal{H}_m^\vee$ such that

$$\delta_{\gamma_1} \star \delta_{\gamma_2}(\Gamma) = \begin{cases} n, & n\gamma_1 \otimes \gamma_2 \text{ is a summand of } \Delta(\Gamma); \\ 0, & \text{otherwise} . \end{cases}$$

Since the loop grading $\mathcal{H}^l$ is preserved under Δ on $\mathcal{H}$, $\Gamma \in \mathcal{H}^{l+m}$.

To see preservation under the co-product, it is sufficient to consider monomials that are the product of two primitive elements, $f = \delta_{\gamma_1} \star \delta_{\gamma_2} \in \mathcal{H}_l^\vee$,

$$\Delta(f)(\gamma' \otimes \gamma'') = \delta_{\gamma_1} \star \delta_{\gamma_2}(\gamma'\gamma'') .$$

By the above argument, if this is non-zero then $\gamma'\gamma'' \in \mathcal{H}^l$. Since the loop grading $\mathcal{H}^l$ is preserved under multiplication on $\mathcal{H}$, $\gamma' \in \mathcal{H}^i$ and $\gamma \in \mathcal{H}^{l-i}$.

Results for more complicated monomials in $\mathcal{H}^\vee$ follow by induction. □

The increasing filtration on $\mathcal{H}$ defines a decreasing filtration on $\mathcal{H}^\vee$ as follows:

$$\mathcal{H}_{(n)}^\vee = \{f | f(\gamma) = 0, \ \forall \gamma \in \mathcal{H}^{(n-1)}\} .$$

I use this filtration to define the topology on $\mathcal{H}^\vee$. This filtration is preserved under $\star$. The grading associated to this decreasing filtration on $\mathcal{H}^\vee$ is defined in the standard way:

$$Gr_n \mathcal{H}^\vee = \mathcal{H}_{(n-1)}^\vee / \mathcal{H}_{(n)}^\vee .$$

Since the indecomposables of $\mathcal{H}^\vee$ lie in $\mathcal{H}_{(1)}^\vee$, we can write

$$\mathfrak{g} \subset \mathcal{H}_{(1)}^\vee .$$

Remark 8. Equation (10) shows that the generators of $\mathfrak{g}$ are in one-to-one correspondence with the indecomposable elements of $\mathcal{H}$, $\mathfrak{g}$ is a graded Lie algebra, by the grading in equation (9). Specifically, $\mathfrak{g}(A) \subset \mathcal{L}(\mathcal{H}, A)$ is a filtered Lie algebra.

Theorem 3.13. *The Lie algebra of $G(A)$ is defined as*

$$\mathfrak{g}(A) = \mathfrak{g} \otimes_k A$$

where A is a k algebra.

For f a generator of $\mathfrak{g}(A)$,

$$\Delta(f) = f \otimes \varepsilon + \varepsilon \otimes f .$$

Proposition 3.14. *Let*

$$f = \sum_i \alpha_i f_i$$

be a formal power series in $\mathcal{H}^\vee$ such that $f_i \in Gr_{n_i}\mathcal{H}^\vee$ and n_i is an increasing sequence. Any such formal series is convergent.

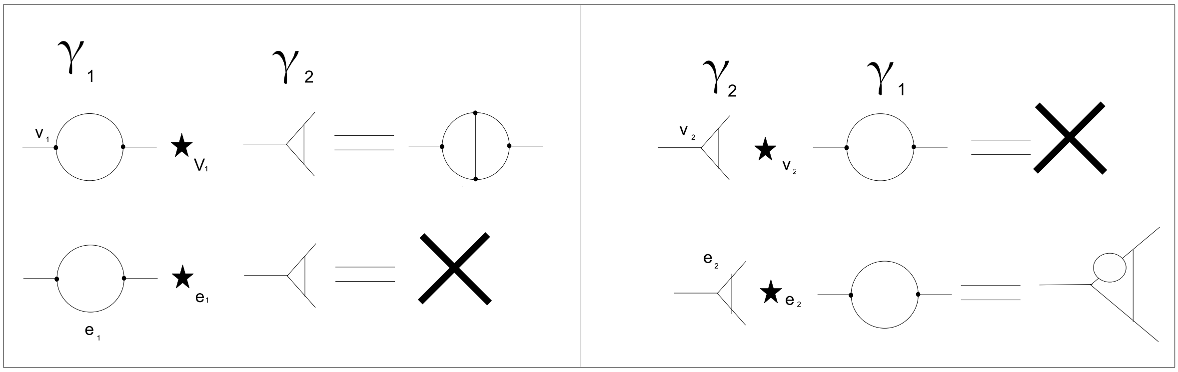


Figure 3: Permissible and impermissible insertions

Proof. Let $F = \sum_0^\infty \alpha_i f_i$ be a formal series in $\mathcal{H}^\vee$, where $\alpha_i \in k$ and f_i is a monomial in $Gr_n\mathcal{H} = \mathcal{H}^\vee_{(n-1)}/\mathcal{H}^\vee_{(n)}$. Consider $F(\Gamma)$ for a monomial $\Gamma \in \mathcal{H}$. Then $\Gamma \in \mathcal{H}^{(n)}$ for some n. Therefore, $f(x) = 0$ for all $f \in \mathcal{H}_{(n+1)}$. Therefore,

$$\sum_{n+1}^{\infty} \alpha_i f_i(\Gamma) = 0 \; .$$

Therefore $F(\Gamma)$ is a finite sum for any $\Gamma \in \mathcal{H}$ and the series is convergent. $\square$

Because of the isomorphism in equation (10), one can also write $\mathfrak{g}$ as a Lie algebra on the 1PI graphs in $\mathcal{H}$. The convolution product in $\mathcal{H}^\vee$ becomes an insertion product in $\mathcal{H}$ written

$$\star \quad : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$$

defined on indecomposable elements γ_1, γ_2 of $\mathcal{H}$ [21].

Definition 19. Let $g \in \gamma_{1,\text{int}}^{[1]} \cup \gamma_1^{[0]}$ be an insertion point of γ_1.

1. If $|\gamma_{2,\text{ext}}^{[1]}| = 2$ and $g \in \gamma_{2,\text{ext}}^{[1]}$ are of the same type of an element of $\gamma_{1,\text{int}}^{[1]}$, then g is a valid insertion point. Insert γ_2 into γ_1 at g. Then sum over all valid insertion points.
2. If $\gamma_{2,\text{ext}}^{[1]} = f_g$ for some $g \in \gamma_1^{[0]}$, insert γ_2 into γ_1 at the vertex g in all valid orientations. Sum over all valid insertion points after modding out the equivalence class of planar embeddings.
3. If no valid insertion point g can be found then $\star$ is not defined.

This is a much more general definition of this insertion product than is required for the Lagrangian in this paper, where the condition in 1 simplifies to "if g is a vertex and γ_2 has three external edges" and the condition in 2 simplifies to "if γ_2 has two external edges and g is an internal line. "

Example 4. In the first panel in Figure 3, γ_2 has three external legs. Therefore it can be inserted into the vertex v_1 of γ_1, but not into the edge e_1. In the second panel, because γ_1 has two external legs, it can only be inserted into an edge e_2 of γ_2 and cannot be inserted into the vertex v_2.

This definition can be extended to products as follows. If $\gamma_1 = \gamma' \coprod \gamma''$, then $\gamma' \coprod \gamma'' \star \gamma_2$ is defined for $g \in \gamma'^{[i]} \cup \gamma''^{[i]}$, with $i \in \{0, 1\}$ according to $\gamma_{2,\text{ext}}^{[1]}$ as defined above. If $\gamma_2 = \gamma' \coprod \gamma''$, then $\gamma_1 \star (\gamma' \coprod \gamma'')$ is defined

$$(\gamma_1 \star \gamma') \star \gamma'' \; .$$

It can be extended to the entire Hopf algebra as an enveloping algebra by linearity.

Remark 9. Loosely speaking, this operation reverses the action of Δ. The coproduct contracts away a subgraph, but looses track of where the graph was contracted from. The insertion product start with information about where to insert a graph and "uncontracts", or inserts a subgraph to form a larger graph. Therefore, a graph can only be inserted at a point where its external structure matches the structure of edges meeting that point.

Definition 20. The insertion product gives a bigraded Lie structure on the indecomposable elements of $\mathcal{H}$.

$$[\gamma_1, \gamma_2] = \gamma_1 \star \gamma_2 - \gamma_2 \star \gamma_1 \ .$$

On can check that this bracket satisfies the Jacobi identity by direct calculation. This Lie algebra is generated by the indecomposable elements of $\mathcal{H}$. Call it $\mathfrak{g}^\vee$.

Lemma 3.15. *The Lie algebra $\mathfrak{g}^\vee$ on the indecomposable elements of $\mathcal{H}$ under the $\star$ operation is isomorphic as a Lie algebra to $\mathfrak{g}$ defined on the primitive elements of $\mathcal{H}^\vee$ under the convolution product $\star$.*

Proof. Let γ_1, $\gamma_2 \in \mathcal{H}$ be generators. There is a Lie bracket on the corresponding primitive elements of $\mathcal{H}^\vee$ defined

$$[\delta_{\gamma_1}, \delta_{\gamma_2}] = (\delta_{\gamma_1} \star \delta_{\gamma_2}) - (\delta_{\gamma_2} \star \delta_{\gamma_1}) = (\delta_{\gamma_1} \otimes \delta_{\gamma_2})\Delta - (\delta_{\gamma_2} \otimes \delta_{\gamma_1})\Delta \ .$$

This is an element in $\mathcal{H}^\vee$. It is only non-zero on the Lie bracket

$$[\gamma_1, \gamma_2]$$

in $\mathcal{H}$. The two terms on the right hand side are non zero only on the 1PI graphs found in the products $\gamma_2 \star \gamma_1$ and $\gamma_1 \star \gamma_2$ respectively. □

4 Birkhoff decomposition and renormalization

As discussed in section 2.2, the Feynman rules injectively map the Feynman diagrams into $S^n(E)$, where $E = \mathbb{C}_c^\infty(\mathbb{R}^6)$. On the 1PI graphs, for ϕ^3 theory in six dimensions, there can only be two or three external legs. The external leg data for a 1PI graph is given by $S^n(E)$ where $n \in \{2, 3\}$.

Since elements of the Hopf algebra $\mathcal{H}$ are polynomials in 1PI graphs of arbitrarily large degree, the Feynman rules map $\mathcal{H}$ to the space of distributions on an arbitrary number of external legs. If $x \in \mathcal{H}$ is a generator, a 1PI graph with 2 or 3 external legs, then it tries to be a distribution on E such that

$$\text{Feynman rules}(x) \in \mathrm{Hom}_{\mathrm{vect}}(S^n(E), \mathbb{C}) \ .$$

The result of these distributions on test functions is supposed to give physically significant values pertaining to the physical interaction of fields that the diagram represents. However, these distributions are often divergent, leading to the need for regularization and renormalization.

The first step to extracting finite values to these divergent integrals is *regularization.* One rewrites the integral in terms of a set of parameters that may yield a sensible value upon reaching a predetermined limit. The second is *renormalization*, where all divergences, now neatly captured by the parameter z, are removed. For the regularization schemes studied in this paper, since elements of the Hopf algebra $\mathcal{H}$ can be written as polynomials of its generators,

$$\text{regularized Feynman rules} \ : \ \mathcal{H} \hookrightarrow \mathrm{Hom}_{\mathrm{vect}}(S(E), \mathbb{C}\{\{z\}\}) \ . \tag{13}$$

Unlike the unregularized Feynman rules, this map is well defined.

Remark 10. Notice that equation (13) is an algebra homomorphism by the definition of the Hopf algebra $\mathcal{H}$.

There are many methods of regularization and of renormalization. In this paper I will consider regularization schemes with one parameter that can be written as a Rota-Baxter algebra. Dimensional regularization is an example of such a scheme, and is worked out in [5], [6] and [8]. In section 7, I work out the example of ζ-function regularization.

Example 5. In dimensional regularization, the Feynman integrals are regularized by performing a change of variables into spherical coordinates capturing the divergent parameter in the dimension, d, of the space over which one integrates. For details, see [40], chapter 9.

After regularization, Feynman integrals are renormalized by minimal subtraction, which uses Cauchy's theorem to calculate the residue at $z = 0$. At the level of the Lagrangian, this process introduces counterterms in the Lagrangian which cancel the divergences. In the case of non-primitive graphs, the subtracted divergences may contain further subdivergences. Therefore, in the general case, one iteratively removes divergences. This prescription was first discovered by Bogoliubov and Parasiuk in 1957, and corrected by Hepp in 1966. In 1969 Zimmermann proved that this prescription gets rid of all subdivergences. This has since become known as the BPHZ renormalization prescription, and is a standard technique. Connes and Kreimer [5] showed that this method of extracting renormalized values from Feynman graphs corresponds exactly to the extraction of finite values by Birkhoff decomposition, as explained below.

Fix the base field for the Hopf algebra of Feynman graphs, $\mathcal{H}$, as $k = \mathbb{C}$ for the remainder of this paper. The regularized Feynman integral can then be expressed as a function of z, whose actions on the test functions result in a Laurent series in z that converges on a suitably small punctured disk Δ around the origin.

Bundle Note 1. The infinitesimal disk Δ at the base of Figure 1 comes from this variation of the regularization parameter.

Example 6. Notice that the example Lagrangian at the beginning of this paper has $d = 6$. The regularization parameter is in the coordinates $z = D - 6$. For ζ-function regularization, the regulation occurs by raising the propagators to the complex power s. The regulator is $r = s - 1$. See section 7.

One can rewrite the infinitesimal disk $\Delta \simeq \text{Spec } \mathbb{C}\{z\}$. The punctured disk Δ^* can be rewritten $\Delta^* \simeq \text{Spec } \mathbb{C}\{\{z\}\}$, the localization of the field of convergent Laurent series in z at 0. Writing $\mathcal{A} = \mathbb{C}\{\{z\}\}$ for short, one can decompose $\mathcal{A}$ into two subalgebras $\mathcal{A}_- \oplus \mathcal{A}_+$, where $\mathcal{A}_+ = \mathbb{C}\{z\}$, and $\mathcal{A}_- = z^{-1}\mathbb{C}[z^{-1}]$ is a non-unital subalgebra that contains the strictly negative powers of z. Finally, define $G = \text{Spec } \mathcal{H}$, a complex Lie group. Then sections of the trivializable G fiber bundle, K^* over the punctured infinitesimal disk, Δ^*, are maps

$$\gamma : \quad \Delta^* \to G \, .$$

Therefore, these sections can be written

$$\gamma : \quad \text{Spec } \mathcal{A} \quad \to \quad \text{Spec } \mathcal{H} \, .$$

There is a natural isomorphism between the group of maps between these two affine spaces and the group of algebra homomorphisms between the corresponding algebras via the contravariant functor F,

$$\begin{array}{rccc} F : & \text{Maps}(\text{Spec } \mathcal{A}, \text{Spec } \mathcal{H}) & \to & \text{Hom}_{\text{alg}}(\mathcal{H}, \mathcal{A}) \\ & \gamma & \mapsto & \gamma^\dagger \, . \end{array}$$

Thus the space of these sections is isomorphic to the group $G(\mathcal{A})$, the group of algebra homomorphisms $\mathcal{H} \to \mathcal{A}$.

Bundle Note 2. If $W \subset K$ is the fiber over 0 in the $K \to \Delta$ bundle, and if $K^* = K \setminus W$, then $G(\mathcal{A})$ is the group corresponding to the sections of $K^* \to \Delta^*$.

Theorem 4.1. *The regularized Feynman rules of a QFT defined by $\mathcal{L}$ are a linear map from $S(E)$ to $G(\mathcal{A})$.*

Proof. The map in (13) shows that

$$\text{regularized Feynman rules } : \mathcal{H} \hookrightarrow \mathrm{Hom}_{\mathrm{vect}}(S(E), \mathcal{A}) .$$

To see that this is an algebra homomorphism, notice that for $x, y \in \mathcal{H}$, regularized Feynman rules for a disjoint union of 1PI graphs x and y is just the product of the regularized Feynman integrals associated to the graph x and to the graph y. Therefore

$$\text{regularized Feynman rules } \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathrm{Hom}_{\mathrm{vect}}(S(E), \mathcal{A})) . \tag{14}$$

The symmetric algebra on E can be written $S(E) = \oplus_n S^n(E)$. The restricted dual of $S^\vee(E) = \oplus_n S^{n\vee}(E)$, where there is an isomorphism from $S^{n\vee}(E) \simeq S^n(E)$. Furthermore, the isomorphism $S^n(E^\vee) \simeq S^{n\vee}(E)$ gives

$$S(E^\vee) \simeq S^\vee(E) \simeq \mathrm{Hom}_{\mathrm{vect}}(S(E), \mathbb{C}) .$$

Therefore,

$$\mathrm{Hom}_{\mathrm{vect}}(S(E), \mathcal{A}) \simeq S(E^\vee) \otimes \mathcal{A} .$$

Since an algebra homomorphism is a linear map, I can write equation (14) as

$$\begin{aligned} \text{regularized Feynman rules } \in \mathrm{Hom}_{\mathrm{lin}}(\mathcal{H}, \mathrm{Hom}_{\mathrm{lin}}(S(E), \mathcal{A})) &\simeq \mathrm{Hom}_{\mathrm{lin}}(\mathcal{H}, S(E^\vee) \otimes \mathcal{A}) \\ &\simeq \mathrm{Hom}_{\mathrm{lin}}(S(E), \mathrm{Hom}_{\mathrm{lin}}(\mathcal{H}, \mathcal{A})) . \end{aligned}$$

Since the regularized Feynman rules are also an algebra homomorphism, we have

$$\text{regularized Feynman rules } \in \mathrm{Hom}_{\mathrm{lin}}(S(E), \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathcal{A})) ,$$

or

$$\text{regularized Feynman rules } \in \mathrm{Hom}_{\mathrm{lin}}(S(E), G(\mathcal{A})) .$$

□

Bundle Note 3. *Certain* sections of the $K^* \to \Delta^*$ bundle correspond to the regularized Feynman diagrams of a QFT defined by $\mathcal{L}$ acting on the test function $f \in S(E)$. While these sections depend on the test functions f, I write $\gamma_{\mathcal{L}}$ to mean any one of this class of sections associated to the Lagrangian $\mathcal{L}$.

The Birkhoff decomposition theorem allows one to uniquely factor the algebra homomorphisms, separating out the divergent parts, (functions not defined at $z = 0$).

Theorem 4.2. ***Birkhoff Decomposition Theorem*** *[38] Let C be a smooth simple curve in $\mathbb{CP}^1 \setminus \infty$ which separates $\mathbb{CP}$ into two connected components. We call the component containing ∞ C_- and the other component C_+. Let G be a simply connected complex Lie group. For any $\gamma : C \to G$, there are holomorphic maps $\gamma_\pm : C_\pm \to G$ such that $\gamma(z)$ decomposes on C as the product $\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$. This decomposition is unique up to the normalization $\gamma_-(\infty) = 1$.*

Remark 11. The Δ in the bundle is an infinitesimal analogue of $\mathbb{CP}^1 \setminus \infty$.

The Lie group G is simply connected. Since I am interested in the algebra homomorphisms not well defined at $z = 0$, I only consider loops C that do not pass through the point $z = 0$. Then $0 \in C_+$, which is homeomorphic to a disk, while C_- is homeomorphic to an annulus. In the Birkhoff decomposition,

$$\gamma_+ : \quad C_+ = \text{Spec } \mathcal{A}_+ \quad \to \quad G = \text{Spec } \mathcal{H}$$

is a holomorphic function which is finite at 0. That is,

$$\gamma_+ : \quad \text{Spec } \mathcal{A}_+ \quad \to \quad \text{Spec } \mathcal{H} \, .$$

Similarly, the map

$$\gamma_- : \quad C_- \quad \to \quad G$$

is holomorphic on near ∞. That is, γ_- can be extended to define a map

$$\gamma_- : \quad \text{Spec } (\mathbb{C}[Z]) \quad \to \quad \text{Spec } \mathcal{H}$$

where $Z = z^{-1}$. The unique Birkhoff decomposition of a section

$$\gamma = \gamma_-^{-1} \gamma_+$$

can be written as the unique factorization of the algebra homomorphism

$$\gamma^\dagger = \gamma_-^{\dagger \star -1} \star \gamma_+$$

where

$$\begin{aligned} \gamma^\dagger : \mathcal{H} \quad &\to \quad \mathcal{A}\, ; \quad \gamma^\dagger \in G(\mathcal{A}) \\ \gamma_+^\dagger : \mathcal{H} \quad &\to \quad \mathcal{A}_+\, ; \quad \gamma^\dagger \in G(\mathcal{A}_+) \\ \gamma_-^\dagger : \mathcal{H} \quad &\to \quad \mathbb{C}[Z]\, ; \quad \gamma^\dagger \in G(\mathbb{C}[Z])\, . \end{aligned}$$

For $x \in \mathcal{H}$, $\gamma^\dagger(z)(x)$ is a Laurent series convergent somewhere away from $z = 0$. The normalization condition at the end of the Birkhoff Decomposition Theorem translates to $\gamma_-^\dagger(Z)(\varepsilon) = 1$ on the algebra homomorphisms. Furthermore, if $x \in \ker \varepsilon$ then $\gamma_-^\dagger(x) \in \mathcal{A}_-$. If $x \notin \ker \varepsilon$, that is, if $x \in \mathbb{C}$, then $\gamma_-^\dagger(Z)(x) \in \mathbb{C}$.

Remark 12. For ease of notation, I will write the homomorphisms $\gamma_+^\dagger(z)$ and $\gamma_-^\dagger(z)$ both as function of z, with the understanding that $\forall x \in \mathcal{H}$,

$$\gamma_-^\dagger(z)(x) = \sum_{-n}^{0} a_i(x) z^i$$

for some constants a_i that depend on x. Furthermore, if the z dependence is not important to the context I will omit it.

This gives the following theorem.

Theorem 4.3. *The homomorphisms $\gamma_-^\dagger(z)$ and $\gamma_+^\dagger(z)$ are both in $G(\mathcal{A})$.*

Connes and Kreimer in [5] find a recursive expression for $\gamma_-^\dagger(z)$ and $\gamma_+^\dagger(z)$ that corresponds to the BPHZ renormalization recursion, displayed below in Theorem 4.4. The explicit forms of $\gamma^\dagger(z)$ and $\gamma_\pm^\dagger(z)$ are calculated in section 5.4.1. The recursive formulas for dimensional regularization can be generalized to using Rota-Baxter algebras. To do this, first define a linear idempotent Rota-Baxter map $P : \mathcal{A} \to \mathcal{A}$. Notice that this is not an algebra homomorphism. Appendix A develops the theory of Rota-Baxter algebras.

Theorem 4.4. *For an indecomposable $x \in \mathcal{H}$, one can recursively define*

$$\gamma_-^\dagger(z)(x) = -P(\gamma^\dagger(z)(x) + \sum_{(x)} \gamma_-^\dagger(z)(x')\gamma^\dagger(z)(x''))$$

and

$$\gamma_+^\dagger(z)(x) = \gamma^\dagger(z)(x) + \gamma_-^\dagger(z)(x) + \sum_{(x)} \gamma_-^\dagger(z)(x')\gamma^\dagger(z)(x'') .$$

Proof. This is a generalization of a formula which first appeared in [5]. It is proved in detail in Appendix A following the arguments in [10]. □

Example 7. In the case of dimensional regularization and ζ-function regularization, the Rota-Baxter map is $\pi : \mathcal{A} \to \mathcal{A}_-$, a projection onto the negative powers of Laurent series.

$$\begin{array}{rccc} \pi & : & \mathcal{A}_+ \oplus \mathcal{A}_- & \to & \mathcal{A}_- \\ & & (x, y) & \mapsto & y . \end{array}$$

Setting $P = \pi$, exactly recovers the Birkhoff decomposition formula of Connes and Kreimer in [5]. In this case, $\gamma_+^\dagger(z)(x)$ is well defined at $z = 0$, but not at $z = \infty$. Similarly, $\gamma_-^\dagger(z)(x)$ is not well defined at $z = 0$. It is called the pole part because it is a Laurent series with only negative powers of z for $x \in \ker(\varepsilon)$.

Remark 13. For $\gamma_{\mathcal{L}}$, a section associated to the Lagrangian $\mathcal{L}$, notice that for any $x \in \ker \varepsilon$, $\gamma_{\mathcal{L}}^\dagger(z)(x) \in \mathcal{A}$ is a Laurent polynomial in the regulator, $\gamma_{\mathcal{L}+}^\dagger(z)(x) \in \mathcal{A}_+$ is a somewhere convergent formal power series in z, and thus well defined for $z = 0$, and $\gamma_{\mathcal{L}-}^\dagger(z)(x) \in \mathcal{A}_-$ is a Laurent expansion with only negative powers of z, and thus undefined at $z = 0$. Therefore $\gamma_{\mathcal{L}}^\dagger(z)$, $\gamma_{\mathcal{L}+}^\dagger(z)$ and $\gamma_{\mathcal{L}-}^\dagger(z)$ are called the unrenormalized, renormalized and counterterm parts of x respectively.

Definition 21. We can define the subgroup of renormalized algebra homomorphisms as

$$G_+(\mathcal{A}) = G(\mathcal{A}_+) = \{\gamma^\dagger \in G(\mathcal{A}) | \gamma^\dagger(z) = \varepsilon \star \gamma_+^\dagger\}$$

and the subgroup of counterterms as

$$G_-(\mathcal{A}) = G(\mathbb{C}[Z]) = \{\gamma^\dagger \in G(\mathcal{A}) | \gamma^\dagger(z) = \gamma_-^{\dagger\star-1} \star \varepsilon\} .$$

Proposition 4.5. *The composition $(\gamma^\dagger \circ S)(z)$ defines the inverse map under $\star$. Specifically,*

$$(\gamma^\dagger \circ S)(z) = \gamma^{\dagger\star-1}(z) .$$

Proof. The equation

$$(\gamma^\dagger \circ S)(z) \star \gamma^\dagger(z) = \gamma^\dagger(z) \star (\gamma_-^\dagger \circ S)(z) = \varepsilon$$

comes directly from equation (8). □

Bundle Note 4. The group $G(\mathcal{A})$ corresponds to the group of sections of the trivializable bundle $K^* \to \Delta^*$, with

$$\gamma : \Delta^* \to \Delta^* \times G .$$

Remark 14. For ease of notation, henceforth, elements of $G(\mathcal{A})$ will be noted by $\gamma^\dagger$. The symbol γ will be reserved for the sections or loops discussed in this section. The letters x and y will represent the elements of the Hopf algebra $\mathcal{H}$.

5 Renormalization mass

The regularization process results in a Lagrangian that is a function of the regularization parameter. Prior to regularization, the Lagrangian of any theory is scale invariant. That is

$$\int_{\mathbb{R}^n} \mathcal{L}(x)\, d^n x = \int_{\mathbb{R}^n} \mathcal{L}(tx)\, d^n(tx) \; .$$

When the Lagrangian is regularized, and written in terms of a regularization parameter, z, it is no longer scale invariant. This implies that the counterterms of the associated theory depend on the scale of the Lagrangian, which violates the physical principle of locality. In order to preserve scale invariance, introduce a regularization mass, which is a function of the regularization parameter and scale factor, to the regularized Lagrangian. The role of the regularization mass is to cancel out any scaling effects introduced by regularization.

In dimensional regularization, the renormalization mass parameter, as a function of the regularization parameter is of the form μ^z. The Lagrangian becomes

$$\int \frac{1}{2}\left[(|d\phi|^2 - \mu^z m^2\phi^2) + \mu^z g\phi^3\right] d^{6+z}x \; ,$$

where m and ϕ are also functions of the regularization parameter z. The coupling constant transforms as

$$g \mapsto g\mu^z \; ,$$

where μ is called the renormalization mass. For a thorough treatment of how this is carried out, see [37] Ch. 9.

In the language of renormalization, the original Lagrangian is called the bare Lagrangian, written

$$\mathcal{L}_B = \frac{1}{2}(|d\phi_B|^2 - m_B^2\phi_B^2) + g_B\phi_B^3$$

where the subscript B indicates the bare, or unrenormalized, quantities. One can write the Lagrangian after the renormalization process as the sum

$$\mathcal{L}_B = \mathcal{L}_{ct} + \mathcal{L}_{fp}$$

where $\mathcal{L}_{ct}$ corresponds to the part containing the counterterms and $\mathcal{L}_{fp}$ the finite parts. These two parts of the renormalized Lagrangian lead to counterterm and finite parts of the Feynman integrals. Following physics conventions, I write the bare quantities in terms of renormalized quantities:

$$\phi_B = \sqrt{1 + A(g_B, z)}\phi_r\, ; \quad m_B = m_r(1 + B(g_B, z))\, ; \quad g_B = g_r\mu^{-z}\,(1 + C(g_r, \mu, z))$$

where $\lim_{z\to 0} A,\, B,\, C = \infty$. For more details on this process see [37], Section 9.4 and [40], chapters 21 and 10. For ease of notation, write $Z_\phi = 1+A$, $Z_m = (1+B(g_B, z))^2 Z_\phi(g_B, z)$, and $Z_g = (1+C(g_r, \mu, z))Z_\phi^{3/2}(g_B, z)$. Then the bare Lagrangian can be written

$$\begin{aligned}
\mathcal{L}_B &= \frac{1}{2}(Z_\phi(g_B, z)|d\phi|^2 - m_r^2 Z_m(g_B, z)\phi^2) + g_r Z_g^{3/2}(g_B, z)\phi^3 \\
&= \frac{1}{2}(|d\phi|^2 - m_r^2\phi^2) + g_r\phi^3 \\
&\qquad + \frac{1}{2}((Z_\phi - 1)|d\phi|^2 - (Z_m - 1)m_r^2\phi^2) + (Z_g^{3/2} - 1)g_r\phi^3 \; .
\end{aligned}$$

The second line is called the renormalized Lagrangian, consisting of finite quantities $\mathcal{L}_{fp}$, and last line is the counterterm $\mathcal{L}_{CT}$. The terms Z_ϕ, Z_g and Z_m appear in a Feynman integral with different weights. For a

1PI graph, x, the weights they get under rescaling are determined by the weight of the corresponding $x \in \mathcal{H}$. Under scaling, this is written $\mu^Y(x)$.

Notice that every term in the bare Lagrangian depends on g_B, and thus on μ. All the renormalization constants are determined by knowing μ. The rest of this section analyzes the renormalization mass μ, and the dependence of g on it, given by the β-function, introduced below. This fact is used to define the renormalization group. For a more detailed overview of the renormalization group see [14].

5.1 An overview of the physical renormalization group

The renormalization group describes how the dynamics of a system depends on the scale at which it is probed. Working in units where $\hbar = c = 1$, the quantities of mass, momentum, energy and frequency (or inverse length) have the same units. Therefore, the introduction of the renormalization mass into the Lagrangian has the same effect as changing the energy scale on the Lagrangian. One would expect that probing at higher energy levels, and thus smaller length scales, reveals more details about a system than at lower energies. For a theory to be renormalizable, one must be able to average over the extra parameters at the higher energy, λ, and rewrite the extra information in terms of a finite number of parameters at a lower energy, μ. The Lagrangian at the energy scale μ obtained by this averaging is called the *effective Lagrangian at* μ, $(\mathcal{L}, \mu)$. For a specified set of fields and interactions the effective Lagrangian at a μ is a Lagrangian with coefficients which depend on the scale, μ.

Formally, let $M \simeq \mathbb{R}_+$ be a non-canonical energy space with no preferred element. Fix a set of fields and interactions. Call S the set of effective Lagrangians (for this set of fields and interactions) in the energy space, M. One can always find the effective Lagrangian at a lower energy scale, $(\mathcal{L}, \mu)$ if the effective Lagrangian at a higher energy scale, $(\mathcal{L}, \lambda)$ is known, by averaging over the extra parameters. That is, for $\lambda, \mu \in M$ such that $\lambda > \mu$, there is a map

$$R_{\lambda,\mu} : \quad S \to S \tag{15}$$

so that the effective Lagrangian at μ is written $R_{\lambda,\mu}\mathcal{L}$ for $\mathcal{L} \in S$. The map in (15) can be written as an action of $(0, 1]$ on $S \times M$:

$$\begin{array}{ccc} (0,1] \times (S \times M) & \to & S \times M \\ t \circ (\mathcal{L}, \lambda) & \mapsto & (R_{\lambda,t\lambda}\mathcal{L}, t\lambda) \ . \end{array} \tag{16}$$

The map $R_{\lambda,\mu}$ satisfies the properties

1. $R_{\lambda,\mu} R_{\mu,\rho} = R_{\lambda,\rho}$.
2. $R_{\lambda,\lambda} = 1$.

Definition 22. The set $\{R_{\lambda,\mu}\}$ forms a semi-group called the renormalization group in the physics literature.

The renormalization group equations can be derived from differentiating the action in (16) as

$$\frac{\partial}{\partial t}(R_{\lambda,t\lambda}\mathcal{L}_B) = 0 \ .$$

This differential equation gives rise to a system of differential equation which describe the t dependence of the bare parameters, m_B, g_B and ϕ_B, in $R_{\lambda,t\lambda}\mathcal{L}_B$.

Recall that all the parameters in $\mathcal{L}_B$ depend on the bare couping constant g_B. Therefore, by solving for g_B, one can solve all the renormalization group equations. *The β-function describes the t dependence of g_B.* After identifying this dependence the t dependence of all the other parameters in the renormalization group equations can be written as function of β. To find the β-function, define a flow on the renormalization group as in (16)

$$\mu \to t\mu \ .$$

Then g_B becomes a function of t and the β-function can be written as

$$\beta(g_B) = t\frac{\partial g_B}{\partial t}$$

with the initial condition $g_B(1) = g_B$. The β-function is a generator of the renormalization group flow.

Remark 15. The counterterm of the Lagrangian of ϕ^3 theory under dimensional regularization does not depend on the energy scale:

$$\frac{\partial}{\partial t}(R_{\lambda,t\lambda}\mathcal{L}_{ct}) = 0\ . \tag{17}$$

This turns out to be a key criterion for defining the β-function in the Connes-Marcolli renormalization bundle.

This development of the renormalization group and renormalization group equations follows [14]. For details on the renormalization group equations for a ϕ^4 theory, QED and Yang-Mills theory, see [40] chapter 21 or [37] Chapter 9.

Example 8. The β-functions are calculated in terms of a power series in the coupling constant. The following are the one loop approximations of the β-functions for various theories. [40] [42]

1. For the scalar ϕ^4 theory in 4 space-time dimensions,

$$\beta(g) = \frac{3g^2}{16\pi^2}\ .$$

2. For QED, the β-function has the form

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)$$

where e is the dimensionless electric charge.

3. For QCD, the β-function has the form

$$\beta(g) = -\frac{1}{48\pi^2}(33 - 2N_f)g^3$$

where N_f is the number of fermions.

4. For a general Yang-Mills theory with symmetry group G, the β-function has the form

$$\beta(g) = -\frac{11g^3}{48\pi^2}C_2(G)$$

where C_2 is the quadratic Casimir operator.

Connes and Marcolli show that for a scalar theory under dimensional regularization, the β-function can be written as an element of the Lie algebra associated to the Hopf algebra associated to the theory. The β-functions above result from evaluating the corresponding Connes-Marcolli β-function on the sum of the one loop diagrams of the theory. This is discussed in section 5.5

The Connes-Marcolli β-function requires a more general construction of the renormalization group and effective Lagrangians. In the renormalization bundle, $M \simeq \mathbb{C}^\times$ and allows a $\mathbb{C}^\times$ action on $S \times M$. This generalized renormalization group is actually a group. For a general regularization scheme, the renormalization group flow is given by

$$t \circ (\mathcal{L}_B, \lambda) = (R_{\lambda,\lambda(t)}\mathcal{L}_B, \lambda(t))$$

with $t \in \mathbb{C}^\times$. For dimensional regularization, this flow is given by $\lambda(t) = t\lambda$ as above. This is defined in more detail in the next section.

Remark 16. The renormalization group is represented by an action of $\mathbb{C}^\times$ on B, the $\mathbb{C}^\times$ principal bundle over Δ. The renormalization flow is represented by the $\mathbb{C}^\times$ action on the sections of the $K^* \to \Delta^*$ bundle. For more information see sections 5.2 and 5.5.

- The action in (16) is defined by the ratio of the Lagrangian at the smaller energy scale, μ, to the original energy scale, λ, where $t = \frac{\mu}{\lambda}$. Geometrically, I represent the regularized Lagrangian at the energy scale λ by the section $\gamma(z)$ and the regularized Lagrangian at energy scale μ by $\gamma_{t^z}(z)$. Then the quantity

$$(R_{\lambda,t\lambda}\mathcal{L}_B, t\lambda) = h_{t^z} = \gamma^{\dagger\star-1}(z) \star \gamma^\dagger_{t^z} \ .$$

- In order to calculate the β-function, I am interested in the original Lagrangian, i.e. at $z = 0$. Therefore, I take the limit of the quantity above

$$F_t(\gamma^\dagger) = \lim_{z\to 0} h_{t^z}(\gamma^\dagger(z)) \ .$$

  Note that the right-hand side still depends nontrivially on t.

- The β-function is defined on sections of the form γ_{t^z} as

$$\beta(\gamma^\dagger) := \frac{dF_t(\gamma^\dagger)}{dt}|_{t=1} \ .$$

- The physical section $\gamma_{\mathcal{L}}$ has the additional property that

$$\frac{d}{dt}\gamma^\dagger_{t^z-}(z) = 0 \ .$$

  The evaluation of the counterterm on a 1PI graph x does not depend on the mass scale. The set of sections that share this property are called local sections.

The following sections detail the implementation of the renormalization mass term, introduce the set of the physically significant algebra homomorphisms, and the renormalization group in the context of the Hopf algebra. I also define and describe properties of the physical sections.

5.2 Renormalization mass parameter

The sections of the bundle $K \to \Delta$ can be adjusted to incorporate the action of the renormalization mass parameter in the Lagrangian. The renormalization group $\mathbb{C}^\times$, which is parametrized by the renormalization mass, acts on the bundle $K \to \Delta$. This forms a new bundle $P \simeq K \times \mathbb{C}^\times \to B \simeq \Delta \times \mathbb{C}^\times$. The group of sections of this bundle is given by the semidirect product of $G(\mathcal{A})$ with $\mathbb{C}^\times$ over the $\mathbb{C}^\times$ action, $G(\mathcal{A}) \rtimes_\theta \mathbb{C}^\times$. This section develops this new group.

Remark 17. I use the notation $t(s) := e^s$, with $s \in \mathbb{C}$. In fact, for many purposes it is useful to parameterize $\mathbb{C}^\times$ by the exponential map

$$\begin{array}{rcl} \exp \ : \ \mathbb{C} & \to & \mathbb{C}^\times \\ s & \mapsto & t(s) = e^s \ . \end{array}$$

This is the notation in much of the literature. When the parameter s is not important, I shorten the notation to t.

Before studying the action of $\mathbb{C}^\times$ on $G(\mathcal{A})$, one first needs to study the filtration on $\mathcal{H}^\vee$. This will allow the introduction of the semi-direct product structure.

5.2.1 The grading operator

The filtration on the Hopf algebra induces a derivation and a one-parameter group of automorphisms on both $\mathcal{H}$ and $\mathcal{H}^\vee$.

Proposition 5.1. *1. The operator,*

$$\begin{array}{cccc} Y: & Gr_n\mathcal{H} & \to & Gr_n\mathcal{H} \\ & x & \mapsto & nx = |x|x \end{array}$$

is a grading preserving derivation.

2. Exponentiating Y yields a one parameter group of grading preserving multiplicative automorphisms

$$\theta_s(x) = e^{Ys}x = e^{ns}(x) \tag{18}$$

for $x \in Gr_n\mathcal{H}$ with $s \in \mathbb{C}$.

Proof. [25] Obviously, the operation Y is grading preserving. If $x \in Gr_n$, then so is nx. To see that Y is a derivation, for $x \in Gr_n$ and $y \in Gr_m$,

$$Y(xy) = (m+n)xy = Y(x)y + xY(y) \ .$$

Similarly, if $x \in Gr_n$, then so is $\theta_s(x) = e^{ns}x$ for $s \in \mathbb{C}$. Thus θ_s is grading preserving. To see that it is a multiplicative automorphism, for $x \in Gr_n$ and $y \in Gr_m$,

$$\theta_s(xy) = e^{s(m+n)}xy = \theta_s(x)\theta_s(y) \ .$$

□

Definition 23. Both of these operators can be defined on $\mathfrak{g}(\mathcal{A})$ as a derivation and one parameter group of automorphisms by $Y(f)(x) = f \circ Y(x)$ and $\theta_s(f)(x) = f \circ \theta_s(x)$ with $f \in \mathcal{L}(\mathcal{H}, \mathcal{A})$, and $x \in \mathcal{H}$.

Lemma 5.2. *For $x \in \ker \varepsilon$,*

$$Y^{-1}(x) = \int_0^\infty \theta_{-s}(x)ds$$

where the integral is taken along the real axis.

Proof. [7] Since $\mathcal{H}$ is positively graded,

$$\int_0^\infty \theta_{-s}(x)ds = \int_0^\infty e^{-Ys}(x)ds = -Y^{-1}e^{-Ys}(x)|_0^\infty = Y^{-1}(x) \ .$$

□

Since $\theta_{-s}(x)$ is a holomorphic function in s, by the Cauchy Integral Theorem, the integral can be taken along any other path in $\mathbb{C}$ with the same endpoints. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a path in $\mathbb{C}$ such that $\rho(0) = 0$ and $\lim_{s\to\infty} \rho(s) = \infty$. Then I can write

$$Y^{-1}(x) = \int_0^\infty \theta_{-s}(x)ds = \int_\rho \theta_{-s}(x)ds$$

where the integral on the left is taken over the real line while the integral on the right is taken over the path ρ. This extends to a definition of Y^{-1} for $\gamma^\dagger \in \mathfrak{g}(\mathcal{A})$, by remark 8:

$$Y^{-1}(\gamma^\dagger(z)) = \int_\rho \theta_{-s}(\gamma^\dagger(z))ds \ . \tag{19}$$

This is well defined because $\mathfrak{g}(\mathcal{A}) \subset \mathcal{H}^\vee_{(1)}(\mathcal{A})$.

5.2.2 Geometric implementation of the renormalization group

Now I have developed the tools to build the bundle that geometrically captures the action of the renormalization group.

Definition 24.

1. Δ is a disk centered at the origin. As previously introduced, this is the infinitesimal disk of a single complex regularization parameter centered at the origin. The parameter $z \in \Delta$.
2. K is a trivializable G bundle over Δ.
3. B is a $\mathbb{C}^\times$ principal bundle over Δ. The space B consists of the dimensional parameter, z, and the mass parameter, t, representing a single renormalization mass parameter. Let $\sigma_t(z) = (z, t^z)$ be a section of this bundle.
4. The bundle $P \to B$ comes from a product of $\mathbb{C}^\times$ with the bundle K. Its sections are of the form (γ, t), where γ is a section of $K \to \Delta$. The result of a $\mathbb{C}^\times$ action on γ, written $\gamma_t = t^Y \gamma$ is a section of P, written $(t^Y \gamma, 1)$. Recall that Y is the grading operator on $\mathcal{H}$.
5. Let $V \subset B$ be the fiber over $0 \in \Delta$ in B and $W \subset K$ be the fiber over $0 \in \Delta$ in K. Then $B^* = B \setminus V$, $P^* = P \setminus (V \times G)$, $K^* = K \setminus W$ and $\Delta^* = \Delta \setminus 0$.

Notice P^* is a trivializable bundle over B^* (and likewise Δ^*). B^* is a trivializable bundle over Δ^*.

The $\mathbb{C}^\times$ component in B plays the role of the renormalization group introduced in section 5.1. Notice that $\mathbb{C}^\times$ is the affine group scheme associated to the Hopf algebra $\mathbb{C}[t, t^{-1}]$ with the co-product $\Delta(t) = t \otimes t$. Thus the $\mathbb{C}^\times$ action on $\mathcal{H}$ given by θ

$$\begin{aligned} \mathbb{C}^\times \times \mathcal{H} &\to \mathcal{H} \\ (t, x) &\mapsto t^Y x \end{aligned}$$

has the associated affine group scheme $\tilde{G} = G \rtimes_\theta \mathbb{C}^\times$. On the level of group schemes, this corresponds to the group homomorphism

$$\rho : \tilde{G} \to \mathbb{C}^\times$$

with kernel G.

The θ_s operator on $\mathcal{H}$ can be extended to $G(\mathcal{A})$ by

$$\theta_s \gamma^\dagger(x) = t^Y \gamma^\dagger(x) = \gamma^\dagger(t(s)^Y x)$$

with $x \in \mathcal{H}$. This action defines an affine group scheme $\tilde{G}(\mathcal{A}) = G(\mathcal{A}) \rtimes_\theta \mathbb{C}^\times$. The natural homomorphism $\tilde{G}(\mathcal{A}) \to \mathbb{C}^\times$ with kernel $G(\mathcal{A})$ corresponds to an automorphism

$$\begin{aligned} \mathbb{C}^\times &\to \mathrm{Aut}(G(\mathcal{A})) \\ t &\mapsto t^Y \gamma^\dagger(z) \, . \end{aligned}$$

Bundle Note 5. The group $\tilde{G}(\mathcal{A})$ corresponds to the group of sections of the $P^* \to B^*$ bundle.

The structure of the group $\tilde{G}(\mathcal{A})$ is as follows:

Definition 25. Let

$$\tilde{G}(\mathcal{A}) := G(\mathcal{A}) \rtimes_\theta \mathbb{C}^\times \, .$$

The $\mathbb{C}^\times$ action on $G(\mathcal{A})$ is given by

$$(\gamma^\dagger(z), t) = \gamma^\dagger(z) t^{Z_0} \tag{20}$$

where Z_0 is a generator of the Lie algebra of $\mathbb{C}^\times$.

Remark 18. Notice the similarity between the action in (20) and the action in (16). For $\gamma_{\mathcal{L}}$, a section of $K^* \to \Delta^*$, corresponds to the regularized Feynman rules for a Lagrangian L. The sections of $P^* \to B^*$, $(\gamma_{\mathcal{L}}, t) \in \tilde{G}(\mathcal{A})$ correspond to the regularized Feynman rules of an effective Lagrangian written as a function of the scaling parameter.

This defines a new Lie algebra on $\tilde{G}(\mathcal{A})$.

Definition 26. Define

$$\tilde{\mathfrak{g}}(\mathcal{A}) := \mathrm{Lie}(\tilde{G}(\mathcal{A})) = \mathfrak{g}(\mathcal{A}) \oplus \mathbb{C}Z_0 \,.$$

with a Z_0 satisfying

$$[Z_0, x] = Y(x)$$

for $x \in \mathcal{H}$ and

$$t^{Z_0} = t$$

for $t \in \mathbb{C}^\times$.

Lemma 5.3. *For $s \in \mathbb{C}$, the operator Z_0 is related to θ_s by the formula*

$$e^{sZ_0}\gamma^\dagger(z)e^{-sZ_0} = \sum_0^\infty \frac{(sY)^n(\gamma^\dagger(z))}{n!} = \theta_s(\gamma^\dagger(z)) \,.$$

Proof. The first equality comes directly from the Baker-Hausdorff formula

$$e^A B e^{-A} = \sum_0^\infty \frac{B_n}{n!}$$

for A, B elements of a non-commutative algebra where $B_0 = B$, $B_n = [A, B_{n-1}]$.

The middle term is simply the operator $e^{sY}\gamma^\dagger$, so the second equality is definitional. ☐

The action of $\mathbb{C}^\times$ in $\tilde{G}(\mathcal{A})$ is really an action of $\mathbb{C}$ that factors through $\mathbb{C}^\times$. The lifted action is given below.

Lemma 5.3 gives

$$\begin{array}{rccl} \theta_s : & \mathbb{C}^\times & \to & \mathrm{Aut}\ (G(\mathcal{A})) \\ & t(s) & \mapsto & t(s)^{Z_0}\gamma^\dagger(z)t(s)^{-Z_0} \,. \end{array}$$

The product on this group is defined by

$$(\gamma^\dagger, t(s)) \star (\gamma^{\dagger'}, t(s')) = (\gamma^\dagger \star \theta_s \gamma^{\dagger'}, t(s)t(s')) \,.$$

The inverse is given by

$$(\gamma^\dagger, t(s))^{\star-1} = (\theta_{-s}\gamma^{\dagger\star-1}, t(-s)) \tag{21}$$

since

$$(\gamma^\dagger, t(s)) \star (\theta_{-s}\gamma^{\dagger\star-1}, t(-s)) = (\gamma^\dagger\theta_s(\theta_{-s})\gamma^{\dagger\star-1}, 1) = (\varepsilon, 1)$$

and

$$(\theta_{-s}\gamma^{\dagger\star-1}, t(-s)) \star (\gamma^\dagger, t(s)) = (\theta_{-s}\gamma^{\dagger\star-1}\theta_{-s}\gamma^\dagger, 1) = (\theta_{-s}(\gamma^{\dagger\star-1} \star \gamma^\dagger), 1) = (\varepsilon, 1) \,.$$

Elements of the form $t^Y\gamma \in G(\mathcal{A})$ are written as $(t^Y\gamma, 1) \in \tilde{G}(\mathcal{A})$.

Remark 19. The sections of the bundle $P \to B$ are equivariant under the action of $C^{\times}$.

Recall from Section 5.1 that the renormalization mass term varies with the complex regularization parameter as μ^z. Generally speaking, a regularization process defines how a mass parameter depends on the regularization parameters. Geometrically, this is written as a pullback of over the section of the $B \to \Delta$ bundle. Specifically, for the Lagrangian in this paper, the action of the renormalization group on sections $\gamma_{\mathcal{L},f}$ are pulled back over sections of the form

$$\begin{array}{rcl} \sigma_p \ : \ \Delta & \to & B \\ z & \mapsto & (z, p^z) \end{array}$$

with $p(s) = e^s$ for $s \in \mathbb{C}$.

Bundle Note 6. Pulling back over σ_p is the same as fixing the scale of the effective Lagrangian given in section 5.1 to p.

Remark 20. The regularization schemes in this paper are dimensional regularization and ζ-function regularization. The regularized Lagrangian for both of these schemes is the one presented in Section 5.1. For different regularization schemes, one can generalize to any section σ of $B \to \Delta$. This is done in Section 6.

At this scale, the section, $\gamma_{\mathcal{L}p^z}$ on the $P^* \to \Delta^*$ bundle defined

$$\sigma_p^*(t^Y \gamma_{\mathcal{L},f}) = p^{zY} \gamma_{\mathcal{L},f} \ . \tag{22}$$

By abuse of notation, I pullback by sections σ_t, and write the corresponding sections of the $P^* \to \Delta^*$ bundle γ_{t^z}.

Fixing the renormalization scale by pulling back over σ_t gives a bijection from $G(\mathcal{A})$ to $G(\mathcal{A})$.

Lemma 5.4. *For each $\gamma^\dagger$, there is a bijection*

$$\begin{array}{rcl} h_{t^z} \ : \ G(\mathcal{A}) & \to & G(\mathcal{A}) \\ \gamma^\dagger(z) & \mapsto & \gamma^{\dagger -1}(z) \star \gamma^\dagger_{t^z}(z) \ . \end{array}$$

Proof. [11] To see that h_{t^z} is a bijection, notice that $h_{t^z}(\gamma^\dagger) \in G(\mathcal{A})$ can be defined recursively for $x \in \mathrm{Gr}_m\mathcal{H}$ in the m^{th} graded component of $\mathcal{H}$ using the definition of the $\star$ product

$$t^{mz}\gamma^\dagger(z)(x) = \gamma^\dagger(z)(x) h_{t^z}(\gamma^\dagger(z))(1) + \gamma^\dagger(z)(1) h_{t^z}(\gamma^\dagger(z))(x) + \sum_{(x)} \gamma^\dagger(z)(x') h_{t^z}(\gamma^\dagger(z))(x'') \ .$$

Recalling that $\gamma^\dagger(z)(1) = h_{t^z}(z)(1) = 1$, this simplifies to the recursive expression

$$h_{t^z}(\gamma^\dagger(z))(x) = \gamma^\dagger(z)(x) - t^{mz}\gamma^\dagger(z)(x) + \sum_{(x)} \gamma^\dagger(z)(x') h_{t^z}(\gamma^\dagger(z))(x'')$$

and $h_{t^z}(\gamma^\dagger(z))$ can be defined recursively over m given $\gamma^\dagger(z)$ and vice-versa. $\square$

Corrollary 5.5. *The map h_{t^z} corresponds to an injection from sections of $K^* \to \Delta^*$ to sections on $P^* \to \Delta^*$. This is a one family parameter in the bundle $P^* \to \Delta^*$.*

Bundle Note 7. The quantity $h_{t^z}(\gamma^\dagger(z))$ is a one parameter family in $G(\mathcal{A})$ that describes how a section of $K^* \to \Delta^*$ changes under an action of the renormalization group. It is the ratio of the section under the renormalization group to the section prior to the action. Viewed as a section of $P^* \to \Delta^*$ it describes the effect of the renormalization group on the regularized theory at a fixed scale.

5.3 Local counterterms

Regularized QFTs that have counterterms invariant under scaling,

$$\frac{d}{dt}(\gamma_{\mathcal{L}\ t^z})_- = 0 \ ,$$

are said to have local counterterms. These play an important role in the physics literature. Similarly, they have a pivotal role in the renormalization bundle. Connes and Marcolli [8] and Manchon [25] show that these are the only sections for which the β-function is defined, see section 5.5. This section examines the subset of $G(\mathcal{A})$ whose counterterms do not depend on the mass scale.

Definition 27. Let $G^\Phi(\mathcal{A}) \subset G(\mathcal{A})$ be the subset of $G(\mathcal{A})$ respecting the physical condition (17) that the counterterms of $\gamma^\dagger_{t^z}$, $(\gamma^\dagger_{t^z})_-(z)$, do not depend on the the mass scale. More precisely,

$$G^\Phi(\mathcal{A}) = \{\gamma^\dagger \in G(\mathcal{A}) | \ \frac{d}{dt}(\gamma^\dagger_{t^z})_- = 0\} \ .$$

That is, for $\gamma \in G^\Phi(\mathcal{A})$, $(\gamma^\dagger_{t^z})_- = \gamma^\dagger_-$.

For this to be truly physical, the mass parameter would be $t \in \mathbb{R}_+$, and the counterterm of the graph $x \in \mathcal{H}$ given by $(\gamma^\dagger_t)_-(z)(x)$ would be independent of the *real* renormalization mass term t.

Remark 21. The sections corresponding to the unrenormalized, renormalized and counterterm part of a regularized Lagrangian, $\gamma^\dagger_{\mathcal{L}}$, $\gamma^\dagger_{\mathcal{L}+}$ and $\gamma^\dagger_{\mathcal{L}-}$ from Remark 13 are elements of $G^\Phi(\mathcal{A})$.

I can now define the collection of algebra homomorphisms corresponding to the physically meaningful evaluations on the counterterms and finite parts of a section.

Definition 28. 1. Define $G^\Phi_-(\mathcal{A}) \subset G^\Phi(\mathcal{A})$ as the subset consisting solely of the pole parts of the Birkhoff decomposition of a $\gamma^\dagger \in G(A)$. That is,

$$G^\Phi_-(\mathcal{A}) = \{\tilde{\gamma}^\dagger(z) \in G^\Phi(\mathcal{A}) | \ \exists \gamma^\dagger(z) \in G^\Phi(\mathcal{A}) \text{ s.t. } \tilde{\gamma}^\dagger(z) = \gamma^{\dagger\star-1}_-\} \ .$$

2. Define $G^\Phi_+(\mathcal{A}) = G_+(\mathcal{A}) \subset G^\Phi(A)$ as the set consisting solely of the holomorphic parts of the Birkhoff decomposition of a $\gamma^\dagger \in G(A)$. That is,

$$G^\Phi_+(\mathcal{A}) = \{\gamma^\dagger(z) \in G^\Phi(\mathcal{A}) | \ \text{ s.t. } \gamma^\dagger_- = \varepsilon\} \ .$$

Remark 22. A point of clarification on Birkhoff Decomposition and counterterms: Birkhoff decomposition is preserved by the action of the renormalization group $\mathbb{C}^\times$. For any $\gamma^\dagger \in G(\mathcal{A})$, with $\gamma^\dagger = \gamma^{\dagger\star-1}_- \star \gamma^\dagger_+$,

$$\gamma^\dagger_t = (\gamma^{\dagger\star-1}_-(z))_t \star (\gamma^\dagger_+(z))_t = (\gamma^\dagger_t)^{\star-1}_-(z) \star (\gamma^\dagger_t)^{\star-1}_+(z) \ .$$

The first equality is given by automorphism, and not by Birkhoff decomposition. The second equality is the Birkhoff decomposition. Because the action of t^Y does not have a z dependence, it cannot effect the pole structure of $\gamma^\dagger_-$. By uniqueness of the Birkhoff decomposition,

$$(\gamma^\dagger_-(z))_t = \gamma^\dagger_{t-}(z) \quad ; \quad (\gamma^\dagger_+(z))_t = \gamma^\dagger_{t+}(z)$$

However, Birkhoff decomposition is not preserved under pullbacks over σ_t. For $\gamma^\dagger_{t^z} \in G(\mathcal{A})$,

$$(\gamma^{\dagger\star-1}_-(z))_{t^z} \neq (\gamma^\dagger_{t^z})^{\star-1}_-(z) \ \text{ and } \ (\gamma^\dagger_+(z))_{t^z} \neq (\gamma^\dagger_{t^z})^{\star-1}_+(z) \ .$$

Explicit calculation shows that $(\gamma^{\dagger\star-1}_-(z))_{t^z}$ is not in $G(\mathcal{A}_-)$. See section 5.6 for details.

For clarity, Table 1 lists the various subsets of and groups related to $G(\mathcal{A})$.

G	Spec $\mathcal{H}$.
$G(\mathcal{A})$	The $\mathcal{A}$ valued points of G, corresponding to loops from Δ^* to G .
$\tilde{G}(\mathcal{A})$	$G(\mathcal{A}) \rtimes_\theta \mathbb{C}^\times$. The affine group scheme corresponding to the actions of the renormalization mass on $G(\mathcal{A})$.
$G^\Phi(\mathcal{A})$	$\{\gamma^\dagger \in G(\mathcal{A}) \| \frac{d}{dt}(\gamma^\dagger_{t^z})_- = 0\}$, corresponding to sections with invariant counterterms.
$G^\Phi_-(\mathcal{A})$	Counterterms of elements of $G^\Phi(\mathcal{A})$.
$G^\Phi_+(\mathcal{A})$	Renormalized parts of elements of $G^\Phi(\mathcal{A})$.

Table 1: Important groups involving G

5.4 Maps between $G(\mathcal{A})$ and $\mathfrak{g}(\mathcal{A})$

This section first establishes the bijection between the Lie algebra $\mathfrak{g}(\mathcal{A})$ and the Lie Group $G(\mathcal{A})$. Then it develops some related tools that will help in understanding the renormalization group, its flow, and generator. Finally it defines and examines β, the beta function.

5.4.1 The $\tilde{R}$ bijection

Manchon [25] defines a bijection from the Lie group $G(\mathcal{A})$ to the Lie algebra $g(\mathcal{A})$ using the grading operator Y.

Theorem 5.6. *For $\gamma^\dagger \in G(\mathcal{A})$, one can rewrite*

$$\gamma^\dagger \circ Y = \gamma^\dagger \star \psi \tag{23}$$

with $\psi \in \mathfrak{g}(\mathcal{A})$. This defines a bijective correspondence

$$\begin{array}{rccl} \tilde{R}: & G(\mathcal{A}) & \rightarrow & \mathfrak{g}(\mathcal{A}) \\ & \gamma^\dagger & \mapsto & \psi = \gamma^{\dagger\star-1} \star Y(\gamma^\dagger) \; . \end{array}$$

Proof. This is a different proof from the one given in [25].

Let $\phi(z)$ and $\psi(z)$ be two elements in $\mathcal{L}(\mathcal{H}, \mathcal{A}, \star)$. The map $\tilde{R}(\phi) = \psi$ is defined as

$$\phi(Y(x)) = \phi \star \tilde{R}(\phi)(x) \tag{24}$$

for any $x \in \mathcal{H}$.

To show that

$$\tilde{R}: \quad G(\mathcal{A}) \rightarrow \mathfrak{g}(\mathcal{A})$$

consider $\phi(z) \in G(\mathcal{A})$. Then $\phi(xy) = \phi(x)\phi(y)$ for x, $y \in \mathcal{H}$, and

$$\tilde{R}(\phi(z))(x) = \psi(z) = \phi^{\star-1} \star Y(\phi(z)) \; . \tag{25}$$

Notice that $\tilde{R}(\phi)(1) = 0$. For an indecomposable element $x \in \mathcal{H}^{i,\cdot}$, equation (25) gives

$$\psi(x) = n\phi(x) + \sum_{(x)} \phi^{\star-1}(x')Y(\phi)(x'')$$

and equation (24) gives

$$n\phi(z)(x) = \psi(x) + \sum_{(x)} \phi(x')\psi(x'') \; . \tag{26}$$

For two indecomposable elements $x \in \mathrm{Gr}_l\mathcal{H}$ and $y \in \mathrm{Gr}_m\mathcal{H}$, and $\phi(z) \in G(\mathcal{A})$, equation (25) can be rewritten

$$\begin{aligned}\psi(z)(xy) = (\phi^{\star-1} \otimes Y(\phi))(\Delta(x)\Delta(y)) = (l+m)\phi(x)\phi(y) + m\phi^{\star-1}(x)\phi(y) + l\phi^{\star-1}(y)\phi(x) + \\ \textstyle\sum_{(x)}(\phi^{\star-1}(y)\phi^{\star-1}(x')Y(\phi)(x'') + \phi^{\star-1}(x')Y(\phi)(x'')\phi(y) + \phi^{\star-1}(x')\phi(x'')m\phi(y)) \\ + \textstyle\sum_{(y)}(\phi^{\star-1}(x)\phi^{\star-1}(y')Y(\phi)(y'') + \phi^{\star-1}(y')Y(\phi)(y'')\phi(x) + \phi^{\star-1}(y')\phi(y'')l\phi(x)) + \\ \textstyle\sum_{(x)(y)} \phi^{\star-1}(x')\phi^{\star-1}(y')(Y(\phi)(x'')\phi(y'') + \phi(x'')Y(\phi(y''))) \ .\end{aligned} \tag{27}$$

The first line of (27) simplifies to

$$Y(\phi(x))\phi(y) + \phi(x)Y(\phi(y)) + \phi^{\star-1}(x)Y(\phi(y)) + \phi^{\star-1}(y)Y(\phi(x)) \ .$$

Using equation (26) and rewriting ε as

$$\phi^{\star-1} \star \phi(x) = \varepsilon(x) = \phi^{\star-1}(x) + \phi(x) + \sum_{(x)} \phi^{\star-1}(x')\phi(x'')$$

the second line of (27) simplifies to

$$(\phi^{\star-1}(y) + \phi(y))(\psi(x) - Y(\phi(x))) + (\varepsilon(x) - \phi^{\star-1}(x) - \phi(x))Y(\phi(y)) \ .$$

Likewise, the third line becomes

$$(\phi^{\star-1}(x) + \phi(x))(\psi(y) - Y(\phi(y))) + (\varepsilon(y) - \phi^{\star-1}(y) - \phi(y))Y(\phi(x)) \ ,$$

and the fourth becomes

$$(\psi(x) - Y(\phi(x)))(\varepsilon(y) - \phi^{\star-1}(y) - \phi(y)) + (\psi(y) - Y(\phi(y)))(\varepsilon - \phi^{\star-1}(x) - \phi(x)) \ .$$

Putting these four together and simplifying gives

$$\psi(xy) = \psi(x)\varepsilon(y) + \psi(y)\varepsilon(x) \ .$$

That is, ψ is an infinitesimal generator of $\mathcal{L}(\mathcal{H}, \mathcal{A})$, and therefore in $\mathfrak{g}(\mathcal{A})$, and

$$\tilde{R}: \quad G(\mathcal{A}) \to \mathfrak{g}(\mathcal{A}) \ .$$

To check that this map is surjective, for any $\psi(z) \in \mathfrak{g}(\mathcal{A})$, once can iteratively define $\tilde{R}^{-1}(\psi)$ by equation (23) with $x \in \mathrm{Gr}_n\mathcal{H}$. Let $\tilde{R}^{-1}(1) = b$, and $\psi(1) = 0$,

$$n\tilde{R}^{-1}(\psi)(x) = b\psi(x) + \sum_{(x)} \tilde{R}^{-1}(\psi)(x')\psi(x'') \ .$$

Therefore, given b, one can uniquely calculate $\tilde{R}^{-1}\psi(x)$ given $\psi(x)$ by recursion on n. To check that $\tilde{R}^{-1}(\psi) \in G(\mathcal{A})$, apply equation 25 to the product xy where $x \in \mathrm{Gr}_l\mathcal{H}$ and $y \in \mathrm{Gr}_m\mathcal{H}$ are two indecomposable elements. Let $\phi \in \{\tilde{R}^{-1}(\psi)\}$.

$$\begin{aligned} Y(\phi(xy)) &= (\phi \star \psi)(\Delta x)(\Delta y) \\ &= \phi(x)\psi(y) + \phi(y)\psi(x) + \phi(x)\sum_{(y)}\phi(y')\psi(y'') + \phi(y)\sum_{(x)}\phi(x')\psi(x'') \end{aligned}$$

since $\psi(xy) = 0$ if l, $m > 0$. This simplifies to

$$\begin{aligned} Y(\phi(xy)) &= \phi(x)(\psi(y) + \sum_{(y)}\phi(y')\psi(y'')) + \phi(y)(\psi(x) + \sum_{(x)}\phi(x')\psi(x'')) \\ &= \phi(x)Y(\phi(y)) + \phi(y)Y(\phi(x)) = Y(\phi(x)\phi(y)) \ . \end{aligned}$$

That is, $\tilde{R}^{-1}(\psi)(xy) = \phi(xy) = \phi(x)\phi(y)$. Furthermore, if $y = 1$, equation (25) becomes

$$Y(\phi(xy)) = Y(\phi(x)) = \phi(y)\psi(x) + \phi(y)\sum_{(x)} \phi(x')\psi(x'') = \phi(1)Y(\phi(x)) .$$

Therefore, there is a unique $b = \tilde{R}^{-1}(1)$ and the map $\tilde{R}$ is a bijection. Since $\tilde{R}^{-1}(1) = 1$ and $\tilde{R}^{-1}(\psi)(xy) = \tilde{R}^{-1}(\psi)(x)\tilde{R}^{-1}(\psi)(y)$, $\tilde{R}^{-1}(\psi) \in G(\mathcal{A})$. □

Since $\tilde{R}$ is a bijective correspondence, define an inverse map

$$\tilde{R}^{-1}: \quad \mathfrak{g}(\mathcal{A}) \to G(\mathcal{A})$$

as follows. Notice that for $\gamma^\dagger \in G(A)$ such that $\gamma^\dagger = \tilde{R}^{-1}(\psi)$, (23) can be rewritten

$$\tilde{R}^{-1}(\psi) = \gamma^\dagger(x) = \begin{cases} (Y^{-1}(\gamma^\dagger \star \psi))(x), & x \in \ker \varepsilon; \\ x, & x \in \mathbb{C}. \end{cases}$$

where Y^{-1} is as defined in Remark (19).

The rest of this section is an exposition on Araki's expansional notation, as given by [25] [7].

Proposition 5.7. *[25] Let $\psi \in \mathfrak{g}(\mathcal{A})$ and $\gamma^\dagger \in G(\mathcal{A})$. Then one can write*

$$\gamma^\dagger = \tilde{R}^{-1}(\psi) = \varepsilon + Y^{-1}(\tilde{R}^{-1}(\psi) \star \psi) \tag{28}$$

where ε is accounts for the fact that $\gamma^\dagger(1) = 1$ for $\gamma^\dagger = \tilde{R}^{-1}(\psi)$.

This expression can be rewritten using the following definition.

Definition 29. [7] Recursively define a set of functions

$$\{d_i\}: \quad \mathfrak{g}(\mathcal{A}) \to \mathcal{H}^\vee_{(1)}$$

as follows:

$$\begin{aligned} Y(d_1(\alpha)) &= \alpha && (29)\\ Y(d_n(\alpha)) &= d_{n-1} \star \alpha && (30) \end{aligned}$$

for $\alpha \in \mathfrak{g}(\mathcal{A})$.

Since Y^{-1} is defined on $\mathfrak{g}(\mathcal{A})$, these functions can be rewritten according to equation (19) as

$$\begin{aligned} d_1(\alpha) = Y^{-1}(\alpha) &= \int_\rho \theta_{-s}(\alpha) ds && (31)\\ d_{n+1}(\alpha) = Y^{-1}(d_n \star \alpha) &= \int_{\rho_n} \theta_{-s_n}(\ldots \int_{\rho_1} (\theta_{-s_1}(\alpha)ds_1) \star \ldots \alpha) ds_n && (32) \end{aligned}$$

where $\rho_i(0) = 0$ and $\lim_{s\to\infty} \rho_i(s) = \infty$, and $\alpha \in \mathfrak{g}(\mathcal{A})$.

Remark 23. Since the function $\theta_{s_i}(\alpha)$ is holomorphic in $s_i \in \mathbb{C}$, by Cauchy's integral formula, the functions $d_j(\alpha)$ do not depend on the paths $\{\rho_1, \ldots, \rho_j\}$. For ease of calculation, and to maintain consistency with Araki who works over the real numbers, the following calculations will be done over the positive real axis, $\mathbb{R}_{\geq 0} \subset \mathbb{C}$.

The proposition above becomes:

Proposition 5.8. *[25] Given a $\gamma^\dagger \in G(\mathcal{A})$, there exists a unique $\psi \in \mathfrak{g}(\mathcal{A})$ such that*

$$\tilde{R}^{-1}(\psi(z)) = \gamma^\dagger(z) = \varepsilon + \sum_1^\infty d_n(\psi) \ .$$

To prove this proposition, one only needs to check that the sum is well defined.

Lemma 5.9. *The sum $\sum_1^\infty d_n(\alpha)$ is well defined.*

Proof. Since $\alpha \in \mathfrak{g}(\mathcal{A})$, it is in $\mathcal{H}^\vee_{(1)}$ Therefore, $d_n(\alpha) \in \mathcal{H}^\vee_{(1)}$. The assertion follows from local finiteness of $\mathcal{H}^\vee$. □

Proof. Expanding (28) gives, for $\psi(z) \in \mathfrak{g}(\mathcal{A})$ and $\gamma^\dagger(z) \in G(\mathcal{A})$,

$$\gamma^\dagger(z) = \tilde{R}^{-1}(\psi(z)) = \varepsilon + \sum_{n=1}^\infty \underbrace{Y^{-1}(\ldots Y^{-1}(}_{n \text{ times}} \psi(z)) \ldots) \ .$$

But

$$\underbrace{Y^{-1}(\ldots Y^{-1}(}_{n \text{ times}} \psi(z)) \ldots) = d_n(\psi(z)))$$

from equation (32), so this is can be rewritten

$$\gamma^\dagger(z) = \varepsilon + \sum_{n=1}^\infty d_n(\psi(z))$$

proving Proposition 5.8. □

Proposition 5.10. *[25] One can write*

$$\tilde{R}^{-1}(\psi(z)) = \gamma^\dagger = \lim_{r\to\infty} e^{-rZ_0} e^{r(Z_0+\psi(z))}$$

with $r \in \mathbb{R}_{\geq 0}$.

Proof. For $\alpha \in \mathfrak{g}(\mathcal{A})$, rewrite equation (32) using Lemma 5.3 and Araki summation notation [1] by integrating along the non-negative real axis, as specified in remark (23),

$$d_n(\alpha) = \int_0^\infty \bigstar_{j=1}^n e^{-(\sum_j^n r_i)Z_0} \alpha e^{(\sum_j^n r_i)Z_0} dr_1 \ldots dr_n$$
$$= \int_0^\infty e^{-(\sum_1^n r_i)Z_0} \alpha \star e^{r_1 Z_0} \alpha \star \ldots \star e^{r_{n-1}} \alpha e^{r_n} dr_1 \ldots dr_n \ .$$

Since $0 \leq r_i \leq \infty$, rewrite this as

$$= \lim_{r\to\infty} \int_{\sum_{i=1}^n r_i \leq r} e^{-(\sum_1^n r_i)Z_0} \alpha \star e^{r_1 Z_0} \alpha \star \ldots \star e^{r_{n-1}} \alpha e^{r_n} dr_1 \ldots dr_n \ .$$

Let $r_0 = r - \sum_1^n r_i$. Then rewrite

$$d_n(\alpha) = \lim_{r\to\infty} e^{-rZ_0} \int_{\sum_{i=1}^n r_i \leq r} e^{r_0 Z_0} \alpha \star \ldots e^{r_{n-1}} \alpha e^{r_n} dr_1 \ldots dr_n \ .$$

Then, using the formula from [1], that given A, B in a unital Banach algebra,

$$e^{r(A+B)} = 1 + \sum_{n=1}^\infty \int_{\sum_{i=0}^n r_i = r,\, r_i > 0} e^{r_0 B} A \star \ldots e^{r_{n-1} B} A e^{r_n B} dr_1 \ldots ds_n$$

and Proposition 5.8 gives the desired identity. □

Connes and Marcolli [7] define an object called the time ordered expansional based on these calculations.

Definition 30. Let $\alpha(r)$ be a smooth $\mathfrak{g}(A)$ valued function, for $r \in \mathbb{R}_{\geq 0}$ and some $\mathbb{C}$ algebra A. Define

$$Te^{\int_a^b \alpha dr} = \varepsilon + \sum_1^\infty \int_{0 \leq r_1 \leq \ldots \leq r_n \leq b} \alpha(r_1) \cdots \alpha(r_n) dr_1 \ldots dr_n \ .$$

This definition can be generalized to integrals over general paths $\rho \in \mathbb{C}$ by writing the above integrals as path integrals.

Definition 31. Let $\alpha(s)$ be a smooth $\mathfrak{g}(A)$ valued function, for $s \in \mathbb{C}$ and some $\mathbb{C}$ algebra A. Let ρ be a curve in $\mathbb{C}$ parameterized by $r \in \mathbb{R}_{\geq 0}$. Define

$$Te^{\int_\rho \alpha ds} = \varepsilon + \sum_1^\infty \int_{0 \leq r_1 \leq \ldots \leq r_n \leq r} [\alpha(\rho(r_1))\rho'(r_1)] \cdots [\alpha(\rho(r_n))\rho'(r_n)] dr_1 \ldots dr_n$$

with $\rho(0) = a$ and $\rho(r) = b$.

In defining $\tilde{R}^{-1}$, recall that $\theta_s(\psi)$ is holomorphic in s. Therefore $\tilde{R}^{-1}$ can be defined over any curve in $\mathbb{C}$.

Theorem 5.11. *Let ρ be a curve in $\mathbb{C}$, parameterized by $r \in \mathbb{R}_{\geq 0}$ such that $\rho(0) = 0$ and $\lim_{r \to \infty} \rho(r) = \infty$. Then*

$$\tilde{R}^{-1}(\psi(z)) = \gamma^\dagger(z) = Te^{\int_\rho \theta_{-s}(\psi(z)) ds} \ .$$

Proof. Let $\alpha(s) = \theta_{-s}(\psi)$. The result follows from Equation (32) and Proposition 5.8. $\square$

Remark 24. The map $Te^{\int_\rho \theta_{-s}(\alpha) ds}$, where $\rho(0) = 0$ and $\lim_{r \to \infty} \rho(r) = \infty$, is the exponential map from the Lie algebra $\mathfrak{g}(\mathcal{A})$ to the Lie group $G(\mathcal{A})$. $\tilde{R}$ is just the inverse of this map. Specifically, for $\gamma^\dagger \in G(\mathcal{A})$,

$$\gamma^\dagger = Te^{\int_\rho \theta_s(\tilde{R}(\gamma^\dagger)) ds} \ .$$

The following properties of $Te^{\int_a^b \alpha dt}$ can be found in [7] and [1].

Theorem 5.12.
1. *For any curve ρ in $\mathbb{C}$ such $\rho(0) = a$ and $\rho(1) = b$, $Te^{\int_\rho \alpha ds} \in G(\mathcal{A})$.*
2. *One has*
$$Te^{\int_a^a \alpha dt} = 1 \ .$$
If η is another curve in $\mathbb{C}$ such that $\eta(0) = b$ and $\eta(1) = c$
$$Te^{\int_\rho \alpha ds} \star Te^{\int_\eta \alpha ds} = Te^{\int_{\rho \cdot \eta} \alpha ds}$$
where $\rho \cdot \eta$ indicates the curve formed by first following ρ and then η.
3. *Define $g(b) = Te^{\int_\rho \alpha ds}$. Then $g(b) \in G(\mathbb{C})$ is the unique solution at $t = b$ to the differential equation*
$$dg(t) = g(t)\alpha(t)dt; \qquad g(a) = 1 \ .$$

4. *Let $\Omega \subset \mathbb{C}^2$ and for $s, t \in \mathbb{C}^2$, let*

$$\omega(s,t) = \alpha(s,t)ds + \beta(s,t)dt$$

be a $\mathfrak{g}(\mathbb{C})$ valued flat connection. That is

$$\partial_t \alpha - \partial_s \beta = [\alpha, \beta] \ .$$

Let ρ be a path in Ω with $\rho(0) = a$ and $\rho(1) = b$. Then $Te^{\int_0^1 \rho^* \omega dt}$ depends only on the homotopy type of the path ρ.

Proof. [7] proves these facts for $\rho(t) = t$. The rest follows from remark (23). $\square$

By its connection to the time ordered expansional, the map $\tilde{R}$ has important properties when restricted to the physically significant sections $G^\Phi(\mathcal{A})$. This is the subject of the next section.

5.4.2 The geometry of $G^\Phi(\mathcal{A})$

Since the sections in $G^\Phi(\mathcal{A})$ satisfy the condition

$$\frac{d}{dt}\gamma^\dagger_{t^z} = 0 \ ,$$

they are the also the sections that define a β-function of the form found in physical theories. Since the β-function of a theory lies in the Lie algebra $\mathfrak{g}(\mathbb{C})$, I first examine the image of $G^\Phi(\mathcal{A})$ in $\mathfrak{g}(\mathcal{A})$. This section follows the arguments in [11].

Theorem 5.13. *The following restrictions of the map $\tilde{R}$ hold:*

1. *$z\tilde{R}$ restricts to a bijective correspondence*

$$z\tilde{R} : \ G^\Phi(\mathcal{A}) \to \mathfrak{g}(\mathcal{A}) \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+) \ .$$

2. *$z\tilde{R}$ further restricts to a bijective correspondence*

$$z\tilde{R} : \ G^\Phi_-(\mathcal{A}) \to \mathfrak{g} := \mathfrak{g}(\mathbb{C}) \ .$$

Proof. See [11]. $\square$

That is, the map $z\tilde{R}$ identifies the physically interesting evaluations, $G^\Phi(\mathcal{A})$, with the the sub-Lie algebra that maps to holomorphic functions. The evaluations of the counterterms are identified with the sub-Lie algebra that maps to constants.

To understand this structure, notice the following relations about h_{t^z} and $\tilde{R}$ from [11]:

Theorem 5.14. 1. *For any $\gamma^\dagger(z) \in G(\mathcal{A})$,*

$$\frac{d}{dt} h_{t^z}(\gamma^\dagger)(z) \mid_{t=1} = z\tilde{R}(\gamma^\dagger(z)) \ . \tag{33}$$

2. *The restriction of h_{t^z} to $G^\Phi(\mathcal{A})$ maps to $G(\mathcal{A}_+)$.*

$$\begin{array}{rccl} h_{t^z} & : \ G^\Phi(\mathcal{A}) & \to & G(\mathcal{A}_+) \\ & \gamma^\dagger(z) & \mapsto & \gamma_+^{\dagger\star-1}(z) \star (\gamma^\dagger_{t^z})_+(z) \ . \end{array}$$

Proof. These calculations are also shown in [11]:

1. Since $\gamma^\dagger_{t^z} = t^{zY}\gamma^\dagger_t$, differentiating gives
$$\frac{d}{dt}\gamma^\dagger_{t^z}(z)|_{t=1} = z(\gamma^\dagger \circ Y)(z) = z\gamma^\dagger(z) \star \tilde{R}(\gamma^\dagger(z)) \ .$$
The definition of h_{t^z} comes from (23)
$$\gamma^\dagger_{t^z}(z) = \gamma^\dagger(z) \star h_{t^z}(\gamma^\dagger)(z) \ .$$
Differentiating both sides,
$$\frac{d}{dt}\gamma^\dagger_{t^z}(z)|_{t=1} = \gamma^\dagger(z) \star \frac{d}{dt}h_{t^z}(\gamma^\dagger)(z)|_{t=1} \ ,$$
so
$$\frac{d}{dt}h_{t^z}(\gamma^\dagger)(z) \ |_{t=1} = z\tilde{R}(\gamma^\dagger(z)) \ .$$

2. Birkhoff decomposition gives
$$\gamma^\dagger_{t^z}(z) = (\gamma^\dagger_{t^z})^{\star-1}_-(z) \star (\gamma^\dagger_{t^z})_+(z) \ .$$
Since $\gamma^\dagger(z) \in G^\Phi$, $(\gamma^\dagger_{t^z})_- = \gamma^\dagger_-$. Therefore
$$\begin{aligned}\gamma^\dagger_{t^z}(z) &= (\gamma^\dagger)^{\star-1}_-(z) \star (\gamma^\dagger_{t^z})_+(z) \\ &= \gamma^\dagger(z) \star (\gamma^\dagger)^{\star-1}_+(z) \star (\gamma^\dagger_{t^z})_+(z) \ .\end{aligned}$$
Since
$$\gamma^\dagger_{t^z}(z) = \gamma^\dagger(z) \star h_{t^z}(\gamma^\dagger(z)) = \gamma^\dagger(z) \star (\gamma^\dagger)^{\star-1}_+(z) \star (\gamma^\dagger_{t^z})_+(z)$$
one has
$$h_{t^z}(\gamma^\dagger)(z) = (\gamma^\dagger)^{\star-1}_+(z) \star (\gamma^\dagger_{t^z})_+(z) \ .$$
Both terms on the right hand side are in $G(\mathcal{A})_+$, so $(h_{t^z}\gamma^\dagger)(z) \in G(\mathcal{A})_+$.

□

The second statement of this theorem is not surprising, given the definition of $G^\Phi(\mathcal{A})$. It states that if a the counterterm of a section does not depend on the renormalization mass parameter (i.e. $\frac{d}{dt}(\gamma_{t^z})_- = 0$), then the action of the renormalization group is defined at $z = 0$.

Remark 25. The first statement of this theorem says that for a regularized theory $\gamma_{\mathcal{L}}$, $z\tilde{R}$ measures the dependence of the unscaled renormalized theory on scaling.

5.5 Renormalization group flow and the β-function

Physically, the β-function is defined as the derivative of the renormalization group flow at the original Lagrangian (when $z = 0$). Geometrically, the flow is related to $h_{t^z}(\gamma)$ at a specified scale, t. Following Connes and Kreimer in [6], define F_t as the limit of h_{t^z} at $z = 0$.

Definition 32. Define an operator F_t as the limit of the operator h_{t^z} on sections of the $P^* \to \Delta^*$ bundle
$$F_t(\gamma^\dagger) = \lim_{z\to 0} h_{t^z}(\gamma^\dagger) = \lim_{z\to 0} \gamma^{\dagger\star-1}(z) \star \gamma^\dagger_t(z) \ .$$

The operator h_{t^z} defined the ratio of the section γ acted upon by t^{zY} with itself for a general z. That ratio, for the unregularized theory $z = 0$, is given by F_t. These limits are not well defined on all of $G(\mathcal{A})$.

Remark 26. Restricted to $G^\Phi(\mathcal{A})$, Theorem (5.14) says that

$$h_{t^z}: \quad G^\Phi(\mathcal{A}) \to G(\mathcal{A}_+) \subset G^\Phi(\mathcal{A}) \ .$$

Therefore,

$$F_t: \quad G^\Phi(\mathcal{A}) \to G(\mathbb{C}) \ .$$

Theorem 5.15. *For a fixed $\gamma^\dagger \in G^\Phi(A)$, the operator F_t defines a one parameter subgroup on $G(\mathbb{C})$*

$$F_u(\gamma^\dagger) \star F_t(\gamma^\dagger) = F_{ut}(\gamma^\dagger) \ .$$

Proof. This proof expands the argument in [11]. Since the limits are defined on $G^\Phi(\mathcal{A})$, I rewrite the statement of the theorem as

$$\lim_{z\to 0} h_{u^z}(\gamma^\dagger) \star h_{t^z}(\gamma^\dagger)) = \lim_{z\to 0} h_{(ut)^z}(\gamma^\dagger).$$

Notice that

$$u^{zY}(\gamma^\dagger_{t^z}) = \gamma^\dagger_{(ut)^z} \ . \tag{34}$$

Since

$$\gamma^\dagger_{(ut)^z} = \gamma^\dagger \star h_{(ut)^z}(\gamma^\dagger) \quad ; \quad u^z(\gamma^\dagger_{t^z}) = u^{zY}(\gamma^\dagger \star h_{t^z}(\gamma^\dagger))$$

applying equation (23) to equation (34) gives

$$h_{(ut)^z}(\gamma^\dagger) = h_{u^z}(\gamma^\dagger) \star (h_{t^z}(\gamma^\dagger))_{u^z} \ .$$

By Theorem 5.14, this can be rewritten

$$h_{(ut)^z}(\gamma^\dagger) = \gamma_+^{\dagger\star-1} \star (\gamma^\dagger_{u^z})_+ \star (\gamma_+^{\dagger\star-1})_{u^z} \star ((\gamma^\dagger_{t^z})_+)_{u^z} \ .$$

When taking the limit on both sides as $z \to 0$, notice that $\lim_{z\to 0} t^{zY}\gamma_+^\dagger(z) = \gamma_+^\dagger(0)$ is not a function of t, but that $\lim_{z\to 0}(\gamma^\dagger_{t^z})_+(z)$ is a polynomial in t. See section 5.6 for an explicit form of $(\gamma^\dagger_{t^z})_+(z)$. Therefore, taking the limit of equation (35) gives

$$F_{tu}(\gamma^\dagger) = F_u(\gamma^\dagger) \star F_t(\gamma^\dagger) \ ,$$

defining a one parameter subgroup of $G(\mathbb{C})$. □

Definition 33. The renormalization group flow generator, or the β-function, is defined on the group $G(\mathcal{A})$ as $\beta = \frac{d}{dt}F_t|_{t=1}$.

Lemma 5.16. *[11] For $\gamma^\dagger(z) \in G^\Phi(\mathcal{A})$, $\beta(\gamma^\dagger) \in \mathfrak{g}$.*

Proof. By local finiteness, $h_{t^z}(\gamma^\dagger)(x)$ is a convergent series for any $x \in \mathcal{H}$. Therefore, one can switch limits and write

$$\beta(\gamma^\dagger) = \lim_{z\to 0} \frac{d}{dt}|_{t=1} h_{t^z}(\gamma^\dagger) \ . \tag{35}$$

By equation (33), $\frac{d}{dt}|_{t=1}h_{t^z}(\gamma^\dagger) \in \mathfrak{g}(\mathcal{A}_+)$. Therefore $\beta(\gamma^\dagger_{t^z}) \in \mathfrak{g}$. □

Corollary 5.17. *Let $V \subset B$ be the fiber over $0 \in \Delta$. Then the β is a vector field on the subset $V \times G^\Phi$ of the fiber $V \times G$ over 0 in the $P \to \Delta$ bundle.*

Proof. For a fixed $\gamma^\dagger$, $h_{t^z}(\gamma^\dagger(z))$ is a one parameter family in $G^\Phi(\mathcal{A})$. Holding z constant shows that $h_{t^z}(\gamma^\dagger(z))$ is a one parameter family in the fibers of $P^* \to B^*$ over a fixed z. Equation (35) shows that for $\gamma^\dagger \in G^\Phi(\mathcal{A})$, $\beta(\gamma^\dagger)$ is the derivative of a one parameter family in $G(\mathcal{A})$ and thus defines a vector field on $V \times G$. $\square$

Remark 27. In [7], [8], or [11], β is defined as an element in $\mathfrak{g}$ for a given element $\gamma^\dagger \in G^\Phi(\mathcal{A})$, instead of as a vector field. This is just the pullback of the vector field over a fixed section γ_{t^z}.

Bundle Note 8. Physically, the β-function defines the adjustments of the terms of the regularized Lagrangian so that it remains scale invariant. The form of the β-function depends on how the regularization scheme affects scaling of the Lagrangian. Geometrically, it is a quantity that is defined on the sections of $P \to \Delta$.

Remark 28. From equation (12) an element $\alpha \in \mathfrak{g}$ has the property that for two elements, $x, y \in \mathcal{H}$,

$$\alpha(xy) = \varepsilon(x)\alpha(y) + \alpha(x)\varepsilon(y)$$

and $\alpha(x) \in \mathbb{C}$. This means that $\beta(\gamma^\dagger)(x) \in \mathbb{C}$. The contribution of x, a generator of $\mathcal{H}$, to the β-function for a theory $\mathcal{L}$, whose renormalization mass scale depends on the regularization mass parameter as t^z is given by $\beta(\gamma^\dagger(x))$. The one loop expansions of the β-function for various theories given in section 5.1 can be written as $\sum_{x \in \mathcal{H}^1} \beta(\gamma^\dagger)(x)$ where the sum is taken over the generators of $\mathcal{H}^1$.

The β-function has an useful form on $G^\Phi(\mathcal{A})$.

Theorem 5.18. *For $\gamma^\dagger \in G^\Phi(\mathcal{A})$*

$$\beta(\gamma^\dagger) = Res(\gamma_-^{\dagger\star -1} \circ Y) = \lim_{z \to 0} z\tilde{R}(\gamma_-^{\dagger\star -1}) \, .$$

Here, $\mathrm{Res}(\gamma^\dagger(z))$ is the residue of the Laurent series $\gamma^\dagger(z)(x)$, for $x \in \mathcal{H}$. That is,

$$\gamma^\dagger(z)(x) = \sum_{-n}^{\infty} a_i(x) z^i$$

with $a_i \in \mathcal{H}^\vee$. Therefore $\mathrm{Res}(\gamma^\dagger(z)(x) = a_{-1}(x)$.

Proof. Before studying the behavior on the entire set $G^\Phi(\mathcal{A})$, consider it's behavior on the counterterms contained in this set, $G_-^\Phi(\mathcal{A})$. Thus I need the following lemma.

Lemma 5.19. *Let $\tilde{\gamma}^\dagger \in G_-^\Phi(\mathcal{A})$. Then*

$$h_{t^z}(\tilde{\gamma}^\dagger(z)) = (\tilde{\gamma}^\dagger_{t^z})_+(z) \, .$$

Proof. [11] By definition, one can rewrite

$$\tilde{\gamma}^\dagger(z) = \gamma_-^{\dagger\star -1}(z) \star \varepsilon \, .$$

The uniqueness of the Birkhoff decomposition gives

$$\tilde{\gamma}^\dagger_+ = \varepsilon \, .$$

Theorem 5.14, statement 2 gives

$$h_{t^z}(\tilde{\gamma}^\dagger(z)) = \tilde{\gamma}_+^{\dagger\star -1}(z) \star (\tilde{\gamma}^\dagger_{t^z})_+(z) \, ,$$

proving the result. $\square$

This fact is used to explicitly write an expression for $\beta(\tilde{\gamma}^\dagger)$.

Lemma 5.20. *For $\tilde{\gamma}^\dagger \in G_-^\Phi(\mathcal{A})$,*

$$\beta(\tilde{\gamma}^\dagger) = Res\ (\tilde{\gamma}^\dagger_{t^z} \circ Y) = z\tilde{R}(\tilde{\gamma}^\dagger)\ .$$

Furthermore, $\beta(\tilde{\gamma}^\dagger) \in \mathfrak{g}$.

Proof. The proof of this lemma follows the proof given in [11]. By definition, $\beta(\tilde{\gamma}^\dagger) = \lim_{z\to 0} \frac{d}{dt} h_{t^z}(\tilde{\gamma}^\dagger)|_{t=1}$. By the previous lemma, $\beta(\tilde{\gamma}^\dagger) = \lim_{z\to 0} \frac{d}{dt}(\tilde{\gamma}^\dagger_{t^z})_+(z)|_{t=1}$. For $x \in \mathcal{H}$ and $\mathbb{I}$ the identity operator, the Birkhoff decomposition of $(\tilde{\gamma}^\dagger_{t^z})_+(z)$ given in Theorem 4.4 is

$$(\tilde{\gamma}^\dagger_{t^z})_+(z)(x) = (\mathbb{I} - \pi)\left(\tilde{\gamma}^\dagger_{t^z}(z)(x) + \sum_{(x)} (\tilde{\gamma}^{\dagger\star-1})(z)(x')(\tilde{\gamma}^\dagger_{t^z})(z)(x'')\right)\ .$$

The t derivative is given by

$$\frac{d}{dt}(\tilde{\gamma}^\dagger_{t^z})_+(z)(x)|_{t=1} = (\mathbb{I} - \pi)\left(z\tilde{\gamma}^\dagger(z)Y(x) + \sum_{(x)} (\tilde{\gamma}^{\dagger\star-1})(z)(x')z(\tilde{\gamma}^\dagger)(z)Y(x'')\right) = n\mathrm{Res}\ (\tilde{\gamma}^\dagger \circ Y)\ .$$

Notice that this can be rewritten

$$\frac{d}{dt}(\tilde{\gamma}^\dagger_{t^z})_+(z)(x)|_{t=1} = (\mathbb{I} - \pi)\left(z\gamma^\dagger(z) \star (\gamma^\dagger \circ Y)(z)(x)\right) = (\mathbb{I} - \pi)\left(z\tilde{R}(\gamma^\dagger)(z)(x)\right) = z\tilde{R}(\gamma^\dagger)(z)(x)\ ,$$

where the last equality comes from Theorem 5.13. ☐

Now I can return to the proof of the theorem and examine the function β on the entire set $G^\Phi(\mathcal{A})$. Still following [11], write

$$\gamma^\dagger_{t^z}(z) = (\gamma_-^{\dagger\star-1}(z))_{t^z} \star (\gamma_+^\dagger(z))_{t^z}$$

Writing $(\gamma_-^{\dagger\star-1}(z))_{t^z} = \gamma_-^{\dagger\star-1}(z) \star h_{t^z}(\gamma_-^{\dagger\star-1}(z))$, one has

$$\gamma^\dagger_{t^z}(z) = \gamma^\dagger(z) \star h_{t^z}(\gamma^\dagger(z)) = \gamma_-^{\dagger\star-1}(z) \star h_{t^z}(\gamma_-^{\dagger\star-1}(z)) \star (\gamma_+^\dagger(z))_{t^z}$$

or

$$h_{t^z}(\gamma^\dagger(z)) = \gamma_+^{\dagger\star-1}(z) \star h_{t^z}(\gamma_-^{\dagger\star-1}(z)) \star (\gamma_+^\dagger(z))_{t^z}\ .$$

Since $\gamma_+^\dagger(0) = 1$, $F(\gamma^\dagger_{t^z}) = \lim_{z\to 0} h_{t^z}(\gamma^\dagger(z)) = \lim_{z\to 0} h_{t^z}(\gamma_-^{\dagger\star-1}(z))$. Notice that $\gamma_-^{\dagger\star-1} \in G_-^\Phi(\mathcal{A})$, so

$$\beta(\gamma^\dagger) := \frac{d}{dt}\lim_{z\to 0} h_{t^z}(\gamma^\dagger)|_{t=1} = \frac{d}{dt}\lim_{z\to 0} h_{t^z}(\gamma_-^{\dagger\star-1})|_{t=1} = \beta(\gamma_-^{\dagger\star-1})\ .$$

The rest of the theorem follows from the previous lemma.

$$\beta(\gamma^\dagger) = \beta(\gamma_-^{\dagger\star-1}) = z\tilde{R}(\gamma_-^{\dagger\star-1}) = \mathrm{Res}\ (\gamma_-^{\dagger\star-1} \circ Y)\ .$$

☐

Theorem 5.21. *For $\gamma^\dagger \notin G^\Phi(\mathcal{A})$, $\beta(\gamma^\dagger)$ is not well defined.*

Proof. By definition

$$\beta(\gamma^\dagger) = \frac{d}{dt}\lim_{z\to 0} h_{t^z}(\gamma^\dagger)|_{t=1} = \frac{d}{dt}\lim_{z\to 0}\gamma^{\dagger\star-1}\star\gamma^\dagger_{t^z} \ ,$$

which Birkhoff decomposes into

$$\beta(\gamma^\dagger) = \frac{d}{dt}\lim_{z\to 0}\gamma_+^{\dagger\star-1}\gamma_-^\dagger\star(\gamma_{t^z}^{\dagger\star-1})_-\star(\gamma^\dagger_{t^z})_+|_{t=1} \ .$$

Since $\gamma_-^\dagger \neq (\gamma_{t^z}^{\dagger\star})_-$, the middle two terms don't cancel, and $\beta(\gamma^\dagger)(x)$ is a Laurent polynomial in z with poles at $z=0$. The limit, and thus $\beta(\gamma^\dagger)$, is not defined. □

One can learn something about the structure of the β-function, however, by looking at the effect of the renormalization group on a regularized section, γ_{t^z}. Consider $\frac{d}{dt}|_{t=1}h_{t^z} = z\tilde{R}(\gamma_{t^z}) \in \mathfrak{g}(\mathcal{A})$. This is a Laurent expansion of the β-function. While the limit at $z=0$ is not well defined, this Laurent series gives information about the structure and type of singularity at the origin.

5.6 Explicit calculations

This section computes explicitly some of the statements proven in theorems from previous sections.

Connes and Marcolli show

Theorem 5.22. *[7] For $\gamma^\dagger \in G^\Phi(\mathcal{A})$,*

$$\gamma_-^{\dagger\star-1} = \varepsilon + \sum_1^\infty \frac{d_n(\beta(\gamma^\dagger))}{z^n} = Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\gamma^\dagger))ds} \ . \tag{36}$$

Proof. Recall that $\gamma_-^{\dagger\star-1} \in G_-^\Phi(\mathcal{A})$.

Lemma 5.20, gives

$$\tilde{R}(\gamma_-^{\dagger\star-1}) = \frac{\beta((\gamma_-^{\dagger\star-1})_{t^z})}{z} = \frac{\beta(\gamma^\dagger)}{z} \ .$$

Theorem 5.11 yields the desired result.

Connes and Kreimer have an explicit proof of this theorem in CK[01]. □

A general term $\gamma^\dagger \in G^\Phi(\mathcal{A})$ has the form

$$\gamma^\dagger = Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\gamma^\dagger))ds}\star\gamma^\dagger_{reg}$$

where $\gamma^\dagger_{reg} = \gamma^\dagger_+ \in G_+^\Phi(\mathcal{A})$ is a regular expression in z.

Corrollary 5.23. *Setting $-\rho$ to be the curve ρ with opposite orientation,*

$$\gamma_-^\dagger = Te^{z^{-1}\int_{-\rho}\theta_{-s}(\beta(\gamma^\dagger))ds} = Te^{-z^{-1}\int_\rho \theta_{-s}(\beta(\gamma^\dagger))ds} \ . \tag{37}$$

Proof. Theorem 5.12 gives

$$Te^{z^{-1}\int_{-\rho}\theta_{-s}(\beta(\gamma^\dagger))ds}Te^{z^{-1}\int_\rho\theta_{-s}(\beta(\gamma^\dagger))ds} = 1 = Te^{z^{-1}\int_{-\rho}\theta_{-s}(\beta(\gamma^\dagger))ds})Te^{z^{-1}\int_\rho\theta_{-s}(\beta(\gamma^\dagger))ds} \ ,$$

so

$$Te^{z^{-1}\int_{-\rho}\theta_{-s}(\beta(\gamma^\dagger))ds} = (Te^{z^{-1}\int_\rho\theta_{-s}(\beta(\gamma^\dagger))ds})^{-1} \ .$$

□

Having explicitly written expressions for the elements of $G^{\Phi}(\mathcal{A})$, $\gamma_-^{\dagger}$, one can study the effect of introducing the renormalization scale and write down explicit expressions for elements in $G^{\Phi}(\mathcal{A})$.

Consider an element $\tilde{\gamma}^{\dagger}_{t^z}(z) \in G^{\Phi}(\mathcal{A}_-)$.

Proposition 5.24. *[7] Birkhoff decomposition gives*

$$\tilde{\gamma}^{\dagger}_{t^z} = Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}_{t^z}))ds} \star Te^{z^{-1}\int_{\eta_{uz}} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds}$$

where $\rho(0) = 0$ and $\rho(\infty) = \infty$ as before and $\eta_{uz}(r)$ is a path in $\mathbb{C}$ such that $\eta_{uz}(0) = -uz$ and $\eta_{uz}(1) = 0$ for $r \in [0, 1]$. Then each part can be rewritten

$$(\tilde{\gamma}^{\dagger}_{t^z})_-^{\star-1} = Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds}$$

and

$$(\tilde{\gamma}^{\dagger}_{t^z})_+ = Te^{z^{-1}\int_{\eta_{uz}} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds} .$$

Proof. Let $\tilde{\gamma}^{\dagger} = \gamma_-^{\dagger\star-1}$. Rewrite $\gamma^{\dagger}_{t^z}(z) = \theta_{uz}(\gamma^{\dagger}(z))$ with $u \in \mathbb{C}$. Then

$$\tilde{\gamma}^{\dagger}_{t^z} = \theta_{uz}(\gamma_-^{\dagger\star-1}) = \theta_{uz}(Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}_{t^z})ds}) .$$

Since θ_t is an automorphism on $G(\mathcal{A})$ and $\theta_t(1) = 1$ one has

$$\tilde{\gamma}^{\dagger}_{t^z} = Te^{z^{-1}\int_\rho \theta_{uz-s}(\beta(\tilde{\gamma}^{\dagger}))ds} = Te^{z^{-1}\int_{\eta_{uz}\cdot\rho} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}_{t^z}))ds} = Te^{z^{-1}\int_{-\rho} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds} \star Te^{z^{-1}\int_{\eta_{us}} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}_{t^z}))ds}$$

where $\eta_{uz} \cdot \rho$ indicates that the path η_{uz} is followed by the path ρ. The path η_{uz} is parameterized by $r \in [0, 1]$ and $\eta_{uz}(0) = -uz$ and $\eta_{uz}(1) = 0$. The second equality is from a change of variables, while the third equality comes from reversing the orientation of the path and statement two of Theorem 5.12.

It only remains to show that $Te^{z^{-1}\int_{\eta_{uz}} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}_{t^z}))ds} \in G(\mathcal{A}_+)$. It is enough to show that $z^{-1}\int_{-uz}^0 \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds$ is holomorphic in z. Calculating,

$$z^{-1}\int_{-uz}^0 \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}_{t^z}))ds = -\frac{1}{zY}\left(1 - e^{uzY}\right)\beta(\tilde{\gamma}^{\dagger}_{t^z}) ,$$

which is holomorphic in z. Thus the pole part, γ_{t^z}, remains invariant under a t^{zY} action. ☐

Remark 29. I have explicitly calculated some properties that previous sections discuss.

1. $\tilde{\gamma}^{\dagger}_{tz}(z) \notin G^{\Phi}_-(\mathcal{A})$.
2. For $\gamma^{\dagger} \in G^{\Phi}(\mathcal{A})$, $(\gamma^{\dagger}_{t^z})_-$ is independent of t.
3. $(\tilde{\gamma}^{\dagger}_{t^z})_+$ is holomorphic in z and depends on t.
4. The β-function satisfies the equation of the renormalization group flow generator

$$t\frac{d}{dt}(\tilde{\gamma}^{\dagger}_{t^z})_+(0) = \beta(\tilde{\gamma}^{\dagger}_{t^z}(0)) .$$

Next, examine the $\gamma^{\dagger}_{t^z}$ for a $\gamma^{\dagger} \in G^{\Phi}(\mathcal{A})$. Connes and Marcolli show:

Corrollary 5.25. *Any $\gamma^{\dagger}_{t^z}(z) = (\gamma^{\dagger}_{t^z})_-^{\star-1}(z) \star (\gamma^{\dagger}_{t^z})_+(z)$ can be uniquely written*

$$\gamma^{\dagger}_{t^z}(z) = Te^{z^{-1}\int_{\eta_{sz}\cdot\rho} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds} \star \theta_{sz}(\gamma^{\dagger}_+(z))$$

where

$$(\gamma^{\dagger}_{t^z})_+(z) = Te^{z^{-1}\int_{\eta_{sz}} \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds} \star \theta_{sz}(\gamma^{\dagger}_+(z))$$

and

$$(\gamma^{\dagger}_{t^z})_- = Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\tilde{\gamma}^{\dagger}))ds} .$$

Proof. This result is shown in [7]. It is derived in the same manner as the previous proposition. ☐

6 Equisingular connections

This section follows Connes and Marcolli's construction of equisingular connections on the renormalization bundle. I show that there is a trivial $\mathbb{C}^\times$ equivariant connection on the $P^* \to B^*$ bundle that is defined on the pullbacks over the sections by the map $\tilde{R}$ defined in Section 5.4.1. When restricted to sections $\gamma \in G^\Phi(\mathcal{A})$, $\tilde{R}$ defines the equsingular connection of Connes and Marcolli. For other sections, it defines a connection on $P^* \to B^*$ which is not well defined in the limit $\lim_{z\to 0}$. These correspond to connections defined by non-scale invariant regularization schemes.

Example 9. The dimensionally regularized Feynman rules associated to the scalar ϕ^3 theory in 6 dimensions correspond to certain sections of K. Let $\gamma_\mathcal{L}$ be one such. The orbit of this section under the renormalization group (i.e. the Feynman rules of the effective Lagrangians under dimensional regularization) is given by $t^{zY}\gamma_\mathcal{L}$, where t is the complex renormalization mass parameter. For a fixed t_0, the section $t_0^{zY}\gamma$ corresponds to the regularized Feynman rules for the effective Lagrangian at the energy scale t_0.

The connection on P^* identified by Connes and Marcolli in [7] corresponds to the pullback of a single global connection on P^* over a section γ_t corresponding to the Lagrangian in equation (2) under dimesional regularization.

A connection, ω, on a G principal fiber bundle, $P^* \to B^*$, can be expressed as a $\mathfrak{g}$ valued one form, $\Omega^1(\mathfrak{g})$ on P^*. Equivalently, ω assigns to each section, (γ, t), of the bundle a $\mathfrak{g}$ valued one form on B^*. These latter one forms are defined by the pullbacks of ω over sections of the bundle. The group of gauge transformations map between pullbacks.

Definition 34. Let $P \to B$ be a G principal bundle with local sections $s_u : U \to P$ and $s_v : V \to P$, with U, $V \subset B$ open sets, such that $s_u(b) = g$ with $b \in B$ and $g \in G$. Then there is a transformation map g_{uv} such that $g_{uv}(b)s_v(b) = s_u(b)$ for $b \in V \cap U$. These maps form the group of gauge transformations.

Since the restricted renormalization bundle $P^* \to B^*$ is trivializable, all sections are global.

Remark 30. Since the sections of $P^* \to B^*$ form a group $\tilde{G}(\mathcal{A})$, the transformation group is the same as the group of sections $\tilde{G}(\mathcal{A})$. The group of gauge transformations of the bundle $P \to B$, is the group of sections of that bundle, $\tilde{G}(\mathcal{A}_+)$. This a subgroup of the gauge transformations on $P^* \to B^*$. The gauge group $\tilde{G}(\mathcal{A}_+)$ has the additional property that it does not affect the type of singularity of the sections at $z = 0$

Let $(\gamma(z), t)$ be a section of $P^* \to B^*$, and let ω be a connection on P. Then the pullbacks of ω by the sections (γ, t) are written $(\gamma, t)^*\omega \in \Omega^1(\mathfrak{g})$. The gauge transformation group acts on these pullbacks by conjugation

$$(f^{\star -1} \star g)^*\omega = g^{-1}dg + g^{\star -1}(f^*\omega)g \tag{38}$$

with g, $f \in \tilde{G}(\mathcal{A})$.

In order to define a connection with such behavior on the pullbacks, first introduce a logarithmic differential operator on P^*.

Definition 35. Let D be a differential operator.

$$\begin{aligned} D : \tilde{G}(\mathcal{A}) &\to \Omega^1(\tilde{\mathfrak{g}}) \\ (\gamma^\dagger(z), t) = (\gamma^\dagger(z), t) &\mapsto (\gamma^\dagger(z), t)^{\star -1} \star d(\gamma^\dagger(z), t) \ . \end{aligned}$$

Recall that $\gamma^{\star -1}(z, t) = t^Y \gamma^\dagger(z) \circ S$ and that $(\gamma^\dagger, t)^{-1} = (t^{-Y}\gamma^\dagger, t^{-1})$.

Recalling that $\gamma^\dagger(xy) = \gamma^\dagger(x)\gamma^\dagger(y)$ with the product the same as that of $\mathcal{A}$, I can check

$$D(\gamma^\dagger(z)(xy), t) = D(\gamma^\dagger(z))(x)(\varepsilon, t)(y) + (\varepsilon, t)(x)D(\gamma^\dagger(z), t)(y) \ .$$

Therefore $D(\gamma^\dagger, t)(x) \in \Omega^1(\mathfrak{g})$, is a $\mathfrak{g}$ valued one form on B^*.

Lemma 6.1. *For $f,\ g \in \tilde{G}(\mathcal{A})$, the differential $D(\gamma(z),t) = (\gamma(z),t)^\star\omega$ defines a connection on sections of $P^* \to B^*$.*

Proof. If D defines a connection, the it must satisfy equation (38). For $f,\ g \in \tilde{G}(\mathcal{A})$,

$$D(fg) = Dg + g^{-1}(Df)g\ .$$

Then

$$D(f^{-1}g) = Dg + g^{-1}D(f^{-1})g\ .$$

Since $df^{-1} = -f^{-1}df f^{-1}$,

$$D(f^{-1}g) = Dg - g^{-1}ff^{-1}dff^{-1}g\ ,$$

or

$$Dg = D(f^{-1}g) + (f^{-1}g)^{-1}Df(f^{-1}g)\ ,$$

which satisfies equation (38). Therefore, the differential $D(f)$ can be written in terms of connections, as $f^*\omega$. □

Proposition 6.2. *[7] The connection ω is $\mathbb{C}^\times$ equivariant,*

$$u^Y\omega(z,t,x) = \omega(z,ut,u^Yx)\ .$$

Proof. It is sufficient to check that $(\gamma(z),t)^*\omega$ is $\mathbb{C}^\times$ equivariant for all sections of the form $(\gamma(z),t)$. This is true since the bundle $P^* \to B^*$ is $\mathbb{C}^\times$ equivariant:

$$u^{Z_0}(\gamma^\dagger(z),t) = (u^Y\gamma^\dagger(z),ut)\ .$$

□

Since $(\gamma^\dagger(z),t) = t^{Z_0}(t^{-Y}\gamma^\dagger(z),1)$, and $(t^Y\gamma^\dagger(z),1)$ is identified with $\gamma_t^\dagger$, it is sufficient to define the connection of sections of the form

$$(t^Y\gamma(z),1)^*\omega = \gamma_t^*\omega\ .$$

Proposition 6.3. *Given any section γ_t, one can directly calculate the corresponding pullback of the connection ω on it.*

$$D(t^Y\gamma^\dagger(z)) = t^Y(\gamma^{\dagger\star-1}(z)\star\partial_z\gamma^\dagger(z))dz + t^Y(\tilde{R}(\gamma^\dagger)(z))\frac{dt}{t}\ .$$

Proof. One has

$$d(t^Y\gamma_t^\dagger(z)) = t^Y\partial_z\gamma^\dagger(z)dz + t^YY\gamma^\dagger(z)\frac{dt}{t}\ .$$

Rewrite this as

$$d(t^Y\gamma^\dagger(z)) = t^Y(\partial_z\gamma^\dagger(z))dz + t^Y\gamma^\dagger(z)\star\tilde{R}(\gamma^\dagger(z))\frac{dt}{t}\ .$$

The logarithmic derivative is given by

$$Dt^Y\gamma^\dagger(z) = t^Y\gamma^{\dagger\star-1}(z)\star t^Y\partial_z\gamma^\dagger(z))dz + t^Y\tilde{R}(\gamma^\dagger)(z)\frac{dt}{t}\ .$$

which I rewrite

$$Dt^Y\gamma^\dagger(z) = t^Y(\gamma^{\dagger\star-1}(z)\star\partial_z\gamma^\dagger(z))dz + t^Y(\tilde{R}(\gamma^\dagger)(z))\frac{dt}{t}\ .$$

□

Since $\omega \in \Omega^1(\tilde{\mathfrak{g}})$, ω has the form

$$(\gamma, t)^*\omega = a_\gamma(z,t)dx + b_\gamma(z,t)\frac{dt}{t}$$
$$(\gamma_t^*)\omega = a_{\gamma_t}(z,1)dx + b_{\gamma_t}(z,1)\frac{dt}{t} \ .$$

The terms a_γ and b_γ are defined as

$$a_{\gamma_t}(z,1) = t^Y(\gamma^{\dagger\star-1}(z) \star \partial_z\gamma^\dagger(z))$$

and that

$$b_{\gamma_t}(z,1) = t^Y(\tilde{R}(\gamma^\dagger)(z)) \ . \tag{39}$$

Remark 31. If $\gamma^\dagger \in G^\Phi(A)$, Theorem 5.18 gives that this means that

$$\text{Res } (b_{\gamma_t}) = t^Y(\beta(\gamma_t^\dagger)) = t^Y(\beta(\gamma_-^{\dagger\star-1})) \ . \tag{40}$$

Notice that if $\gamma \in G^\Phi(\mathcal{A})$, Theorem 5.18 shows that b_γ has a simple pole.

Theorem 6.4. *Let ω be a connection on the bundle $P^* \to B^*$ defined on sections of the bundle by the differential equation $\gamma_t^*\omega = D\gamma_t^\dagger(z)$. The connection is defined by $\tilde{R}(\gamma^\dagger) \in \mathfrak{g}(\mathcal{A})$.*

Proof. Defining the bundle by the pullback of its sections, we see that $\gamma^*\omega$ is uniquely defined by γ. The section γ is uniquely defined by the map $\tilde{R}(\gamma^\dagger) \in \mathfrak{g}(\mathcal{A})$. □

Remark 32. Theorem 6.4 is different from the result of Connes and Marcolli in [7] and [8]. They use the fact that for a flat of the connection, $Te^{\int_\rho \omega(z,t)ds}$ is determined only by the homotopy class of the path, ρ, over which the integral is taken, to uniquely define the pullbacks $\gamma^*\omega$.

Remark 33. Defining the connection on $P^* \to B^*$ by the map $\tilde{R}$ as in Theorem 6.4 loses the geometric intuition for the connection found in Connes and Marcolli's definition. However, unlike the Connes-Marcolli definition, this construction lets me define a connection globally on $P^* \to B^*$. The Connes-Marcolli equisingular connection corresponds to a particular pullback of this global connection. Theorem 6.4 defines a connection for all regularization schemes represented by the bundle $P^* \to B^*$.

I now follow Connes and Marcolli's development of the equisingular connection. The first requirement of their unique definition of the connection to establish the flatness of ω.

Proposition 6.5. *The connection ω is flat.*

Proof. It is sufficient to check that all the pullbacks satisfy

$$[a_{\gamma_t}(z,1), b_{\gamma_t}(z,1)] = \partial_t(a_{\gamma_t}(z,1)) - \partial_z(b_{\gamma_t}(z,1)) \ . \tag{41}$$

For any section γ_t, the left hand side of equation (41) can be written

$$[a_{\gamma_t}(z,1), b_{\gamma_t}(z,1)] = \frac{t^Y}{t}\left(\gamma_t^{\dagger\star-1} \star \partial_z\gamma_t^\dagger \star \gamma_t^{\dagger\star-1} \star nY(\gamma_t^\dagger) - \gamma_t^{\dagger\star-1} \star nY(\gamma_t^\dagger) \star \gamma_t^{\dagger\star-1} \star \partial_z\gamma_t^\dagger\right) \ .$$

The right hand side Equation (41) can be written

$$\frac{t^Y}{t}\left(Y(\gamma_t^{\dagger\star-1}) \star \partial_z\gamma_t^\dagger + \gamma_t^{\dagger\star-1} \star Y(\partial_z\gamma_t^\dagger) - \left(-\gamma_t^{\dagger\star-1} \star \partial_z\gamma_t^\dagger \star \gamma_t^{\dagger\star-1} \star Y(\gamma_t^\dagger) + \gamma_t^{\dagger\star-1} \star Y(\partial_z\gamma_t^\dagger)\right)\right)$$

which simplifies to

$$\frac{t^Y}{t}\left(Y(\gamma_t^{\dagger\star-1}) \star \partial_z\gamma_t^\dagger + \gamma_t^{\dagger\star-1} \star \partial_z\gamma_t^\dagger \star \gamma_t^{\dagger\star-1} \star Y(\gamma_t^\dagger)\right) \ .$$

But this can be rewritten

$$Y(\gamma_t^{\dagger\star-1}) = Y(\gamma_t^{\dagger\star-1}\star\gamma_t^\dagger\star\gamma_t^{\dagger\star-1}) = Y(\gamma_t^{\dagger\star-1}) + \gamma_t^{\dagger\star-1}\star Y(\gamma_t^\dagger)\star\gamma_t^{\dagger\star-1} + Y(\gamma_t^{\dagger\star-1}) = -\gamma_t^{\dagger\star-1}\star Y(\gamma_t^\dagger)\star\gamma_t^{\dagger\star-1} \ .$$

Plugging this into the previous equation gives

$$\partial_t(a_{\gamma_t}(z,1)) - \partial_z(b_{\gamma_t}(z,1)) = \frac{t^Y}{t}(-\gamma_t^{\dagger\star-1}\star Y(\gamma_t^\dagger)\star\gamma_t^{\dagger\star-1})\star\partial_z\gamma_t^\dagger + \gamma_t^{\dagger\star-1}\star\partial_z\gamma_t^\dagger\star\gamma_t^{\dagger\star-1}\star Y(\gamma_t^\dagger) \ .$$

Since ω is flat on all sections, it is flat. □

With this definition of this connection on $P^* \to B^*$, I can now define an equivalence class on these pullbacks defined by the group of gauge transformations $G(\mathcal{A}_+)$. The sections of the form $\gamma_-^{\star-1}\star\epsilon$ corresponding to counterterms are left invariant when passing to the quotient defined by this equivalence relation.

Definition 36. Two pullbacks of the connection $\gamma_t^*\omega$ and $\gamma_t'^*\omega$ are equivalent if and only if one pullback can be written in terms of the action of $G(\mathcal{A}_+)$ on the other as in definition 34.

$$\gamma_t'^*\omega = D\psi_t + \psi_t^{\star-1}\star\gamma_t^*\omega\star\psi_t$$

for $\psi_t \in G(\mathcal{A}_+)_t$, the group of sections that are regular in z and t. I write this equivalence as $\gamma_t'^*\omega \sim \gamma_t^*\omega$.

Remark 34. Proposition 6.1 shows that two connections $\gamma_t^*\omega$ and $\gamma_t'^*\omega$ are equivalent if and only if they can be written as

$$\gamma_t^*\omega = D\gamma_t^\dagger \quad \text{and} \quad \gamma_t'^*\omega = D(\gamma_t^\dagger\star\psi_t)$$

for some $\psi_t \in G(\mathcal{A}_+)$.

Proposition 6.6. *Two sections are equivalent, $\gamma_t^*\omega \sim \gamma_t'^*\omega$, if and only if*

$$\gamma_-^{\dagger\star-1} = \gamma_-^{\dagger\prime\star-1} \ .$$

Proof. Birkhoff Decomposition gives

$$t^Y(\gamma_-^{\dagger\star-1}(z)\star\gamma_+^\dagger(z)) = t^Y(\gamma_-^{\dagger\prime\star-1}(z)\star\gamma_+^{\dagger\prime}(z)) \ .$$

Then

$$\gamma_-^{\dagger\star-1}(z) = \left(\gamma_-^{\dagger\prime\star-1}(z)\star\gamma_+^{\dagger\prime}(z)\right)\star\gamma_+^{\dagger\star-1}(z) \ .$$

The term $\gamma_+^{\dagger\prime}(z)\star\gamma_+^{\dagger\star-1}(z)$ is regular over B, so $\gamma_t^*\omega \sim \gamma_t'^*\omega$ if and only if

$$\gamma_-^{\dagger\star-1}(z) = \gamma_-^{\dagger\prime\star-1}(z) \ .$$

□

Remark 35. The quotient of the pullbacks of ω on the bundle $P^* \to B^*$ by the gauge equivalence is isomorphic to $\tilde{G}(\mathcal{A}_-)$. The gauge equivalence on the connection classifies pullbacks by the counterterms of the corresponding sections.

The connection ω on the $P^* \to B^*$ bundle can be pulled back to a connection ω^* on the $P^* \to \Delta^*$ bundle. The connection ω^* can be expressed on its pullbacks as

$$\sigma_{t^z}^*\omega^* = \sigma_{t^z}^*(\gamma_t^*\omega) = \gamma_{t^z}^*\omega \ .$$

I write $\gamma_{t^z}^*\omega$ to indicate these pullbacks of the connection on $P^* \to \Delta^*$.

Corollary 6.7. *Consider the pullbacks of $\gamma_t^*\omega$ over the sections σ_t, $\gamma_{t^z}^*\omega$. The gauge equivalence $\gamma_{t^z}^{\dagger *}\omega \sim \gamma_{t^z}^{\dagger\prime *}\omega$ in definition 36 is equivalent to*

$$(\gamma_{t^z}^{\dagger}(z))_- = (\gamma_{t^z}^{\dagger\prime}(z))_- \ .$$

Proof. This is shown by the same calculations as above. ☐

This condition defines a property on the pullbacks of the connection called equisingularity.

To see this formally, define:

Definition 37. The pullback $\gamma_t^*\omega$ along on $P^* \to B^*$ is equisingular when pulled back to the bundle $P^* \to \Delta^*$ if and only if

- ω is equivariant under the $\mathbb{C}^\times$ action on the section of $P^* \to B^*$.
- For every pair of sections σ, σ' of the $B \to \Delta$ bundle, $\sigma(0) = \sigma'(0)$, the corresponding pull backs of the connection ω, $\sigma^*(\gamma_t^*\omega)$ and $\sigma'^*(\gamma_t^*\omega)$ are equivalent under the action of $G(\mathcal{A}_+)$.

Remark 36. Let $\gamma^\dagger(z) \in G^\Phi(\mathcal{A})$. The set $G_-^\Phi(\mathcal{A})$ is the fixed points of $G(\mathcal{A})$ under the $\mathbb{C}^\times$ action. The quotient of the set of pullbacks of ω on the bundle $P^* \to \Delta^*$ by the gauge equivalence is isomorphic to $G_-^\Phi(\mathcal{A})$.

Proposition 6.8. *The connection ω is equisingular along γ_t if and only if $\gamma^\dagger \in G^\Phi(\mathcal{A})$.*

Proof. Equisingularity means that for any two sections σ and $\sigma'(z)$, such that $\sigma(0) = \sigma'(0)$,

$$(\sigma^*\gamma_t)^*\omega \sim (\sigma'^*\gamma_t)^*\omega \ .$$

This means that

$$(\sigma^*\gamma_t)_- = (\sigma'^*\gamma_t)_- \ .$$

In other words, the counterterms do not depend on the renormalization mass. This is exactly the definition of sections in $G^\Phi(\mathcal{A})$. ☐

Corollary 6.9. *If $\gamma_t^*\omega \sim \gamma_t'\omega$, and $\gamma^\dagger \in G^\Phi(\mathcal{A})$ then so is $\gamma'^\dagger$.*

For sections corresponding to $G^\Phi(\mathcal{A})$, Connes and Marcolli prove:

Theorem 6.10. *Let $\gamma_{t^z}^\dagger \in G^\Phi(\mathcal{A})$, and $(\tilde{\gamma})^\dagger = \gamma_-^{\dagger\star-1}$. Let ω be a flat connection on $P^* \to B^*$ defined on the pullbacks as $\gamma_{t^z}^*\omega = D\gamma_{t^z}$. These pullbacks are uniquely defined by $\beta(\tilde{\gamma})$ up to the equivalence class defined by $G(\mathcal{A}_+)$. That is*

$$\gamma_{t^z}^*\omega \sim \left(Te^{z^{-1}\int_\rho \theta_{-s}(\beta(\tilde{\gamma}^\dagger))ds}\right)^*\omega = \tilde{\gamma}^*\omega \ .$$

Proof. This is an outline of the proof given by Connes and Marcolli. For details, see [7] and [8].

Notice that

$$(\gamma(z), t(s))^*\omega(z, t(s)) = t(s)^{Z_0}\gamma_{t(-s)}^*(\omega(z,1)) \ .$$

For $\tilde{\gamma} \in G_-^\Phi(\mathcal{A})$, since ω is flat, the expression

$$Te^{\int_\rho(\theta_{-s}\tilde{\gamma})^*(\omega(z,1))ds} = Te^{\int_\rho \theta_{-s}(\tilde{\gamma}^{\dagger\star-1}\star\partial_z\tilde{\gamma}^\dagger)dz + \theta_{-s}\frac{\beta(\gamma^\dagger)}{z})ds}$$

depends only on the homotopy class of ρ.

Connes and Marcolli then show that B^* has trivial monodromy, and choose a path so that

$$Te^{\int_\rho \theta_{-s}(\tilde{\gamma}^{\dagger\star -1}\star\partial_z\tilde{\gamma}^\dagger)dz} = 0 \; .$$

Therefore,

$$Te^{\int_\rho (\theta_{-s}\tilde{\gamma})^*(\omega(z,1))ds} = Te^{z^{-1}\int_\rho \theta_{-s}\beta(\tilde{\gamma}^\dagger)ds} = \gamma_-^\dagger \; .$$

Because

$$DTe^{\int_\rho (\theta_{-s}\gamma)^*(\omega(z,1))ds} = (\gamma_t)^*\omega \; ,$$

this uniquely defines the connection. This shows that for $\gamma \in G^\phi(\mathcal{A})$, $\gamma_{t^z}^*\omega$ is uniquely defined by $\beta(\gamma^\dagger)$ up to the equivalence class defined by $G(\mathcal{A})$. □

Remark 37. This proof works for pullbacks along any two sections $\sigma^*(\gamma_t^*\omega)$, and $\sigma'^*(\gamma_t^*\omega)$ such that $\sigma(0) = \sigma'(0)$. It is convenient to work with sections of the form σ_t because the notation has been developed for these pullbacks and because of their physical significance.

The pullbacks $\gamma_{t^z}^*\omega$, for $\gamma^\dagger \in G^\Phi(\mathcal{A})$, are equisingular pullbacks. For a general section of $P^* \to B^*$, where the counterterm may depend on the renormalization mass, one can still define, by Theorem 6.4 an element of $\mathfrak{g}(\mathcal{A})$ that defines the counterterm of the connection.

Theorem 6.11. *Let ω be a flat connection on $P^* \to B^*$ defined on the pullbacks as $\gamma_{t^z}^*\omega = D\gamma_{t^z}$. For $\gamma^\dagger \in G(\mathcal{A})$, the pullbacks are uniqely defined by $\tilde{R}(\gamma_{t^z}^\dagger)$ up to the equivalence class defined by $G(\mathcal{A}_+)$.*

7 Renormalization bundle for a curved background QFT

Connes and Marcolli construct the renormalization bundle for a Hopf algebra of Feynman graphs in a QFT over a flat Minkowski space. Their construction can be extended to create a renormalization bundle for a QFT on a general manifold M with Riemannian metric tensor g_{ij} and conformal changes of metric. To do this, certain conventions used for calculating QFTs in flat space need altering for a general background.

Since the general background space will have a Riemannian metric, I need to change from a Lorentzian metric to a Riemannian one. The process of switching between metrics for the purpose of regularization is well understood on a flat background. A good reference on the process of translating between the two metrics is Simon's book [39]. It shows that a quantum field theory in $\mathbb{R}^n$ can be written in terms of an Euclidean metric by imbedding Minkowski space into a complex space with an Euclidean metric,

$$\mathbb{R}^n \hookrightarrow \mathbb{C}^n \; ,$$

such that time is a purely imaginary dimension, $t = ix_0$, and the spatial dimension are purely real. In this embedding the Lorentz metric and Euclidean metric give the same results,

$$ds_L^2 = dt^2 - dx_1^2 - \ldots - dx_{n-1}^2 = -dx_0^2 - dx_1^2 - \ldots - dx_{n-1}^2 = ds_M^2 \; .$$

Notice that this is a negative definite metric. This embedding is called Wick rotation, and provides a way to solve problems in a Lorentzian metric by rewriting them in an Euclidean metric. Wick rotation changes the Lagrangian density:

$$\mathcal{L} = |d\phi|_{Lor}^2 - m^2 + \lambda\phi^3 \to \mathcal{L} = -|d\phi|_{Euc}^2 - m^2 + \lambda\phi^3 \; ,$$

where the first norm is the in the Lorentzian metric with signature $(+-----)$ and the second norm is the Euclidean norm. The Lagrangian in a Euclidean metric is

$$L = \int_M \phi(\Delta - m^2)\phi + \lambda\phi^3 \, d^6x \; .$$

There is also a body of work, most notably by Nelson [29], that shows that Markov fields, which are useful in performing field theory calculations in Euclidean space, under certain boundary conditions locally give rise to the Wightman axioms for field theory in Minkowski space. Thus results in Euclidean space are consistent with results in Minkowski space. When considering field theories over curved space-time, both these methods of translating between Lorentzian and Riemannian metrics only work on local coordinate patches. A global means of translating between the two metrics has yet to be developed.

In flat space, a QFT is defined by the Lagrangian

$$L = \int_{\mathbb{R}^n} \mathcal{L}\, d^n x$$

which defines a probability amplitude for the field ϕ by

$$e^{iL}$$

which in turn defines a measure on the space of fields

$$d\mu = e^{iL}\mathcal{D}\phi(x)\ ,$$

where $\mathcal{D}(\phi(x))$ is interpreted as the infinite dimensional measure $\prod_x d\phi(x)$ [41]. This notation changes when working on a curved background. Let $|g| = \det g_{ij}$. On an n-dimensional Riemannian manifold, (M, g), with metric g, the Lagrangian becomes

$$L_M = \int_M \mathcal{L}\sqrt{|g(x)|}d^n x\ ,$$

and the measure becomes [13]

$$d\mu = e^{iL_M}\prod_x d\phi(x)\ .$$

The regularization scheme on a manifold is also different. Recall that in flat space-time, dimensional regularization is performed by analytically continuing the dimensions of the space-time theory over, and then introducing terms to the Lagrangian to keep it dimensionless. For instance, for a scalar ϕ^3 theory,

$$L = \int_{\mathbb{R}^6}\left[\frac{1}{2}\phi(\Delta - m^2)\phi + g\phi^3\right]d^6x \to L(z) = \int_{\mathbb{R}^{6+z}}\left[\frac{1}{2}\phi(\Delta - m^2)\phi + g\mu^z\phi^3\right]d^{6+z}x\ .$$

While dimensional regularization can be used to locally regularize QFTs on a curved background space, it does not work globally as a regularization scheme because of an ambiguity of how to distribute the imaginary dimensions over the manifold. In 1977, Hawking [16] proposed ζ-function regularization as a global regularization scheme for QFTs over a Riemannian manifold to resolve this ambiguity. Instead of changing the dimension of the manifold, the number of derivatives taken in the Lagrangian density is allowed to vary, the Laplacian on the manifold is raised to a complex power, and then terms are introduced to the Lagrangian to keep it dimensionless as before. Again, for ϕ^3 theory,

$$L_M = \int_M\left[\frac{1}{2}\phi(\Delta - m^2)\phi + \lambda\phi^3\right]\sqrt{g}d^6x \to L_M(r) = \int_M\left[\frac{1}{2}\phi(\Delta - m^2)^{1+r}\phi + \lambda\Lambda^{-2r}\phi^3\right]\sqrt{(\Lambda^{\frac{2r}{3}}g)}\,d^6x\ .$$

This section develops QFT on a curved manifold in a parallel manner to QFT in Minkowski space. The Connes-Marcolli renormalization bundle for a particular QFT in Minkowski space has a section that is a geometric representation the regularized theory. The β-function for this theory can be uniquely calculated from a connection on this bundle, pulled back over this section. In the following, I develop the mechanics of ζ-function regularization and construct the corresponding renormalization bundle of ϕ^3 theory in curved space-time under this regularization scheme. The new regularization bundle also has a section that represents the ζ-function regularization of ϕ^3. There is a connection on this bundle, which, when pulled back over this section, provides an expression for the β-function of the theory over a compact manifold. The expressions for the β-function will depend on the geometry of the manifold.

7.1 Feynman rules in configuration space

One can develop QFT on a manifold locally using coordinate patches and partitions of unity. However, QFT in flat space is global phenomenon. Therefore, I wish to express QFT in curved space-time globally. Primarily, this means that I cannot perform the Fourier transforms necessary to write the Feynman rules in terms of momenta. Instead, I work in with Feynman rules in configuration space. Feynman integrals for any theory are written in terms of a convolution product of Green's functions of a Laplacian. As a result, the Feynman integrals themselves are called Green's functions in the literature. In the following sections, I am interested in kernel formed by the convolution product, and the operator it defines. Since there is a one to one correspondence between these kernels and operators, I switch between the two points of view freely.

The Feynman integrals associated to Feynman graphs are generalized convolution products of Green's functions written in terms of the positions on the manifold. I first review the details of the Feynman rules in configuration space for a flat background with a Lorentzian metric for ϕ^3 theory in six dimensions. While this information is presented in many standard textbooks, such as [41] and [40], I summarize the results in this section. Feynman integrals in phase space are Fourier transforms of the Feynman integrals in configuration space. In phase space, the integrals are distributions on the space of test functions on the incoming and outgoing momenta of the interaction, subject to conservation of momentum. In configuration space, the test functions are the smooth functions on $\mathbb{R}^n$. Evaluated on these spaces of test functions the Feynman integrals evaluate to the same physical quantity. The Fourier transform translates between two points of view: in phase space the integrals are concerned with the momenta of the particles involved in the interaction, in configuration space the integrals are interested in the positions of the interacting particles. The Feynman graphs and rules corresponding to the two types of integrals reflect this change of perspective. In both pictures, the legs of the diagrams are labeled with momenta, and the vertices with positions. In phase space, Feynman graphs have internal legs with two end points, and external legs with one endpoint. All vertices were treated equally. The Feynman rules require imposing conservation of momentum, and integrals are taken over the momenta associated to each loop of the graph. In configuration space, all edges are treated equally, and there are internal and external vertices. External vertices have valence 1. They can be viewed as vertices attached to the ends truncated external legs from the phase space picture. Internal vertices are simply the vertices with valence > 1. Conservation of momentum is expressed in the way the convolution product is written. Each type of edge in a field theory is associated to a type of Green's function, or propagator, $G_i(x, y)$ where x and y are the endpoints of the edge. Each type of vertex, determined by the valence, n_v and the types of edges meeting at it is associated to a coupling constant g_v.

In 6 dimensional ϕ^3 theory, with a Lorentzian metric, the Feynman rules over position space are relatively simple. There is only one type of edge in the theory, associated to the Green's kernel

$$G(x, y) = \int_{\mathbb{R}^6} \frac{d^6 p}{(2\pi^6)} \frac{i e^{-ip\cdot(x-y)}}{p^2 - m^2} \tag{42}$$

of the Laplacian $\Delta - m^2$. There is only one coupling constant and the only possible valence at a vertex is 3.

As is the case for the Feynman rules in momentum space, these rules do not yeild well defined integrals. The resulting integrals need to be regularized in order to get well defined expressions. In $\mathbb{R}^n$, the Feynman rules for a theory with with multiple types edges and vertices with different valences n_v and coupling constants g_v, map between graphs and distributions, or Feynman integrals, in the following manner:

- If a graph Γ has I edges, write down the I fold product of propagators, of various types according to the type of edges,

$$\prod_1^I G_i(x_i, y_i)$$

where x_i and $y_i \in M$ are the endpoints for the i^{th} edge.

- For each internal vertex, v_i of valence n_{v_i}, define a measure

$$\mu_i = -ig_v \delta(x_1, x_2) \ldots \delta(x_{n_{v_i}-1}, x_{n_{v_i}})$$

where the x_i are the endpoints of the edges that are incident on the vertex in question in the graph.

- Integrate the product of propagators from above against this measure for each of the V internal vertices of the graph

$$\int_{(\mathbb{R}^n)^{\sum n_{v_i}}} \prod_1^I G_i(x_i, y_i) \prod_1^V d\mu_j \ .$$

- Divide by the symmetry factor of the graph.

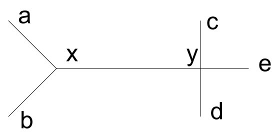
If V is the number of internal vertices of a graph Γ, the Feynman rules gives an expression that can be written as an integral over $(\mathbb{R}^n)^{\sum n_{v_i}}$ with measures $d\mu_1 \otimes \ldots \otimes d\mu_V$. The variables that are not integrated over are the external vertices. The Feynman rules are defining a convolution product of Greens functions, convolved over the internal vertices. The types of Green's kernels involved gives the types of edges of the graph. This analysis can equivalently be done in the language of symbols, as in [30].

Remark 38. Given such a convolution product of the correct type of kernels, one can determine the corresponding graph. Notice that this convolution product does not correspond to a composition of operators. In fact, this convolution product is not commutative. The order in which the convolution product is taken determines the shape of the graph.

Example 10. For example, in a scalar QFT defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(|d\phi|^2 - m^2)\phi + g_3\phi^3 + g_4\phi^4 \ ,$$

in $\mathbb{R}^4$, the tree level graph



has the Feynman integral

$$\begin{aligned} I_\Gamma(a,b,c,d,e) \quad = \quad & -g_3 g_4 \int_{(\mathbb{R}^4)^2} \int_{\mathbb{R}^4} \frac{d^4p_1}{(2\pi^4)} \frac{ie^{-ip_1\cdot(a-x)}}{p_1^2 - m^2} \int_{\mathbb{R}^4} \frac{d^4p_2}{(2\pi^4)} \frac{ie^{-ip_2\cdot(b-x)}}{p_2^2 - m^2} \int_{\mathbb{R}^4} \frac{d^4p_3}{(2\pi^4)} \frac{ie^{-ip_3\cdot(x-y)}}{p_3^2 - m^2} \\ & \cdot \int_{\mathbb{R}^4} \frac{d^4p_4}{(2\pi^4)} \frac{ie^{-ip_4\cdot(c-y)}}{p_4^2 - m^2} \int_{\mathbb{R}^4} \frac{d^4p_5}{(2\pi^4)} \frac{ie^{-ip_5\cdot(d-y)}}{p_5^2 - m^2} \int_{\mathbb{R}^4} \frac{d^4p_6}{(2\pi^4)} \frac{ie^{-ip_4\cdot(e-y)}}{p_6^2 - m^2} \, \mathrm{d}^4x \mathrm{d}^4y \ . \end{aligned}$$

In six dimensional ϕ^3 theory, the Feynman rules simplify considerably, as there is only one type of edge, and the only possible valence is $n_v = 3$. For a graph, Γ, with V internal vertices and I edges in this theory, the Feynman integral in configuration space is

$$I_\Gamma = (-ig)^V \int_{(\mathbb{R}^6)^V} \prod_1^I G_i(z_k, z_l) \, \mathrm{d}z_1 \ldots \mathrm{d}z_V \ ,$$

where $z_k, z_l \in \{z_1, \ldots, z_k\}$. Substituting the Green's function from equation (42) gives the exact form of Feynman rules for this theory. Working out the details, one see that these rules are exactly the inverse Fourier transform of the rules stated in section 2.2. The integral I_Γ has $(2I - 3V)$ free variables, corresponding to the $2I-3V$ external vertices of Γ. As a distribution I_Γ acts space of test functions $S^{2I-3V}(\hat{E})$, where $\hat{E} = C_c^\infty((\mathbb{R}^6)$ is the space of test functions in configuration space. These test functions are the external data for the graph. Notice that the test functions in configuration space are the inverse Fourier transforms of the test functions in phase space defined in section 4. Working in configuration space, there is no conservation of momentum condition to keep track of. In flat space, conservation of momentum is imposed once the appropriate Fourier transforms are taken. On a curved background, imposing conservation of momentum in trickier.

Recall from section 2.1.1 that the building blocks for the Feynman diagrams of a field theory are the 1PI diagrams, and that a renormalizable ϕ^3 theory in six space-time dimensions has only two or three external legs, and the distributions associated to the Feynman diagrams act on $S^2(\hat{E}) \oplus S^3(\hat{E})$. The results about Feynman rules from section 4 all continue to hold. Most importantly, Theorem (4.1) holds. For any $x \in \mathcal{H}$, let I_x be the unrenormalized Feynman integral associated to x. If $I_x(\sigma) < \infty$ for all appropriate $\sigma \in S(\hat{E})$, the diagram x is said to be convergent. If $I_x(\sigma)$ is not well defined for some $\sigma \in S(\hat{E})$, then the integral needs to be regularized, similar to its phase space counterparts. When the regularized Feynman integral I_x^{reg} acts on σ,

$$I_x^{\mathrm{reg}}(\sigma) : \; x \to \mathbb{C}\{\{z\}\} \; .$$

As in phase space, regularization is a linear map from $S(\hat{E})$ to $G(\mathcal{A}) = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathcal{A})$. The regularization process defines certain sections of of the bundle $K^* \to \Delta^*$ by assigning them a test function. On flat space, the Feynman rules in configuration space and phase space differ only by a Fourier transform, so it stands to reason that they correspond to the same sections of the same renormalization bundle. The goal of the rest of this paper is to develop they Feynman rules and ζ-function regularization on a general manifold, and show that the structures of the corresponding renormalization bundle are the same.

7.2 Feynman rules on a compact manifold

In curved space-time, working with a Riemannian metric, The Feynman rules assign distributions to Feynman graphs in a similar manner to the Feynman rules in flat configuration space. To do this, first define the Laplacian, Δ_M on a closed compact manifold (M, g). Then find the Green's kernel of the Laplacian on (M, g), Δ_M^{-1}.

What follows is a sketch of a derivation for the Laplacian on a manifold. For more details see [36]. Let (M, g) be a compact Riemannian manifold of dimension $n = 6$ with metric tensor g. While this argument is generalizable, I continue to work with the ϕ^3 theory in six, now curved, space-time dimensions.

Definition 38. Let $|g| = \det g$, g_{ij} be the ij^{th} entry of the metric tensor and $\partial_i = \frac{\partial}{\partial x_i}$.

In order to define integrals on a manifold, one needs to define the volume element:

Definition 39. The volume of n dimensional manifold is given by

$$\int_M \mathrm{dvol}(x) := \int_M \sqrt{|g|} dx_1 \wedge \ldots \wedge dx_n \; .$$

While the goal of this analysis is to study QFT globally on the manifold, calculations are done in coordinate patches. In local coordinates, ∇^i is the i^{th} term of the gradient on M

$$\nabla^i = g^{ij} \partial_j$$

and the divergence of a vector field X on (M, g) is defined by

$$\mathrm{div}(X) = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} X^i) \; .$$

Definition 40. Let $\Delta = \mathrm{div} \circ \nabla$, be the Laplace operator on M.

Notice that Δ is dependent on the metric g.

Lemma 7.1. *The Laplacian Δ on (M, g) can be explicitly written in local coordinates as*

$$div \circ \nabla = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j) \ .$$

Proof. This expression comes from the statement of Stoke's Theorem

$$\int_M |d\phi^2| \ \mathrm{dvol}(x) = - \int_M \phi \Delta \phi \ \mathrm{dvol}(x) \ .$$

Proof of this construction can be found in any standard differential geometry text. For example, see [36]. □

The role of the Laplacian is the same for QFT over a manifold M as over flat space. To simplify notation, I write $\Delta_M = \Delta - m^2$. Notice that this is a negative definite operator. Using this relation, the Lagrangian density of a scalar field theory on M can be written:

$$\mathcal{L} = \frac{1}{2} \phi(x) (\Delta_M) \phi(x) + \lambda \phi(x)^3 \ .$$

The Lagrangian is then just

$$\int_M \mathcal{L} \ \mathrm{dvol}(x) \ .$$

By the spectral theorem, any self-adjoint elliptic operator acting on $E_M = C^\infty(M)$, such as the Laplacian, on a compact manifold, has a discrete a spectrum of eigenvalues $\{\lambda_i\}_{i=0}^\infty$. Since $-\Delta_M$ is a positive semi-definite operator, the spectrum is non-negative, $\lambda \geq 0$. Counted with multiplicity, I arrange the eigenvalues so that $\lambda_{i+1} \geq \lambda_i$. Furthermore, there is an orthonormal eigenbasis $\{\phi_i\}$ such that

$$\int_M \phi_i(x) \phi_j(x) dx = \delta_{i,j} \ ,$$

with $\delta_{i,j}$ the Kronecker delta function. Thus I can write

$$C^\infty(M) = E_0 \oplus E_+ \tag{43}$$

where $E_0 = \ker(-\Delta_M)$ and $E_+ = \oplus_i E_i$ is the direct sum of the eigenspaces with positive eigenvalues. Notice the E_0 is a finite dimensional vector space.

The Green's kernel is a distribution on $(M \times M)$, $G_M \in (C^\infty(M \times M))^\vee$, such that

$$\Delta_M G_M(x, y) = \delta(x, y) \ . \tag{44}$$

Here $\delta(x, y)$ is the Dirac delta function. If E_0 has dimension greater than 0, then Δ^{-1} is not well defined. However, Δ has a left inverse given by he function $G_M(x, y)$, according to equation 44.

On the other hand, $\Delta_M|_{E_+}$ is invertible. By defining $G_M(x, y)$ on E_+, I can write

$$G_M(x, y) = \sum_{i=0}^\infty \frac{\phi_i(x) \phi_i(y)}{\lambda_i}$$

where $\lambda_i > 0$. This definition satisfies condition in equation (44) and is a two sided inverse of $\Delta_M|_{E_+}$. In the sequel, I call $\Delta_M = \Delta_M|_{E_+}$ and $G_M(x, y) = (\Delta_M|_{E_+})^{-1}$.

The Green's function $G_M(x, y)$ corresponds to a propagator for Feynman integrals on a compact manifold. Feynman integrals for diagrams in general correspond to convolution products of these kernels. The Schwartz kernel theorem gives a correspondence between operators and kernels.

Theorem 7.2. ***Schwartz Kernel Theorem.*** *[17] Let M and N be Riemannian manifolds. Let $f \in C_c^\infty(M)$ and $g \in C_c^\infty(N)$. Let A be a continuous linear map from $C_c^\infty(N)$ to distributions on M, $\mathcal{D}(M)$. For every such A, there is a $K \in \mathcal{D}(M \times N)$ such that*

$$\langle Af, g\rangle = K(g \otimes f) \ . \tag{45}$$

Conversely, for every $K \in \mathcal{D}(M \times N)$, there is such an A. The K is called the kernel of the linear operator A.

Example 11. The Green's function $G_M(x, y)$ is a distribution on $M \times M$. It is not well defined as a function along the diagonal of $M \times M$,

$$G(x, x) \text{ dvol}(x) = \infty \ ; \ G(x, y)|_{x \neq y} \text{ smooth}$$

The operator associated to this Green's function is

$$\Delta_M^{-1} f(x) = \int_M G_M(x, y) f(y) \text{ dvol}(y) \in C^\infty(M) \ . \tag{46}$$

which is well defined as an operator from $C^\infty(M)$ to $C^\infty(M)$.

One can write the Feynman rules over a curved manifold by replacing the rules on a flat manifold by replacing the Green's functions, G_i with $G_{M,\,i}$ and $\mathbb{R}^n$ by M. The Feynman rules on a general background manifold are:

- If a graph Γ has I edges, write down the I fold product of propagators, of various types according to the type of edges,

  $$\prod_1^m G_{M,\,i}(x_i, y_i)$$

  where x_i and $y_i \in M$ are the endpoints of each edge.
- For each internal vertex, v_i of valence n_{v_i}, define a measure on $M^{\times n_{v_i}}$ locally

  $$\mu_i = -i\lambda_v \delta(x_1, x_2) \ldots \delta(x_{n_{v_i}-1}, x_{n_{v_i}})$$

  where the x_i are the endpoints of the edges incident on the vertex in question in the graph, and $\delta(x_i, x_j)$ is the Dirac delta function. .
- Integrate the product of propagators from above against this measure

  $$\int_{(M)^{\sum n_{v_i}}} \prod_1^I G_i(x_i, y_i) \prod_i^V d\mu_i \ .$$

- Divide by the symmetry factors of the graph.

This construction runs parallel to the construction of a Feynman integral in flat configuration space. As in the previous Feynman integrals, the distribution associated to a Feynman diagram is a convolution of the kernels associated to the propagators over the internal vertices. The measures μ_i mean that the operators associated to Feynman integrals can be written in terms functions involving Tr Δ_M^{-1}. The Feynman integral defined in this way are not well defined. They need to be regularized first before they can be made sense of. The appropriate regularization scheme for field theories over compact manifolds is the topic of the next section.

7.3 Regularization on a compact manifold

In flat space, the Feynman rules need to be regularized because the integration occurrs over an unbounded domain. On a compact manifold, unlike the integrals in flat space-time, the Feynman integrals are defined over compact spaces. However the operator Δ_M^{-1} is not trace class, because

$$\mathrm{Tr}\Delta_M^{-1} = \int_M G_M(x,x)\ \mathrm{dvol}(x) = \sum_1 \frac{1}{\lambda_i} = \text{diverges} .$$

This leads to divergences in operators associated to the Feynman integrals, since the Feynman integrals are convolution product of the kernels G_M. Therefore Feynman integrals need to be regularized.

Lemma 7.3. *The operator Δ^{-1} is not trace class on a six dimensional manifold.*

Proof. Let

$$N(\lambda) = |\{\lambda_k | \sqrt{\lambda_k} \leq \lambda\}|$$

counting multiplicity. By Weyl's law for an n-dimensional closed manifold M,

$$\lim_{\lambda\to\infty} N(\lambda) = \frac{\mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)}\lambda^n + O(\lambda^n) . \tag{47}$$

Specifically, the series $\{\lambda_k\}$ grows as $k^{\frac{2}{n}}$. However, by equation (47), if

$$\dim M > 1 \Rightarrow \int_M G(x,x)\ \mathrm{dvol}(x) = \sum_i \frac{1}{\lambda_i}$$

is unbounded because $\frac{2}{n} \leq 1$. □

The regularization process should take non convergent, non-distribution valued kernels, such as the Feynman integrals to meromorphic functions over an infinitesimal punctured disk Δ^* with coefficients in $\mathcal{D}(M\times M)$. Thus, the Feynman integral should be regularized to as globally defined, distribution valued, somewhere convergent Laurent polynomials. By the Schwartz Kernel theorem, these regularized Feynman integrals, K_Γ^{reg} define regularized Feynman operators A_Γ^{reg} such that, for $f,g \in S(E_M)$, the inner product

$$\langle A_\Gamma^{Reg} f, g\rangle \in \mathbb{C}\{\{z\}\}$$

gives a somewhere convergent complex valued Laurent polynomial. The form of A_Γ^{reg} is worked out in example 12 below.

The Feynman operators A_Γ are regularized by raising the propagators, Δ_M^{-1} to complex powers. An operator raised to a power is defined on its eigenspaces as

$$\begin{aligned} Af &= \lambda f \\ A^s f &= \lambda^s f . \end{aligned}$$

Thus I can define a one parameter family of operators for $s\in\mathbb{C}$ as

$$\Delta_M^{-s} : E_M \to E_M .$$

The Green's kernel associated to Δ_M^{-s} is

$$G_M^s(x,y) = \sum_i \frac{\phi_i(x)\phi_i(y)}{\lambda_i^s}$$

If $s\notin\mathbb{Z}$, then Δ_M^{-s} is trace class.

Theorem 7.4. *[Seeley] [38] Let M be an n-dimensional compact smooth manifold, $E_M = \mathbb{C}^\infty(M)$, and*

$$\Delta^{-s} : E_M \to E_M$$

with $s \in \mathbb{C}$ be a one parameter family of operators and kernel $G^s_M(x,y)$. Then $G^s_M(x,x)$, the is meromorphic in s with simple poles at $s = k - \frac{n}{2}$, where $k \in \mathbb{Z}_{\geq 0}$.

One can calculate the poles for Tr Δ_M^{-s}. Let M be a general n dimensional Riemannian manifold. First consider the heat operator, $e^{t\Delta_M}$, with $t \in \mathbb{R}_{>0}$. It is related to the complex powers of Laplacian, Δ_M by a Mellin transform:

$$\Gamma(z+1)(-\Delta_M)^{-z-1} f(x) = \int_0^\infty \exp(t\Delta_M) f(x) t^z \, dt \, , \tag{48}$$

where $f(x) \in C^\infty(M)$, and $z \in \mathbb{C}$. By the Schwartz kernel theorem, the heat operator has a unique kernel $G(t,x,y)$. This kernel is continuous on $M \times M$, and smooth away from the diagonal. Along the diagonal, the kernel of the heat operator has the asymptotic expansion

$$G(t,x,x) \sim (4\pi)^{-n/2} \sum_{k \geq 0} u_k(x) t^{\frac{k-n}{2}} \, ,$$

for small t [36]. For large t, this expansion goes to 0 as

$$\lim_{t \to \infty} G(t,x,x) \sim e^{-ct} \, .$$

The coefficients $u_k(x)$ of the asymptotic expansion can be written as polynomials in terms of the metric dependent curvatures of (M,g). For odd k, $u_{k=2n+1} = 0$.

Plugging in the asymptotic expansion, equation (48) can be rewritten as

$$(-\mathrm{Tr}\Delta_\mathrm{M})^{-z-1} \sim \frac{1}{\Gamma(\mathrm{z}+1)} \int_\mathrm{M} \int_0^\infty (4\pi)^{-\mathrm{n}/2} \sum_{\mathrm{k} \geq 0} \mathrm{u_k(x)} \mathrm{t}^{\mathrm{z}+(\mathrm{k}-\mathrm{n})/2} \, \mathrm{dt} \, \mathrm{dvol(x)} \, .$$

Work by Wodzicki [43], Guilleman [15], Konstevich and Vichi [20] finds explicit expressions for the poles of $\mathrm{Tr}(-\Delta_\mathrm{M})^{-\mathrm{z}-1}$.

Theorem 7.5. *The poles of Tr $(-\Delta_M)^{-z-1}$ occur at $z+1 \in \{\frac{n-k}{2} \notin \mathbb{Z}_+\}$, with $k \in \mathbb{Z}_{\geq 0}$ and are simple. The residues at $s = z+1 = \frac{n-k}{2}$ is given by*

$$Res \ (\mathrm{Tr}(-\Delta_\mathrm{M})^{-\mathrm{z}-1})|_{z+1=\frac{\mathrm{n}-\mathrm{k}}{2}} = \frac{2\mathrm{u_k(x)}}{\Gamma(\frac{\mathrm{k}-\mathrm{n}}{2})}|_{\mathrm{t}=1} \, .$$

Specifically, for $z = 0$ and $n = 6$,

$$\mathrm{Res} \ (\mathrm{Tr}(-\Delta_\mathrm{M})^{-1})|_{\mathrm{k}=4} = \frac{2\mathrm{u_4(x)}}{\Gamma(1)}|_{\mathrm{t}=1} \, .$$

By expressing the other poles of $\mathrm{Tr}(-\Delta_\mathrm{M})^{-z-1}$ as Taylor series around $z = 0$, one can write $\mathrm{Tr}(-\Delta_\mathrm{M})^{-z-1}$ as a Laurent polynomial in z with a simple pole at $z = 0$.

Given this regularization scheme, one can begin to make sense of the regularized Feynman integrals globally on a manifold M.

Proposition 7.6. *The regularized Feynman rules for a graph Γ is a Schwartz kernel $K_\Gamma^{reg} \in \mathcal{D}(M^{E_\Gamma})\{\{z\}\}$ that can be written as a somewhere convergent Laurent polynomial with finite poles at 0 and distribution valued coefficients.*

Proof. The regularized Feynman integral for a graph Γ, is a covolution product of $G^{z+1}(x,y)$ taken over the internal vertices of Γ for a small complex parameter z. Along the diagonal, $G^{1+z}(x,x)$ is meromorphic in z

$$G^{1+z}(x,x) = \sum_{-1}^{\infty} g_i(x) z^i \ ,$$

where $g_i(x)$ is a smooth function over M. Away from the diagonal, $G^{1+z}(x,y)$ is entire in z,

$$G^{1+z}(x,y) = \sum_{0}^{\infty} h_i(x,y) z^i \ ,$$

where $h_i(x,y)$ is a smooth function over M. Therefore, I can write $G^{1+z}(x,y) = \sum_{-1}^{\infty} f_i(x,y) z^i$, where $f_i(x,y)$ are distibution valued coefficients.

The Schwartz kernel associated to a Feynman integral K_Γ^{reg} is a convolution product of some number of these Greens functions. For a fixed $z \neq 0$, the kernel of $K_\Gamma^{reg}(z) \in \mathcal{D}(M^{\times E_\Gamma})$, where E_Γ is the number of external legs of Γ, is a smooth well defined quantity. It can be written as a Laurent polynomial with a finite number of poles, since each Green's function contributes at most one pole. $\square$

For 1PI graphs, the external data comes in the form of test functions on two or three copies of the manifold. Since M is compact, the space of test functions are $E_M = C^\infty(M)$, and the Feynman integrals are distributions acting on the symmetric algebra of these test functions, $S(E_M)$. Specifically, $K_\Gamma^{reg} \in S^{E_\Gamma \vee}(E_M) \otimes \mathbb{C}\{\{z\}\}$.

Corrollary 7.7. *The operator, $A_\Gamma(z)$ associated to K_Γ^{reg} can be written as a somewhere convergent Laurent polynomial with finite poles at 0 with operator coefficients.*

This gives a globally well defined statement of the Feynman rules on a curved background. Paycha and Scott have a local expression for the symbols associated to the operators associated to Feynman graphs in [31].

There is a Hopf algebra associated to the Feynman rules on a general manifold.

Definition 41. Let

$$\mathcal{H} = \mathbb{C}[\text{1PI graphs on } M]$$

be the Hopf algebra generated by 1PI graphs on the manifold M with external vertices and the same product and co-product structure as in Section 3.

Since there are local diffeomorphisms from M to $\mathbb{R}^6$, the conditions that define a renormalizable theory do not change. Section 2.1.1 shows that the 1PI diagrams for a renormalizable ϕ^3 QFT over a six dimensional space have two or three external legs. Therefore, Corrollary 7.7 provides another way of proving Theorem 4.1. This is worked out in the following example.

Remark 39. By equation (46), one can ignore the propagators associated to external legs and define the Schwartz kernel of the Feynman integral as the product of propagators associated to internal edges. Notice that this construction can be simplified to parallel the construction in flat phase space.

Example 12. Let K_Γ^{reg} be the Feynman integral for a 1PI graph Γ. By the Feynman rules, this is written

$$K_\Gamma^{reg}(z)(x_1, \dots x_{E_\Gamma}) = \int_{M^V} \prod_1^I G_i^{z+1}(x_{i_1}, x_{i_2}) \ \mathrm{dvol}(x_1, \dots, x_V)$$

with I the number of internal legs of Γ, E_Γ the number of external vertices, V the number of internal vertices, and $x_{i_1}, x_{i_2} \in \{x_1, \dots, x_{V+E_\Gamma}\}$. The integral is taken over the internal vertices of the diagram. This is a one parameter family of distributions on $M^{\times E_\Gamma}$. The associated one parameter family of operators $A^{reg}_\Gamma(z)$ of the Feynman graph Γ gives the map

$$A^{reg}_\Gamma(z) : (M^{\times i}, g^{\otimes i}) \to (M^{\times j}, g^{\otimes j}) ,$$

for a fixed $z \neq 0$, where $i + j = E_\Gamma$. For the Feynman diagrams, equation (45) can be written:

$$\langle A^{reg}_\Gamma(z) f, g\rangle = \int_{M^j} A^{reg}_\Gamma(z)(f(x_1, \dots, x_i))g(x_1, \dots, x_j) \text{ dvol}(x_1, \dots, x_j) =$$

$$\int_{M^{E_\Gamma}} K^{reg}_\Gamma(z)(x_1, \dots, x_i, y_1 \dots y_j)g(y_1, \dots, y_j)f(x_1, \dots, x_i) \text{ dvol}(x_1, \dots x_j, y_1 \dots, y_j)$$

for $f \in S^i(E_M)$ and $g \in S^j(E_M)$. This inner product gives a Laurent polynomial in z, $\langle A^{\text{ren}}_\Gamma f, g\rangle \in \mathbb{C}\{\{z\}\}$.

Remark 40. The calculations for the poles of $G^{z+1}(x, x)$ have all been done locally. However, Theorem 7.4 defines the meromorphic function associated to Tr Δ_M^{-z-1} in terms of global functions on M. Therefore these calculations can be extended to a global regularization scheme of a QFT over a compact curved background space-time.

Having established z is the regularization parameter, one needs to check the effect of this regularization parameter on the Lagrangian for the field theory. As on a flat background, one wants the Lagrangian

$$\int_M \left[-\frac{1}{2}\phi(x)\Delta_M\phi(x) + \lambda\phi(x)^3\right] \text{ dvol}(x)$$

to remain dimensionless under regularization.

Using the convention where $c = \hbar = 1$, and the notation $[x]$ = length units of the physical quantity x, give the following unit identifications 1 = [length] = [mass]$^{-1}$ = [energy]$^{-1}$. Thus

$$\begin{array}{cc} [\phi(x)] & = -2 \\ [\Delta_M] & = -2 \\ [\lambda] & = 0 \\ [\text{ dvol}] & = 6 \ . \end{array} \tag{49}$$

The conformal dimension of the Laplacian raised to a power is $\left[\Delta_M^{1+z}\right] = -2(1 + z)$. This induces a scaling of the regularized Lagrangian by a scale factor Λ^{2z} where $[\Lambda] = 1$,

$$L(z) = \int_M \left[\frac{1}{2}\phi(x)(-\Delta)^{1+z}_M\phi(x) + \lambda\Lambda^{-2z}\phi(x)^3\right] \Lambda^{2z} \text{ dvol}(x) \ . \tag{50}$$

The term Λ^{2z} dvol$(x) = \Lambda^{2z}\sqrt{|g|}d^6x$ corresponds to a scaling of the metric, for instance

$$g \to \Lambda^{\frac{2z}{3}} g \ .$$

The renormalization mass parameter is given by Λ^{2z} under ζ-function regularization.

Remark 41. This scaling factor Λ plays the role of the renormalization mass in dimensional regularization. Notice that sign of the exponent of λ is opposite the sign of the exponent in dimensional regularization. This is because λ is actually a length scale and not a mass scale as previosly defined.

Now I can regularize a Feynman integral.

Theorem 7.8. *Let A_Γ be the operator associated to the Feynman diagram Γ. Then, for $f \in C^\infty(M^{\times i})$ and $h \in C^\infty(M^{\times j})$, wit $i + j = E_\Gamma$, $\langle A^{reg}_\Gamma(z)f, h\rangle$ depends only on the metric g of M, and the combinatorics of the graph Γ.*

Proof. Let E_Γ be the number of external vertices of the graph Γ, and V the total number of vertices. It is sufficient to regularize $\langle A_\Gamma \phi_e, \phi_f\rangle$ where $\phi_e = \prod_{i=1}^{j} \phi_{e_i}(x_{e_i})$, $\phi_f = \prod_{j}^{E_\Gamma} \phi_{f_i}(x_{f_i})$ are products of eigenfunctions of Δ_M. These functions correspond to the external data at vertex e_i or f_i. Write

$$\langle A_\Gamma \phi_e, \phi_f\rangle = \int_{M^V} \phi_e \phi_f \prod_{i=1}^{I} \sum_{k=0}^{\infty} \frac{\phi_{k_i}(x_{i_1})\phi_{k_i}(x_{i_2})}{\lambda_{k_i}} \,\mathrm{dvol}(x_1, \ldots, x_V)\ .$$

This can be regularized using the Mellin transform

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(1+s)} \int_0^\infty t^s e^{-t\lambda} dt$$

then

$$\langle A_\Gamma(z)\phi_e, \phi_f\rangle = \frac{1}{\Gamma(1+z)^I} \int_{M^V} \phi_e\phi_f \sum_{k_1 \ldots k_I = 0}^{\infty} \phi_{k_i}(x_{i_1})\phi_{k_i}(x_{i_2}) \int_0^\infty e^{-\sum_i t_i \lambda_{k_i}} \prod_{i=1}^{I} t_i^z dt_i \,\mathrm{dvol}(x_1, \ldots, x_V)\ .$$

Conservation of momentum is applied at each trivalent vertex by the relation

$$\int \sum_{i,j,k} \phi_i(y)\phi_j(y)\phi_k(y) dy = \int \sum_{i,j,k} \phi_i(y) \sum_l a_l^{jk} \phi_l(y) dy = \sum_{j,k,i} a_i^{jk}\ .$$

Since the quantity a_i^{jk} is symmetric on i, j, and k I write it instead as $a(i,j,k)$. The quantity $a(i,\, j,\, k)$ is tensorial, and depends only on the metric of M. Define a function $\varepsilon'(v)$ that gives the set of edges incident on the vertex v. Applying conservation of momentum gives

$$\langle A_\Gamma(z)\phi_e, \phi_f\rangle = \frac{1}{\Gamma(1+z)^I} \int_{M^V} \int_0^\infty \sum_{k_1 \ldots k_I = 0}^{\infty} e^{-\sum_i t_i \lambda_{k_i}} \prod_{i=1}^{I} t_i^z dt_i \prod_{v=1}^{V-E_\Gamma} a(\varepsilon'(v)) \,\mathrm{dvol}(x_1, \ldots, x_V)\ .$$

Working out the $a(\varepsilon'(v)$s re-indexes the eigenvalues in terms of the graphs loop number, L and loop indices, l_i. From here, one can apply the Schwinger trick, and carry out calculations in a manner similar to [18], Chapter 6. The operator $A_\Gamma(z)$ is a convolution product of the Δ_M^{-1}, twisted by the quantities $a(\varepsilon'(v))$. Since the trace of Δ_M^{-s} and $a(\varepsilon'(v))$ depend only on the metric of M, so does $\langle A_{M,\Gamma}\phi_e, \phi_f\rangle$. □

Remark 42. If M is a flat manifold, then

$$a(i,\, j,\, k) = \begin{cases} 1 & \text{if } i+j+k = 0, \\ 0 & \text{else.} \end{cases}$$

In this case, $a(i,\, j,\, k) = \delta(i+j+k)$, imposes conservation of momentum at each vertex.

Corrollary 7.9. *There is a graph polynomial associated to each Feynman graph on M. The terms of the polynomial are similar to the terms found in [18], chapter 6. The coefficients of the terms, however, are functions of $a(i,\, j,\, k)$.*

7.4 The renormalization bundle for ζ-function regularization

Given a Lagrangian, $\mathcal{L}$, on a general manifold M, and a Hopf algebra $\mathcal{H}_\mathcal{L} = \mathbb{C}[\{$1PI Feynman graphs of $\mathcal{L}\}]$, one can construct a renormalization bundle for ζ-function regularization.

Define the affine group scheme $G_\mathcal{L} = \mathrm{Spec}\ \mathcal{H}_\mathcal{L}$, with Lie algebra $\mathfrak{g}_\mathcal{L}$. The group of algebra homomorphisms $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}_\mathcal{L}, \mathcal{A})$ is written $G_\mathcal{L}(\mathcal{A})$, where $\mathcal{A} = \mathbb{C}\{\{z\}\}$ as before. Let Δ be the complex infinitesimal disk around the origin. The regulator z is in this space, defined by the complex power to which the Laplacian Δ_M in the Lagrangian is raised.

Remark 43. The symbol Δ stands for both the infinitesimal disk and the Lagrangian. The meaning should be clear from context.

Remark 44. Notice that over a flat background space, ζ-function regularization and dimensional regularization are equivalent. They can be represented as sections of the same renormalization bundle. Direct calculation shows that they differ only by a holomorphic function. Therefore $\gamma_{\text{dim reg}} \sim \gamma_{\zeta \text{ reg}}$ under the $G(\mathcal{A}_+)$ gauge equivalence introduced in section 6.

Notice that by Theorem 4.1,

$$\text{regularization} \in \text{Hom}_{\text{vect}}(S(\hat{E}), G_{\mathcal{L}}(\mathcal{A})) .$$

That is, the regularization process maps from the the background manifold M to the group underlying Feynman graphs $G(\mathcal{A})$. This corresponds to sections of the bundle

$$\begin{tikzcd} K \times M \arrow[d] \\ \Delta \times M \arrow[d] \\ M \end{tikzcd} \tag{51}$$

since the operators associated to the Feynman graphs depend on the metric g on M, and thus the position in M.

Remark 45. In previous sections, I work over flat background. There, as with any constant metric, the operators associated to the Feynman integrals are independent the position in M, and this product with M could be ignored. For non-constant metrics, it needs to be introduced.

Theorem 7.10. *The renormalization bundle for dimensional regularization is a special case of the renormalization bundle for ζ-function regularization over a general background manifold. Sections of the renormalization bundle in curved space-times depend on position over M.*

$$P^* \times M \to B^* \times M \to \Delta^* \times M \to M .$$

By Theorem 7.10 all the analysis on the ζ-function renormalization bundle parallels the analysis of the Connes-Marcolli renormalization bundle. The counterterms of a theory under ζ function regularization are conformally invariant [13]. That is, varying the length scale of the manifold, the counterterms stay constant. If $\gamma_{\mathcal{L}}$ is one of the sections defined by regularization,

$$\frac{d}{dt}(\gamma_{\mathcal{L}t^z})_- = 0 .$$

That is, $\gamma_{\mathcal{L}} \in G^{\Phi}(\mathcal{A})$. One can calculate $\beta(\gamma_{\mathcal{L}})$.

Proposition 7.11. *The function $\beta(\gamma_{\mathcal{L}})$ is given by $z\tilde{R}(\gamma_{\mathcal{L}})$. Since the physically interesting action of the renormalization group on $\gamma_{\mathcal{L}}$ is given by $t^Y\gamma_{\mathcal{L}}$, $\beta(\gamma_{\mathcal{L}}) = z\tilde{R}(\gamma_{\mathcal{L}-})$. This is completely determined by the counterterm, $\gamma_{\mathcal{L}-}$.*

Proof. This is a direct application of Theorem 5.18. □

For $\pi : \mathcal{A} \to \mathcal{A}_-$, and a primitive 1PI graph $x \in \mathcal{H}$, the counterterm for $\gamma_{\mathcal{L},f}$ is

$$\gamma_{\mathcal{L}\, f-} = \pi(A_x(f)) .$$

By Theorem 7.8, this depends only on the metric of M and the structure of x. In fact, for any $x \in \ker(\varepsilon)$, the counterterm is written

$$\gamma_{\mathcal{L},f\,-}(x)(z) = \sum_{-n}^{-1} a_i(f) z^i \ .$$

All the coefficients depend only on x and the metric g. Specifically,the β-function of this theory, $\beta(\gamma_{\mathcal{L}}) = \mathrm{Res}\,(\gamma_{\mathcal{L}\,-})$ depends only on the metric on M.

7.5 Non-constant conformal changes to the metric

Finally, I can extend this analysis to ζ-function regularization under a non-constant regularization mass parameter. To do this, I need to view the scalar fields as densities on a manifold M and adjust the Laplacian by adding a suitable multiple of the scalar curvature.

7.5.1 Densities

While compactness and orientability are not necessary for the arguments of the following sections, I will keep with the conventions of the previous sections and let M be a smooth, compact, oriented Riemannian n-manifold. It has a principal $\mathrm{Gl}_n(\mathbb{R})$-bundle $\mathrm{Frame}(M)$ of frames, whose fibers $\mathrm{Frame}_x(M)$ are ordered bases $\{v_1, \ldots, v_n\}$ for the tangent space T_xM. The structure group $\mathrm{Gl}_n(\mathbb{R})$ acts freely and transitively on the fibers by rotating the frames.

A representation $\rho : \mathrm{Gl}_n(\mathbb{R}) \to \mathrm{Aut}(V)$ of the structure group as automorphisms of a vector space V defines a vector representation

$$V \times_{\mathrm{Gl}_n(\mathbb{R})} \mathrm{Frame}(M) \to M$$

over M. For a general (not necessarily orientable) manifold, M, the determinant of the structure group defines a bundle

$$\det : \mathrm{Gl}_n(\mathbb{R}) \to \mathbb{R}^\times = \mathrm{Gl}_1(\mathbb{R}) \ .$$

Definition 42. For an orientable manifold and for any $r \in \mathbb{R}$ the representation

$$|\det|^{r/n} : \mathrm{Gl}_n(\mathbb{R}) \to \mathbb{R}_+^\times$$

defines a line bundle which I denote $\mathbb{R}(r)$.

The sections of the bundle $\mathbb{R}(r) \to M$ are called r densities on M. The bundle can be trivialized by choosing a metric, g, for M. Let ϕ be a section of $\mathbb{R}(r) \to M$. Given a choice of g, it can be written uniquely as

$$\phi = f|g|^{\frac{r}{2n}}$$

for some $f \in C^\infty(M)$.

Example 13. The canonically trivial bundle $\mathbb{R} \times M$ is naturally isomorphic to $\mathbb{R}(0)$. That is, there is an isomorphism on the space of sections of the trivial line bundle

$$\Gamma(M, \mathbb{R} \times M) \simeq C^\infty(M) \ .$$

Sections of $\mathbb{R}(n)$ can be trivialized by a choice of g and written

$$\phi(x) = f|g|^{\frac{1}{2}} \quad ; \quad f \in C^\infty(M)$$

as defined above. Similarly, the bundle $\Lambda^n T^*(M)$ of n-forms is naturally isomorphic to $\mathbb{R}(n)$. The space of sections

$$\Gamma(M, \mathbb{R}(n)) = \{f\, dx_1 \wedge \ldots \wedge dx_n | f \in C^\infty\} \simeq \mathbb{C}^\infty(\Lambda^n T^*(M)) = \Omega^n \ .$$

Example 14. Lebesgue theory canonically assigns an integral

$$\omega \mapsto \int_M \omega \ : \ \Omega^n(M) \to \mathbb{R}$$

to a smooth section of the bundle of forms, and hence to a measurable section of $\mathbb{R}(n)$. A Riemannian metric g on M defines a nowhere-vanishing n-form

$$\mathrm{dvol}(g) = |g|^{1/2} dx_1 \wedge \cdots \wedge dx_n \ .$$

This allows one to integrate functions (0-forms) on M:

$$f \mapsto \int_M f \ \mathrm{dvol}(g) \ .$$

Remark 46. In section 7.3, where g is fixed, I use the notation $\mathrm{dvol}(x)$ to emphasize the dependence of the volume element on the variable x. In this section, where the metric is not fixed, I use the notation $\mathrm{dvol}(g)$ to emphasize the dependence of the volume element on the metric g.

If ϕ is a continuous section of $\mathbb{R}(r)$, then for any $s > 0$, $|\phi|^s$ is a continuous section of $\mathbb{R}(rs)$. In particular, when $n \geq r > 0$,

$$||\phi||^{n/r} := \int_M |\phi|^{n/r}$$

defines an analog of a classical Banach norm. This becomes apparent under a trivialization

$$||\phi||^{n/r} = \int_M (|f||g|^{\frac{r}{2n}})^{n/r} = \int |f|^{n/r} \ \mathrm{dvol}(g) \ .$$

When $r = 0$, the norm is given by the classical essential supremum. I can now consider sections of $\mathbb{R}(r)$, and write $\mathbb{L}(r)$ for the Lebesgue space of r densities, with these norms. In this terminology, n-forms become n-densities, the Banach space dual of $\mathbb{L}(d)$ is $\mathbb{L}(n-d)$, and Hölder's inequality becomes the assertion that the point-wise product

$$\mathbb{L}(d_0) \otimes \mathbb{L}(d_1) \to \mathbb{L}(d_0 + d_1)$$

is continuous with respect to the natural norms.

Notice that sections of $\mathbb{R}(\frac{n}{2})$ define a Hilbert space $\mathbb{L}(\frac{n}{2})$ with inner product

$$\langle \phi, \psi \rangle = \int_M \phi\psi \ .$$

This inner product is independent of the Riemannian metric. A choice of g defines an isometry with the classical Lebesgue space $L^2(M, g)$. Let $\phi = f|g|^{\frac{1}{2}}$ and $\psi = h|g|^{\frac{1}{2}}$. The inner product is

$$\langle \phi, \psi \rangle_g = \int_M f|g|^{\frac{1}{4}} h|g|^{\frac{1}{4}} dx_1 \wedge \ldots \wedge dx_n \ .$$

Finally, there is a linear operator

$$\phi \mapsto |g|^{\frac{d_1 - d_0}{2n}} \phi \tag{52}$$

that maps smooth sections of density d_0 to those of density d_1. When $d_1 \geq d_0$ it defines a continuous linear map from $\mathbb{L}(d_0)$ to $\mathbb{L}(d_1)$.

7.5.2 Effect of conformal changes on the Lagrangian

I can use this formalism to study how the Lagrangian varies under conformal changes to the metric

$$g \to e^{f(x)} g \quad ; \quad f(x) \in C^\infty(M) \ .$$

For ease of notation, let $u = e^f$. The Lagrangian density for renormalizable scalar field theory on an n-dimensional Riemannian is given by

$$\mathcal{L} = \frac{1}{2}\phi(x)(-\Delta_M)\phi(x) + \lambda\phi^{\frac{2n}{n-2}}(x) \ ,$$

where I consider ϕ, m and λ to be densities of different weights. Notice that ϕ is raised to an integral power only when $n \in \{3, 4, 6\}$.

To emphasize the Laplacian's dependence on the metric g on M, write

$$\Delta_g = \mathrm{div} \circ \nabla = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j) \ .$$

Yamabe's theorem [44] states:

Theorem 7.12. [Yamabe] *Let $\phi \in C^\infty(M)$, and let g be a metric on M. Then the quantity*

$$\int_M \phi \left[-\Delta_g + \frac{1}{4}\frac{n-2}{n-1}R(g)\right] \phi \cdot \ dvol(g) \tag{53}$$

is invariant under the conformal rescaling $g \mapsto \bar{g} = e^{2f(x)}g$, $\phi \mapsto \bar{\phi} = e^{\frac{n-2}{2}f}\phi$, where $f \in C^\infty(M)$.

Proof. The proof uses Yamabe's identity [44]

$$u^2 R(\bar{g}) = R(g) - 4\frac{n-1}{n-2}u^{-(\frac{n}{2}-1)}\Delta_g u^{\frac{n}{2}-1} \ .$$

It is perhaps most easily verified by checking that

$$\int_M [|\bar{\phi}|^2_{\bar{g}} + \frac{1}{4}\frac{n-2}{n-1}R(\bar{g})\bar{\phi}^2] \cdot \ \mathrm{dvol}(\bar{g}) = \int_M [|d\phi|^2_g + \frac{1}{4}\frac{n-2}{n-1}R(g)\phi^2] \cdot \ \mathrm{dvol}(g) \ ,$$

where $f \in C^\infty(M)$ and the inner product of the one form $d\phi$ with itself is

$$\int_M |d\phi|^2_g \ \mathrm{dvol}(g) = \int_M -\phi\Delta_g\phi \ \mathrm{dvol}(g)$$

by integration by parts. $\square$

Notice that I choose a metric in the above proof. Instead of choosing a metric, I can interpret Theorem 7.12 as a statement about invariant densities ϕ of some weight r. To determine the weight of ϕ, notice that the function $\bar{\phi}$ is invariant under the conformal scaling $g \mapsto u^2 g$. Therefore ϕ is a $(\frac{n-2}{2})$ - density trivialized by a choice of Riemannian metric so that

$$\phi = (|g|^{\frac{1}{2}})^{\frac{(n-2)}{2n}} h \ .$$

Then equation (53) is a linear map from $\mathbb{L}(\frac{n-2}{2}) \to \mathbb{R}$.

Definition 43. For ease of notation, define the conformally invariant Laplacian

$$\Delta_{[g]} = \Delta_g - \frac{1}{4}\frac{n-2}{n-1}R(g)\ .$$

As an operator

$$\Delta_{[g]}\ :\ \mathbb{L}(\frac{n-2}{2}) \to \mathbb{L}(\frac{n+2}{2})$$

it is a quadratic form on $\mathbb{L}(\frac{n-2}{2})$. The $[g]$ subscript indicates that the Laplace operator depends only on the conformal equivalence class of g.

The Lagrangian for a renormalizable scalar theory in terms of this conformally invariant operator $\Delta_{[g]}$ and densities ϕ is

$$L = \int_M \phi(-\Delta_{[g]} + m^2)\phi + \lambda\phi^{\frac{2n}{n-2}}$$

where m is a 1-density, λ is a (0)-density, and M is a 6-manifold. The free part of this Lagrangian

$$L_F = \int_M \phi(-\Delta_{[g]} + m^2)\phi$$

is invariant under the transformations $g \to u^2 g$.

Theorem 7.13. *There is a meromorphic family of quadratic forms on $(\frac{n-2}{2})$-densities,*

$$\tilde{Y}_g(r) = |g|^{\frac{1}{2n}} Y_g^{1+r} |g|^{\frac{1}{2n}}\ ,$$

that defines the self-adjoint operator that represents the free term in the Lagrangian of a scalar field theory.

Proof. In order to carry out the arguments from section 7.3, $-\Delta_{[g]} + m^2$ must be a self-adjoint operator acting on the Hilbert space $\mathbb{L}(\frac{n}{2})$. By equation (52),

$$|g|^{\frac{1}{2n}}\phi \in \mathbb{L}(\frac{n}{2})\ .$$

Rewrite

$$L_F = \int_M \phi|g|^{\frac{1}{2n}}|g|^{\frac{-1}{2n}}(-\Delta_{[g]} + m^2)|g|^{\frac{-1}{2n}}|g|^{\frac{1}{2n}}\phi\ .$$

Now I can define an operator

$$Y_g := |g|^{-\frac{1}{2n}}(-\Delta_{[g]} + m^2)|g|^{-\frac{1}{2n}}$$

that acts on the Hilbert space $\mathbb{L}(\frac{n}{2})$.

Proceeding as before, I raise Y_g to a complex power. This gives a family of Lagrangians

$$L(z,g) = \int_M \phi|g|^{\frac{1}{2n}} Y_g^{1+z} |g|^{\frac{1}{2n}}\phi\ .$$

As before, $\mathrm{Tr}\, \mathrm{Y}_{\mathrm{g}}^{1+\mathrm{z}}$ has simple poles in z. Following the same arguments as in Corrollary 7.7, I can expand around $z = 0$ and write this as a Laurent series

$$Y_g^{1+z} = \sum_{i=-1}^{\infty} a_i z^i \tag{54}$$

where a_i are operators.

However, since $\phi \in \mathbb{L}(\frac{n-2}{2})$, the self-adjoint operator in the Lagrangian must be a quadratic form on $\mathbb{L}(\frac{n-2}{2})$. To define such an operator, use equation (52) to get

$$\tilde{Y}_g(z) = |g|^{\frac{1}{2n}} Y_g^{1+z} |g|^{\frac{1}{2n}} \ .$$

□

Remark 47. Notice that by raising Y_g to a complex power, the expression $\phi\tilde{Y}_g(z)\phi$ is now a $n-2z$ density.

Under the conformal change of metric

$$g \mapsto u^2 g = \bar{g}$$

Y_g transforms as

$$Y_g \to u^{-1} Y_g u^{-1} = Y_{\bar{g}}$$

The operator $\tilde{Y}_{\bar{g}}(z)$ can be written,

$$\tilde{Y}_{\bar{g}}(z) = |g|^{\frac{1}{2n}} u \left(u^{-1} Y_g u^{-1}\right)^{1+z} u |g|^{\frac{1}{2n}} \ . \tag{55}$$

The kernel of this operator is defined by a family of pseudo-differential operators with top symbol

$$\xi \mapsto |\xi|_g^{2+2z} \ .$$

Remark 48. When u is constant, equation (55) corresponds exactly to equation (50).

Proposition 7.14. *For a general $u = e^f$, $\tilde{Y}_{\bar{g}}$ can be expanded as a Taylor series in f as*

$$\tilde{Y}_{\bar{g}}(f, z) = e^{-2fz} \tilde{Y}_g(z) \ .$$

Proof. Recall that $u = e^{f(x)}$. The terms of the Taylor series of $\tilde{Y}_{\bar{g}}(r)$at $f = 0$ are given as follows

order 0 The 0^{th} order term is given by evaluating $\tilde{Y}_{\bar{g}}(r)$ at $f = 0$. This gives

$$|g|^{\frac{1}{2n}} Y_g^{1+r} |g|^{\frac{1}{2n}} = \tilde{Y}_g(r)$$

order f Taking the derivative of $\tilde{Y}_{\bar{g}}(r)$ in terms of f gives

$$\left[2e^f |g|^{\frac{1}{2n}} (e^{-f} Y_g e^{-f})^{1+r} |g|^{\frac{1}{2n}} e^f + e^f |g|^{\frac{1}{2n}} (1+r)(e^{-f} Y_g e^{-f})^r (-2e^{-f} Y_g e^{-f}) |g|^{\frac{1}{2n}} e^f\right] \ . \tag{56}$$

To simplify matters later, write

$$(e^{-f} Y_g e^{-f})^r (-2e^{-f} Y_g e^{-f}) = -2(e^{-f} Y_g e^{-f})^{r+1} \ .$$

Evaluating (56) at $f = 0$ gives

$$2\tilde{Y}_g(r) - 2(1+r)\tilde{Y}_g(r) \ .$$

order f^2 Taking the derivative of (56)

$$\begin{aligned}&[4e^f|g|^{\frac{1}{2n}}(e^{-f}Y_g e^{-f})^{1+r}|g|^{\frac{1}{2n}}e^f] - 4e^f|g|^{\frac{1}{2n}}(1+r)(e^{-f}Y_g e^{-f})^{r+1}|g|^{\frac{1}{2n}}e^f \\ &-4e^f|g|^{\frac{1}{2n}}(1+r)(e^{-f}Y_g e^{-f})^{r+1}|g|^{\frac{1}{2n}}e^f + 4e^f|g|^{\frac{1}{2n}}(1+r)^2(e^{-f}Y_g e^{-f})^{r+1}|g|^{\frac{1}{2n}}e^f]\ .\end{aligned} \tag{57}$$

The first two terms come from the derivative of the first term of (56). The third comes from taking the derivatives of outer terms of the second term of (56), and the fourth term comes from the derivative of the middle term of the second term of (56).

Evaluating (57) at $f = 0$ gives

$$4\tilde{Y}_g(r) - 8(r+1)\tilde{Y}_g(r) + 4(r+1)^2\tilde{Y}_g(r)\ .$$

This simplifies to

$$2^2(1-(r+1))^2\tilde{Y}_g(r)\ .$$

order f^n Continuing along these lines shows that the n^{th} derivative of $\tilde{Y}_{\bar{g}}(r)$ with respect to f, evaluated at $f = 0$ is

$$2^n(1-(r+1))^n\tilde{Y}_g(r) = 2^n(-r)^n\tilde{Y}_g(r)\ .$$

Writing this out as a Taylor expansion

$$\tilde{Y}_{\bar{g}}(r) = \sum_{n=0}^{\infty}\frac{(-2rf)^n}{n!}\tilde{Y}_g(r) = (\sum_{n=0}^{\infty}\frac{(-2rf)^n}{n!})\tilde{Y}_g(r) = e^{-2fr}\tilde{Y}_g(r)\ .$$

□

Remark 49. This construction defines a family of effective Lagrangians

$$\mathcal{L}(r,[g]) = \int_M u^{-2r}\left[\phi\tilde{Y}_g(r)\phi + u^{2r}\lambda\phi^{\frac{2n}{n-2}}\right]\ ,$$

where $[g]$ denotes a conformal class of metric. This is the natural analog, in a flat background, of the classical family of effective Lagrangians

$$L(r) = \int_{\mathbb{R}^6}\left[\phi(-\Delta+m^2)^{1+r}\phi + \lambda\Lambda^{-2r}\phi^{\frac{2n}{n-2}}\right]\Lambda^{2r}\,d^6x\ ,$$

with $\tilde{Y}_g(r)$ corresponding to $(-\Delta+m^2)^{1+r}$. In the first display, ϕ is a $\frac{n-2}{2}$ density. In the second display, ϕ is a function. This Lagrangian is not conformally invariant when $u \notin \mathbb{C}^\times$.

Using the conformally corrected Laplacian $\tilde{Y}_g$ does not change the Feynman graphs or the divergence structure of the theory. The bundle B is now a fiber over the manifold M. Interpreting the mass parameter $u \in B$ as 1 density over M, $u = e^{f(x)} \in B \times M$ I can define a bundle $\mathcal{B}(M) = \Delta \times \mathbb{C}^\times(1) \times_{GL_n(\mathbb{C})} M$ that contains the renormalization mass. The entire renoromalization bundle can be written

$$\mathcal{P} := P \times_{GL_n(\mathbb{R})} \mathrm{Frame}(M) \to \mathcal{B}(M)$$

which is diffeomorphic, but not naturally, to the product of M with the Connes-Marcolli renormalization bundle. The action of the rescaling group $\mathbb{C}^\times$ extends to an action of the infinite dimensional group of local conformal rescalings.

The equivariant connection on $P \to B$ becomes an equivariant connection on $\mathcal{P} \to B \times_{GL_n(\mathbb{R})} M$, and the β-function becomes a map from M to $\mathfrak{g}$. I leave its explicit calculation for future work.

A Rota-Baxter algebras

As mentioned in section 4, the fact that $\gamma_-^\dagger$ and $\gamma_+^\dagger$ are both in $G(\mathcal{A})$ depends on the fact that the projection map $\pi : \mathcal{A} \to \mathcal{A}_-$ transforms $\mathcal{A}$ into a Rota-Baxter algebra. In this section I define these algebras, prove the above claim, and show how Rota-Baxter algebras can be used to generalize the regularization scheme currently under consideration. Everything in this section can be found in [9], [10], and [12].

A.1 Definition and Examples

Let k be a commutative ring, and A a (not necessarily associative) k algebra.

Definition 44. A Rota Baxter algebra of weight $\theta \in k$ is the pair (A, R) where R is a linear operator

$$R : A \to A$$

such that, for $x,\ y \in A$,

$$R(x)R(y) + \theta R(xy) = R(R(x)y) + R(xR(y)) \ .$$

Remark 50. This implies that $R(A)$ is closed under multiplication. Therefore, $R(A)$ is an algebra.

Definition 45.

1. Let $\tilde{R} = \theta - R$.
2. A Rota-Baxter ideal is an ideal $I \subset A$ such that $R(I) \subset I$.
3. Let (A, R) and (B, P) be two Rota-Baxter algebras of equal weight. A Rota-Baxter homomorphism is an algebra homomorphism such that $f :\ A \to B$ and $P \circ f = f \circ R$.

Proposition A.1. *The operator $\tilde{R} = \theta - R$ is also a Rota-Baxter operator of weight θ.*

Proof. By direct calculation

$$(\theta - R)(x)(\theta - R)(y) + \phi(\theta - R)(xy) = (\theta - R)((\theta - R)(x)y) + (\theta - R)(x(\theta - R)(y))$$

where ϕ is the weight for $(\theta - R)$. Expanding this equation shows that $\tilde{R}$ is a Rota-Baxter algebra if and only if $\phi = \theta$. ☐

Notice that by the definition, one can write $\tilde{R}(A) = A/(\ker R)$ and $R(A) = A/(\ker \tilde{R})$.

Corrollary A.2. *Writing $R(x)R(y) = R(z)$, with $z = -\theta xy + (R(x)y) - (xR(y))$ gives $\tilde{R}(x)\tilde{R}(y) = -\tilde{R}(z)$.*

Remark 51. The $\tilde{R}$ here is different from the $\tilde{R}$ defined in section 5.4.1. I keep this unfortunate nomenclature in order to be consistent with the existing literature. Only $\tilde{R} = \theta - R$ will be used in the rest of this section.

Example 15. Integration by parts Let $A = \mathrm{Cont}(\mathbb{R}, \mathbb{R})$, the algebra of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define the Rota-Baxter operator

$$I(f)(x) = \int_0^x f(t)dt \ .$$

Let $G(x) = I(g)(x)$ and $F(x) = I(f)(x)$. One sees that

$$F(x)G(x) = \int_0^x f(t)G(t)dt + \int_0^x F(t)g(t)dt \ ,$$

or that

$$I(f)(x)I(g)(x) = I(fI(g)(t))(x) + I(I(f)(t)g)(x) \ ,$$

which shows that $I(f)(x)$ is a Rota-Baxter operator of weight 0.

The case where R is an idempotent linear operator on A gives rise to a special case of Rota Baxter algebras.

Lemma A.3. *If R is an idempotent Rota Baxter operator, then so is $\tilde{R}$. Both R and $\tilde{R}$ have weight 1.*

Proof. First notice that $1-R$ is also an idempotent operator. Therefore, A can be decomposed into submodules $A = R(A) \oplus (1-R)(A)$. Therefore, one can write

$$ab = (R(a) + (1-R)(a))(R(b) + (1-R)(b)) = R(a)b + aR(b) - R(a)R(b) + (1-R)(a)(1-R)(b) \ .$$

Therefore,

$$R(ab) = R(R(a)b) + R(aR(b)) - R(R(a)R(b)) + R((1-R)(a)(1-R)(b)) = R(R(a)b) + R(aR(b)) - R(a)R(b) + 0 \ .$$

The third term comes from the fact that R is idempotent, and that $R(A)$ is a submodule. The last from the decomposition. Therefore, R has weight 1, and $\tilde{R}$ is also idempotent, and the decomposition is actually a subalgebra decomposition. □

In general, $R(A)$ and $\tilde{R}(A)$ are always subalgebras of A. Lemma A.3 shows that if R is idempotent, one can decompose $A = R(A) \oplus \tilde{R}(A)$. I use the convention that if A is unital $R(1) = 0$.

Definition 46. Define a new product in A as

$$x *_R y = R(x)y + xR(y) - \theta xy$$

for x, $y \in A$. This gives a new algebra $A' = (A, *_R)$. If this has a Rota-Baxter operator R' associated to it, one can build up a hierarchy of Rota-Baxter algebras and operators.

Lemma A.4. *If A is an associative Rota-Baxter algebra, then so is $(A, *_R)$ and*

$$x_1 *_R \ldots *_R x_n = \frac{1}{\theta}(\prod_1^n R(x_i) - (-1)^n \prod_1^n \tilde{R}(x_i)) \ .$$

Proof. The lemma follows by induction. The $n = 1$ case is obvious. For $n = 2$, first notice that by the definition of this product,

$$R(x *_R y) = R(x)R(y) \qquad \text{and} \qquad \tilde{R}(x *_R y) = -\tilde{R}(x)\tilde{R}(y) \ . \tag{58}$$

It follows that

$$x *_R y = \frac{1}{\theta}(R(x)R(y) - \tilde{R}(x)\tilde{R}(y)) \ .$$

For $n > 2$, let $z = x_1 *_R \ldots *_R x_{n-1}$. By induction on equation (58), one has that

$$R(z) = \prod_1^{n-1} R(x_i) \qquad \text{and} \qquad \tilde{R}(z) = (-1)^{n-1} \prod_1^{n-1} \tilde{R}(x_i) \ .$$

Then

$$z *_R x_n = \frac{1}{\theta}(R(z)R(y) - \tilde{R}(z)\tilde{R}(y)) = \frac{1}{\theta}\left(\prod_1^n R(x_i) - (-1)^n \prod_1^n \tilde{R}(x_i)\right) \ .$$

□

Definition 47. Given a Lie algebra, A, with the bracket $[x, y] = xy - yx$, one has a Rota-Baxter Lie algebra

$$[R(x), R(y)] + \theta R[x, y] = R[R(x), y] + R[x, R(y)] \ .$$

A.2 Spitzer's identities

Spitzer's non-commutative identity is the main tool for developing the group properties of renormalization.

A.2.1 Spitzer's commutative identity

Spitzer's Commutative identity comes from a generalization of the solution to the differential equation

$$\frac{dy}{dt} = y(t)a(t), \qquad y(0) = 1$$

which has the integral form

$$y = 1 + \int_0^t y(t)a(t)dt$$

with y, and $a \in A$. Generalizing from integration by parts, which is a Rota-Baxter algebra of weight 0, to any Rota-Baxter algebra of any weight gives

$$y = 1 + R(ya) \ . \tag{59}$$

The solution to equation (59) is as follows:

$$e^{-R(\frac{\log(1-\theta a x)}{\theta})} = \sum_0^\infty x^n R(R(\ldots R(a)a\ldots a)a)$$

is *Spitzer's commutative identity.* Proof of this can be found as a special case of the proof of the non-commutative Spitzer's Identity.

A.2.2 Spitzer's non-commutative identity

Let (A, R) be a non-commutative complete filtered algebra (with a decreasing filtration), and a filtration preserving Rota Baxter operator of weight $\theta \neq 0$.

Definition 48. One can define a metric on A as a filtered algebra. Let

$$l(a) = \begin{cases} \{\max k | a \in A^{(k)}\}, & a \notin \cap_n A^{(n)}; \\ \infty, & \text{else.} \end{cases}$$

The metric is given by

$$d(a,b) = 2^{-l(b-a)}$$

Completion of A is defined with respect to this metric.

Definition 49. Let $a \in A_1$. Define $\breve{a} := (1+\theta a)^{-1} - 1 \in A_1$.

Spitzer's Non-Commutative Identity. *Given (A, R), a complete, filtered, non-commutative Rota-Baxter algebra of weight θ, and $a \in A_1$, Spitzer's non-commutative identity is*

$$e^{R(\chi(\frac{\ln(1-\theta a)}{\theta}))} = \sum_0^n (-1)^n R(R\ldots(R(a)a)\ldots a) \ .$$

In order to prove this, and define the terms involved, one needs some tools about non-commutative algebras.

Baker-Cambell-Hasdorff Formula. *In a non-commutative algebra, A, the following identity holds:*

$$e^x e^y = e^{x+y+BCH(x,y)}$$

where the first few terms are

$$BCH(x,y) = \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \ldots$$

For a closed form, see [34].

One has the following recursive definition:

Definition 50. Let

$$\chi_0(u) = u$$

and

$$\chi_n(u) = u - BCH(R(\chi_{n-1}(u)), \tilde{R}(\chi_{n-1}(u))) \ .$$

Because R preserves the filtration, take the limit and write

$$\lim_{n\to\infty} \chi_n = \chi(u) = u - \frac{1}{\theta} BCH(R(\chi(u)), \tilde{R}(\chi(u))) \ . \tag{60}$$

Lemma A.5. *This definition lets one write*

$$e^{\theta\chi(u)+BCH(R(\chi(u),\tilde{R}(\chi(u))))} = e^{\theta u} = e^{R(\chi(u))} e^{(\theta - R)(\chi(u))} \ . \tag{61}$$

Lemma A.6. *If R is an idempotent Rota-Baxter operator which is also an algebra homomorphism, one has:*

1. $R(\chi(u)) = R(u)$ *and* $\tilde{R}(\chi(u)) = \tilde{R}(u)$.
2. *Equation (61) becomes*

$$e^u = e^{R(u)} e^{(u-R(u))}$$

or

$$e^{\tilde{R}(u)} = e^{-R(u)} e^u = e^{\tilde{R}(u)+BCH(-R(u),u)}$$

Proof. [10] □

Definition 51. If $u \in A_1$, then write $e^{\theta u} = 1 + \theta a$, with $a \in A_1$ and $\theta \neq 0 \in k$. The expression $e^{\theta u}$ is well defined due to completeness:

$$u = \frac{\log(1+\theta a)}{\theta} \ . \tag{62}$$

.

I now return to the proof of Spitzer's non-commutative identity.

Proof of Spitzer's Non-commutative Identity. One can write

$$e^{-R(\chi(u))} = 1 + R\left(\sum_1^\infty \frac{(-1)^n (\chi(u))^{*_R n}}{n!}\right) = 1 + R\left(\sum_1^\infty \frac{(-1)^n}{n!\theta}(R(\chi(u)^n - (-1)^n \tilde{R}(\chi(u))))\right) \ ,$$

where the first equality comes from equation (58), the second from Lemma A.4. Using Equation (58) again

$$e^{-R(\chi(u))} = 1 + R\left(\frac{1}{\theta}(e^{-R(\chi(u))} - e^{\tilde{R}(\chi(u))})\right) = 1 + R\left(\frac{1}{\theta}(e^{-R(\chi(u))} - e^{-R(\chi(u))}(1+\theta a))\right)$$

$$= 1 - R\left(e^{-R(\chi(u))}a\right) .$$

The fourth from the fact that

$$e^{\theta u} = e^{R(\chi(u))}e^{\tilde{R}(\chi(u))}$$

from equation (61) and definition 51. Thus $e^{-R(\chi(u))}$ satisfies the recursion relation

$$y = 1 - R(ya) = \sum_1^n (-1)^n R(R \ldots (R(a)a) \ldots a) .$$

□

To generalize this result one defines the following relations:

1. Define
$$X = e^{R(\chi(\frac{1+\theta a}{\theta}))} \quad ; \quad Y = e^{\tilde{R}(\chi(\frac{1+\theta a}{\theta}))}$$
These are Spitzer's non-commutative relations. They satisfy the recursive equations
$$X = 1_A - R(Xa) \quad ; \quad Y = 1_A - \tilde{R}(aY)$$
with a defined as above.

2. The inverses are defined as
$$X^{-1} = 1_A - R(\check{a}X^{-1}) \quad ; \quad Y^{-1} = 1_A - \tilde{R}(Y^{-1}\check{a}))$$

Lemma A.7. $X(1+\theta a)Y = 1$.

Using this notation, $e^{-R(\chi(u))}$ satisfies the conditions for X^{-1}. Similarly, one sees $Y = e^{\tilde{R}(\chi(u))}$. Also, X and Y are clearly elements of the algebra A.

A.3 Application to the Hopf algebra of Feynman graphs

This section returns to the complete filtered unital algebra of linear maps $(\mathrm{Hom}(\mathcal{H}, A), \star)$ for some Rota-Baxter algebra (A, P), of weight $\theta \neq 0$. Define $e = \eta \circ \epsilon$ be the unit map. Ebrahimi-Fard, Guo and Kreimer rewrite the Birkhoff decomposition of elements in $G(A)$ in terms of Spitzer's non-commutative relations.

Theorem A.8. *For $\gamma^\dagger \in G(A)$. One can write $\gamma^\dagger = e^{\star a}$ with $a \in Hom(\mathcal{H}, A)^{(1)}$. $u = \gamma^\dagger - e \in Hom(\mathcal{H}, A)^{(1)}$.*

1. *$R : Hom(\mathcal{H}, A) \to Hom(\mathcal{H}, A)$ is a Rota-Baxter operator given by $R = P \circ f$. $(Hom(\mathcal{H}, A), R)$ is a filtered non-commutative, associative, unital Rota-Baxter algebra.*

2. *The renormalization equations are*
$$\gamma_-^\dagger = -P(\gamma^\dagger + \sum_\gamma \gamma_-^\dagger \gamma^\dagger) = e - R(\gamma_-^\dagger \star u)$$
and
$$\gamma_+^\dagger = \tilde{P}(\gamma^\dagger + \sum_\gamma \gamma_-^\dagger \gamma^\dagger) = e - \tilde{R}(\gamma_+^\dagger \star (\gamma^{\dagger \star -1} - e) .$$
Therefore, one can write $X = \gamma_-^\dagger$ and $Y^{-1} = \gamma_+^\dagger$ where X, and Y are defined as above.

3. *Given the definition of P and $\tilde{P}$, X and Y are algebra homomorphisms. Therefore $\gamma_-^\dagger$ and $\gamma_+^\dagger \in G(\mathcal{A})$.*

4.
$$\gamma^\dagger = e^a = e^{R(\chi(a))} \star e^{\tilde{R}(\chi(a))} \ .$$

If $\mathcal{A} = \mathbb{C}\{z\}[z^{-1}]$, this corresponds to the Birkhoff Decomposition expression
$$\gamma^\dagger = \gamma_-^{\dagger\star -1} \star \gamma_+^\dagger = X^{-1}Y^{-1} \ .$$

5. The renormalization equations can be rewritten in terms of the Bogoliubov operator:
$$\bar{R}[\gamma^\dagger] = \gamma^\dagger + \sum_\gamma \gamma_-^\dagger \star \gamma^\dagger$$

or as $\gamma_-^\dagger = -R(\bar{R}[\gamma^\dagger])$. Thus one has $\bar{R}[\gamma^\dagger] = e^{*_R - \chi(a)} = \gamma_-^\dagger - e$. Similarly, $\tilde{R}(\bar{R}[\gamma^\dagger]) = 2e - \gamma^\dagger$.

This shows that dimensional regularization and renormalization by minimal subtraction can be formulated as a factorization problem in a non-commutative algebra with idempotent Rota-Baxter map. The decomposition of the Hopf algebra homomorphisms are determined by the decomposition map associated to the Rota-Baxter operator on $\mathcal{A}$. The properties of this decomposition implies that $\gamma_-^\dagger$ and $\gamma_+^\dagger \in G(\mathcal{A})$. In fact, both the set of counterterms and the set of renormalized algebra homomorphisms form subgroups of $G(\mathcal{A})$. Furthermore, since the Rota-Baxter operator in $\mathcal{A}$ is idempotent, it gives a direct decomposition of $\mathcal{A}$, which establishes the uniqueness of the Birkhoff Decomposition in this case.

Finally, this theorem implies that it doesn't matter what the algebra A is, as long as it is a unital Rota-Baxter algebra with idempotent operator. Under these conditions, one has a unique Birkhoff decomposition determining the counterterms and the renormalized algebra homomorphisms. Since choosing A physically amounts to choosing a class of regularization schemes, this gives the physicist significant freedom in choosing the regularization scheme.

B Rings of polynomials and series

Let k be a ring. Let x be a formal indeterminate over k.

$k[x]$	The ring of polynomials in x with k coefficients	$f(x) = \sum_1^n a_i x^i$
$k[x, x^{-1}]$	The ring of polynomials in x and x^{-1}	$f(x) = \sum_{-n}^m a_i x^i$
$k(x)$	The ring of rational functions in x.	$\frac{f}{g}$ with f, $g \in k[x]$; $g \neq 0$
$k[[x]]$	Formal power series in x.	$f(x) = \sum_0^\infty a_i x^i$
$k((x))$	Also written $k[[x]][x^{-1}]$ or $k[[x, x^{-1}]$ Formal power series in x with finitely many powers of x^{-1}.	$f(x) = \sum_{-n}^\infty a_i x^i$
For the following two rings, let $k = \mathbb{C}$.		
$\mathbb{C}\{x\}$	Formal power series in x with non-zero radii of convergence.	$\exists r > 0$ such that $f(x) = \sum_0^\infty a_i x^i$ converges for $\lvert x\rvert < r$.
$\mathbb{C}\{\{x\}\}$	Also written $k\{x\}[x^{-1}]$. Convergent formal power series with finitely many powers of x^{-1}.	If $f(x) = \sum_{-n}^\infty a_i x^i$, $\exists r > 0$ such that $\sum_0^\infty a_i x^i$ converges for $\lvert x\rvert < r$.

There are the following inclusions on these sets:

$$k[x] \subset k[x, x^{-1}] \subset k(x)$$

$$k[x, x^{-1}] \subset k\{\{x\}\} \subset k((x))$$

$$k[x] \subset k\{x\} \subset k[[x]]$$

If $k = \mathbb{C}$ then $\mathbb{C}\{x\}$ is the set of meromorphic functions that do not have a pole at 0. Also, there is a map $\mathbb{C}(x) \hookrightarrow \mathbb{C}\{\{x\}\}$ that assigns to each rational function its meromorphic expansion about 0.

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