TOLEDO INVARIANTS ON 2-ORBIFOLDS

by

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Abstract

To each connected component in the space of semisimple representations from the orbifold fundamental group of the base orbifold of a Seifert fibered homology 3-sphere into the Lie group $\mathrm{U}(2,1)$, we associate a real number called the "orbifold Toledo invariant." Using the theory of Higgs bundles, we explicitly compute all values this invariant takes on.

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Chapter 1

Introduction

In this dissertation, we investigate the space

$$\mathcal{R}^{+}_{\mathrm{U}(2,1)}(O) = \frac{\mathrm{Hom}^{+}(\pi_1^{\mathrm{orb}}(O),\ \mathrm{U}(2,1))}{\mathrm{U}(2,1)}$$

of semisimple representations from the orbifold fundamental group of a certain 2-orbifold O into the Lie group $\mathrm{U}(2,1)$, modulo conjugation. To each connected component in $\mathcal{R}^{+}_{\mathrm{U}(2,1)}(O)$, we associate a number that we call the "orbifold Toledo invariant." Our main result (Thm. 8.1.1) explicitly computes all values that the orbifold Toledo invariant takes on. One thereby obtains (Cor. 8.1.2a) a lower bound for the number of connected components in $\mathcal{R}^{+}_{\mathrm{U}(2,1)}(O)$. The orbifolds we consider are quotients of certain 3-manifolds Y—namely, Seifert fibered homology 3-spheres—by the action of S^1. Our results also yield (Cor. 8.1.2b) a lower bound for the number of connected components in $\mathcal{R}^{*}_{\mathrm{PU}(2,1)}(Y) = \frac{\mathrm{Hom}^{*}(\pi_1(Y),\mathrm{PU}(2,1))}{\mathrm{PU}(2,1)}$, the space of irreducible representations from the fundamental group of Y into $\mathrm{PU}(2,1)$, modulo conjugation.

In [36], Toledo introduces an invariant τ for representations of the fundamental group of an oriented 2-manifold M into $\mathrm{PU}(p,1)$. This invariant can be viewed as a map $\tau : \mathrm{Hom}(\pi_1(M), \mathrm{PU}(p,1)) \to \mathbb{R}$. As discussed in Chapter 2, the construction of the Toledo invariant is quite general: one may replace M by an arbitrary topological space and $\mathrm{PU}(p,1)$ by any topological group G. Moreover, if conditions are favorable, then representations which take on distinct Toledo invariants necessarily lie in distinct components of the corresponding representation space. In the case where M is a compact Riemann surface of genus $g > 1$, previously established results about the space $\mathcal{R}_G^+(M) = \frac{\mathrm{Hom}^+(\pi_1(M), G)}{G}$ of semisimple representations of $\pi_1(M)$ into G, modulo conjugation, include:

- The Toledo invariant gives a bijection between the set of all $\tau \in \frac{2}{3}\mathbb{Z}$ with $|\tau| \leq 2g-2$ and the set of all connected components in $\mathcal{R}^+_{\mathrm{PU}(2,1)}(M)$ [16], [39].

- If τ is sufficiently large and c is any integer, then the subset of $\mathcal{R}^+_{\mathrm{PU}(p,p)}(M)$ corresponding to representations with Toledo invariant τ and Chern class c is connected [27].

- The Toledo invariant gives a bijection between the set of even integers τ with $|\tau| \leq\ 2(g-1)$ and the set of connected components in $\mathcal{R}^+_{\mathrm{U}(p,1)}(M)$ [40].

- The subset $\mathcal{R}(\tau, c)$ of $\mathcal{R}^+_{\mathrm{PU}(p,q)}(M)$ corresponding to representations with Toledo invariant τ and Chern class c is non-empty if and only if

  $$\tau = \frac{|qa - p(c-a)|}{p+q} \leq (g-1)\cdot \min\{p,q\}$$

  for some integer a. Moreover, if this inequality is satisfied and $p+q$ and c are coprime, then $\mathcal{R}(\tau, c)$ is connected [6].

Other results concerning Toledo invariants can be found in [7], [15], [16], [19], [20], [36], and [37].

To our orbifold O we associate a complex surface X, called a Dolgachev surface, whose fundamental group is isomorphic to $\pi_1^{\mathrm{orb}}(O)$. The reason for doing so is that for complex algebraic manifolds M, we have a correspondence between representations of $\pi_1(M)$ and certain algebro-geometric objects on M called Higgs bundles. (A

Higgs bundle on M consists of a holomorphic vector bundle plus some extra data; see Chapter 6 for the definition and basic properties.) The relationship between representations of $\pi_1(M)$ and holomorphic vector bundles on M has been developed over the last forty years by Narasimhan and Seshadri [28], Atiyah and Bott [1], Hitchin [22], Donaldson [10], Corlette [8], Simpson [31], and others.

In Chapter 7, we obtain detailed information about those Higgs bundles on the Dolgachev surface X that correspond to semisimple representations $\rho : \pi_1(X) \to \mathrm{U}(2,1)$. In [39], Xia computes the Toledo invariant of such a representation in terms of the associated Higgs bundle. This computation, together with the results of Chapter 7, enables us to determine all Toledo invariants that arise from semisimple representations ρ. In Chapter 5, we define "orbifold Toledo invariants" for representations of the orbifold fundamental group of the 2-orbifold O into $\mathrm{PU}(2,1)$ and show that these are in one-to-one correspondence with Toledo invariants on the Dolgachev surface X. In Chapter 8, we put the pieces together, obtaining the following numerical conditions which completely determine whether or not a real number τ represents an orbifold Toledo invariant:

Main Theorem. *Let O be the base orbifold of Seifert fibered homology* 3*-sphere Y such that $\pi_1^{\mathrm{orb}}(O)$ is infinite. Let n equal the number of cone points that O has, and let $m_1, \ldots, m_n$ denote the orders of these cone points. Let $\tau \in \mathbb{R}$. Then there exists a semisimple representation $\rho : \pi_1^{\mathrm{orb}}(O) \to \mathrm{U}(2,1)$ such that τ is the orbifold Toledo invariant of ρ if and only if $\tau = \pm(y + \sum \frac{y_k}{m_k})$ for some integers $(y, y_1, \ldots, y_n)$ with $0 \leq y_k < m_k$ such that at least one of the numerical conditions* (i)–(iv) *holds:*

(i) *There exist integers $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$ with $0 \leq a_k, b_k < m_k$ such that $b \leq -2$, and $a + \#\{k \mid a_k \neq 0\} \geq 2$, and $2A < B$, and $A < 2B$, and* ($\star$) *below holds.*

(ii) *There exist integers* $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$ *with* $0 \leq a_k, b_k < m_k$ *such that* $-B < A < \frac{1}{2}B$, *and* $d_2 \leq -2$, *and* $b \leq -2$, *and* $(\star)$ *below holds, and* $d_1+1 \leq \min(-d_2-1, -d_3-1)$ *for every* $(n+1)$*-tuple of integers* $(c, c_1, \ldots, c_n)$ *such that* $0 \leq c_k < m_k$ *for all* k *and* $d_1 \geq 0$ *and* $C \geq \frac{2}{3}(A+B)$.

(iii) $y + \sum \frac{y_k}{m_k} > 0$, *and* $2y + \#\{y_k \geq \frac{m_k}{2}\} \leq -2$.

(iv) $y = y_k = 0$ *for all* k.

$(\star)$ $3y + \sum \lfloor \frac{3y_k}{m_k} \rfloor = a + b$, *and* $3y_k - \lfloor \frac{3y_k}{m_k} \rfloor m_k = a_k + b_k$ *for* $k = 1, \ldots, n$.

Here we have used the notations $A = a + \sum \frac{a_k}{m_k}$; $B = b + \sum \frac{b_k}{m_k}$; $C = c + \sum \frac{c_k}{m_k}$; $d_1 = b - c - \#\{b_k < c_k\}$; $d_2 = a - b - \#\{a_k < b_k\}$; *and* $d_3 = a - c - \#\{a_k < c_k\}$.

As a corollary, we obtain a lower bound for the number of connected components in $\mathcal{R}^{+}_{\mathrm{U}(2,1)}(O)$. In Chapter 3, we show that irreducible $\mathrm{PU}(2,1)$ representations of $\pi_1(Y)$ are in one-to-one correspondence with irreducible $\mathrm{PU}(2,1)$ representations of $\pi_1^{\mathrm{orb}}(O)$. The main theorem therefore also furnishes a lower bound for the number of connected components in $\mathcal{R}^{*}_{\mathrm{PU}(2,1)}(Y)$.

The space of irreducible $\mathrm{SU}(2)$ representations of $\pi_1(Y)$ has been studied in detail by Fintushel and Stern [11], Bauer and Okonek [4], Kirk and Klassen [25], Furuta and Steer [14], Bauer [3], and Boden [5]. (The motivation of these authors was the study of the $\mathrm{SU}(2)$ Casson's invariant and Floer homology for such spaces Y.) In many of these papers, the method is to associate to Y an auxiliary object whose fundamental group is closely related to that of Y. In [14] and [5], the auxiliary object is a 2-orbifold. In [4], the auxiliary object is a Dolgachev surface.

One motivation for studying $\mathrm{PU}(2,1)$ representations of the fundamental groups of 3-manifolds comes from spherical CR geometry. A spherical CR structure on a 3-manifold M is a system of coordinate charts into S^3 so that the transition functions are elements of $\mathrm{PU}(2,1)$. (Here we regard $\mathrm{PU}(2,1)$ as the isometry group of the complex ball in $\mathbb{C}^2$ and the conformal group of its boundary S^3.) The space $\frac{\mathrm{Hom}(\pi_1(M),\mathrm{PU}(2,1))}{\mathrm{PU}(2,1)}$ provides a local model for the deformation space of spherical CR structures on M [23]. Moreover, for surface groups, certain values of the Toledo invariant are always achieved by discrete faithful representations [16], [7]. One might hope to find a similar result for a Seifert manifold Y and thereby find a uniformized spherical CR structure on Y. This paper does not address these questions, but we hope to study the matter further.

Our lower bound for the number of components in $\mathcal{R}^*_{\mathrm{PU}(2,1)}(Y)$ takes into account only those $\mathrm{PU}(2,1)$ representations which lift to $\mathrm{U}(2,1)$ representations. Moreover, for $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$, we conjecture that the number of components is in general strictly greater than the number of orbifold Toledo invariants that occur. We plan to continue investigating these representation spaces, with the goal of precisely determining the number of components in them.

Chapter 2

Toledo invariants

Given a manifold (or topological space) M and a topological group G, one may wish to study the representation variety $\mathcal{R} = \frac{\mathrm{Hom}(\pi_1(M), G)}{G}$. The goal of this chapter is to define a family of invariants, called Toledo invariants, that can be used to distinguish components of $\mathcal{R}$. We then describe one such Toledo invariant more specifically in the case where $G = U(2,1)$.

2.1 The "abstract nonsense" of Toledo invariants

Let B be a solid topological space [34]. (Euclidean space $\mathbb{R}^n$ is solid, for example.) Let G be a topological group acting continuously on B on the left. Define $L_g : B \to B$ by $L_g(x) = g \cdot x$ for $g \in B$. If ω represents a cohomology class in $H^*(B, \mathbb{C})$, then we say ω is *G-invariant* if $L_g^* \omega = \omega$ for all $g \in G$. (If B is a manifold, we may regard ω as a closed singular cochain or as a closed differential form, depending which is more convenient.) We now take ω to be a fixed G-invariant representative of a cohomology class in $H^*(B,\mathbb{C})$.

Remark. We have chosen $\mathbb{C}$ as our coefficient group for the sake of convenience; one may choose other coefficient groups as well. □

Let M be a C^∞ manifold. (More generally, we might take M to be any topological space with a universal cover.) We define a map $\tau^{B,G,\omega}$ from $\mathrm{Hom}(\pi_1(M),G)$ to $H^*(M,\mathbb{C})$ as follows. Let $\rho \in \mathrm{Hom}(\pi_1(M),G)$. Let $\tilde{M}$ be the universal cover of M. Note that $\pi_1(M)$ acts on $\tilde{M} \times B$ by $\gamma \cdot (m,x) = (\gamma \cdot m, \rho(\gamma) \cdot x)$. Let E_ρ be the flat B-bundle on M obtained by taking $\tilde{M} \times B$ modulo the action of $\pi_1(M)$. Let $\pi_B : \tilde{M} \times B \to B$ be the projection map onto the second factor, and let φ be the natural map from $\tilde{M} \times B$ to E_ρ. Since $\pi_1(M)$ acts freely on $\tilde{M}$ and ω is G-invariant and closed, the pullback $\pi_B^*\omega$ descends to E_ρ, where it represents a cohomology class $[\varphi_*\pi_B^*\omega] \in H^*(E_\rho,\mathbb{C})$. Since the fibre B is solid, E_ρ has a section; moreover, any two sections are homotopic [34, Theorem 12.2]. Consequently, $[s^*\varphi_*\pi_B^*\omega]$ is a well-defined cohomology class in $H^*(M,\mathbb{C})$.

Definition 2.1.1. *The* Toledo invariant $\tau^{B,G,\omega}(\rho)$ *is defined by*

$$\tau^{B,G,\omega}(\rho) = [s^*\varphi_*\pi_B^*\omega]$$

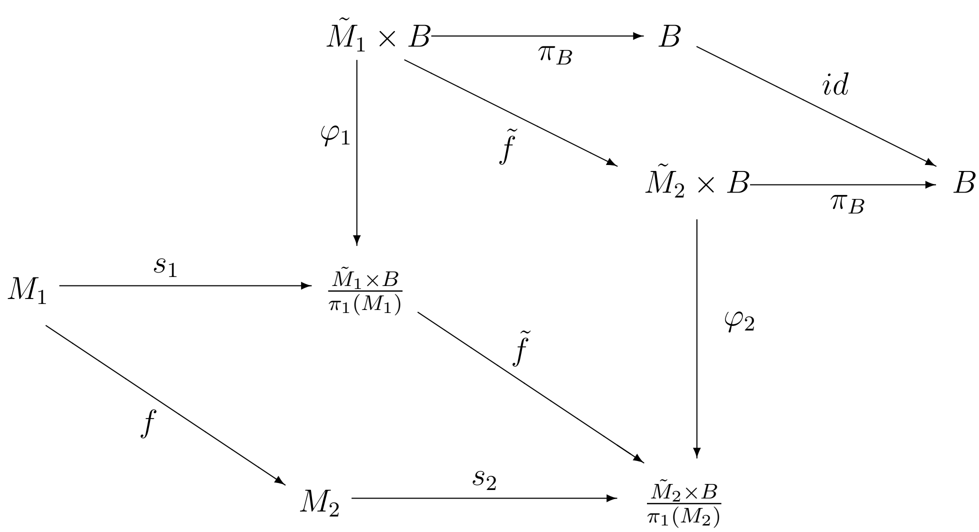
for any section s of E_ρ.

Remark. Let $\mathcal{C}$ be the category whose objects are pairs (M,m) with M a C^∞ manifold and $m \in M$. Let $\mathcal{G}$ be the category of groups. Let π_1 denote the covariant functor from $\mathcal{C}$ to $\mathcal{G}$ that maps a manifold to its fundamental group with base point m. (From now on, we omit mention of the base point m.) Let F_0 be the contravariant functor from $\mathcal{G}$ to $\mathcal{G}$ that maps H to $\mathrm{Hom}(H,G)$. Let $F = F_0 \circ \pi_1$. Let H^* be the contravariant functor on $\mathcal{C}$ that maps a manifold M to $H^*(M,\mathbb{C})$, its cohomology ring with complex coefficients. We now show that the Toledo invariant $\tau^{B,G,\omega}$ defines a natural transformation from F to H^*.

We must prove that for any C^∞ map $f : M_1 \to M_2$, the following square commutes, where the vertical arrows are given by $F(f)$ and $H^*(f)$:

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(M_2), G) & \stackrel{\tau^{B,G,\omega}}{\longrightarrow} & H^*(M_2, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\pi_1(M_1), G) & \stackrel{\tau^{B,G,\omega}}{\longrightarrow} & H^*(M_1, \mathbb{C}) \end{array}$$

Let $\rho \in \mathrm{Hom}(\pi_1(M_2), G)$. Then $\pi_1(M_2)$ acts on $\tilde{M}_2 \times B$ via ρ and $\pi_1(M_1)$ acts on $\tilde{M}_1 \times B$ via $f_* \circ \rho$. The map f lifts to a map $\tilde{f} : \tilde{M}_1 \to \tilde{M}_2$. The lift $\tilde{f}$ induces a map $\tilde{M}_1 \times B \to \tilde{M}_2 \times B$ which descends to a map $\frac{\tilde{M}_1 \times B}{\pi_1(M_1)} \to \frac{\tilde{M}_2 \times B}{\pi_1(M_2)}$; we denote both of these induced maps again by $\tilde{f}$. Moreover, the fibre bundle $E_1 = \frac{\tilde{M}_1 \times B}{\pi_1(M_1)}$ over M_1 may be regarded as the pullback via f of the fibre bundle $E_2 = \frac{\tilde{M}_2 \times B}{\pi_1(M_2)}$. Consequently, a section s_2 of E_2 pulls back to a section s_1 of E_1. In summary, the following diagram commutes:



Chasing this diagram, we find that:

$$\tau^{B,G,\omega}(f_* \circ \rho) = f^*(\tau^{B,G,\omega}(\rho)).$$

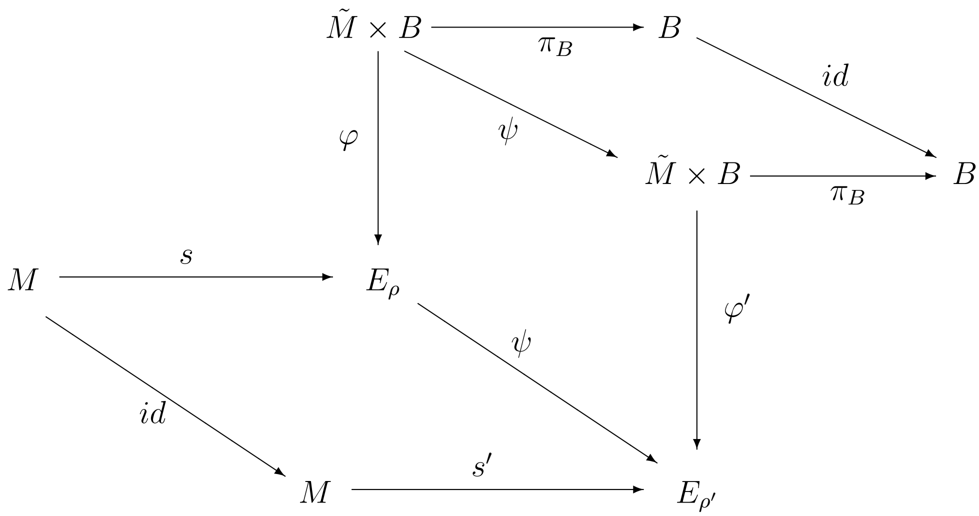
In other words, our square commutes, making $\tau^{B,G,\omega}$ a natural transformation of functors. □

Remark. Suppose there exists a point $b \in B$ such that for all $\gamma \in \pi_1(M)$, we have that $\rho(\gamma) \cdot b = b$. (In other words, suppose that the image of ρ fixes a point in B.) Then the point b defines a "constant" section of the bundle E_ρ. It follows that $\tau^{B,G,\omega}(\rho) = 0$ in this case. (We assume here that ω does not have degree 0.)

In particular, if the action of G fixes a point in B, then all Toledo invariants vanish. This is the case, for example, if G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ acting on $B = \mathbb{C}^n$ via linear transformations. □

Lemma 2.1.2. *Let M be a C^∞ manifold, let $\rho \in \mathrm{Hom}(\pi_1(M), G)$, let $g \in G$, and define $\rho' : \pi_1(M) \to \pi_1(M)$ by $\rho'(\gamma) = g\rho(\gamma)g^{-1}$. Then $\tau^{B,G,\omega}(\rho) = \tau^{B,G,\omega}(\rho')$. In other words, the Toledo invariant is invariant under conjugation.*

Proof. We define a map $\psi : \tilde{M} \times B \to \tilde{M} \times B$ by $\psi(x, b) = (x, g \cdot b)$. Let $E_\rho = \frac{\tilde{M} \times B}{\pi_1(M)}$ (where the action is induced by ρ), and let $E_{\rho'} = \frac{\tilde{M} \times B}{\pi_1(M)}$ (where the action is induced by ρ'). Then ψ descends to a map from E_ρ to $E_{\rho'}$; we denote this new map by ψ as well. If s is a section of E_ρ, then $s' = \psi \circ s$ is a section of $E_{\rho'}$. In summary, we have that the following diagram commutes.



The lemma follows from chasing this diagram. □

Question. Lemma 2.1.2 shows that the Toledo invariant is invariant under the action of the group of inner automorphisms of G. Is the Toledo invariant invariant under the action of the full automorphism group $\mathrm{Aut}(G)$? In other words, does $\tau^{B,G,\omega}(\rho) = \tau^{B,G,\omega}(\alpha \circ \rho)$ for all $\alpha \in \mathrm{Aut}(G)$? □

Let G act on $\mathrm{Hom}(\pi_1(M), G)$ by conjugation. Lemma 2.1.2 shows that $\tau^{B,G,\omega}(\rho)$ defines a map from $\mathrm{Hom}(\pi_1(M), G)/G$ to $H^*(M, \mathbb{C})$. Topologize $\mathrm{Hom}(\pi_1(M), G)/G$ by giving $\mathrm{Hom}(\pi_1(M), G)$ by the point-open topology and giving $\mathrm{Hom}(\pi_1(M), G)/G$ the quotient topology.

Lemma 2.1.3. *Suppose that B and M are C^∞ manifolds, that M is compact, that G is a Lie group, and that ω is a closed G-invariant k-form on B. Topologize $H^k(M, \mathbb{C})$ as a finite-dimensional vector space. Then $\tau^{B,G,\omega}$ defines a continuous function from $\frac{\mathrm{Hom}(\pi_1(M),G)}{G}$ to $H^k(M, \mathbb{C})$.*

Proof. It suffices to show that $\tau^{B,G,\omega}$ is continuous on $\mathrm{Hom}(\pi_1(M), G)$.

Let $t_1, \ldots, t_n$ be generators for $\pi_1(M)$. Note that $\mathrm{Hom}(\pi_1(M), G)$ is homeomorphic to the closed subspace $\{(x_1, \ldots, x_n) \in G^n \mid r_\alpha(x_1, \ldots, x_n) = 1\}$ of G^n, where the

r_α's range over all relations between the t's.

Let $C = \text{Hom}(\pi_1(M), G) \times \tilde{M} \times B$. An action of $\pi_1(M)$ on C is given by

$$\gamma \cdot (\rho, m, x) = (\rho, \gamma \cdot \ m, \rho(\gamma) \cdot x).$$

Then $\frac{C}{\pi_1 M}$ is a fibre bundle over $\text{Hom}(\pi_1(M), G) \times M$ with fibre B. Since B is solid and since $\text{Hom}(\pi_1(M), G)$ is a subspace of G^n, there exists a section $s : \text{Hom}(\pi_1(M), G) \times M \to \frac{C}{\pi_1 M}$. Lift s to a map $\tilde{s} : \text{Hom}(\pi_1(M), G) \times \tilde{M} \to C$. By Tietze extension and by inclusion of C into $G^n \times \tilde{M} \times B$, we have that $\tilde{s}$ extends to a map $\tilde{s}$ from $G^n \times \tilde{M}$ to $G^n \times \tilde{M} \times B$.

Let $\pi_B : G^n \times \tilde{M} \times B \to B$ denote projection onto the third factor. Given $\rho \in G^n$, define $\iota_\rho : \tilde{M} \to G^n \times M$ by $\iota_\rho(m) = (\rho, m)$.

Then $\iota_\rho^* \tilde{s}^* \pi_B^* \omega$ defines a closed $\pi_1(M)$-invariant singular k-cochain on $\tilde{M}$, which therefore defines a closed cochain $\tau_0(\rho)$ on M. Let $\tau(\rho)$ be the associated cohomology class in $H^k(M, \mathbb{C})$. Note that $\tilde{s} \circ \iota_\rho$ defines a section of E_ρ, the fibre bundle from the definition of the Toledo invariant. Therefore, we have that $\tau^{B,G,\omega}(\rho) = \tau(\rho)$. We now show that $\tau(\rho)$ varies continuously with ρ.

Let r be the dimension over $\mathbb{C}$ of $H^k(M, \mathbb{C})$. Given open sets $U_1, \ldots, U_r$ of $\mathbb{C}$ and closed cochains $\sigma_1, \ldots, \sigma_r \in C^k(M, \mathbb{C})$ such that the associated cohomology classes $[\sigma_1], \ldots, [\sigma_r]$ are linearly independent, let

$$IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r) = \{\sum a_\ell \sigma_\ell \mid a_\ell \in U_\ell\}.$$

Then τ is continuous if and only if $\tau_0^{-1}(IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r))$ is open for all $\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r$.

Now fix r closed cochains $\sigma_1, \ldots, \sigma_r \in C^k(M, \mathbb{C})$ such that the associated cohomology classes $[\sigma_1], \ldots, [\sigma_r]$ are linearly independent. Let $C_1, \ldots C_r$ be singular

k-simplices in M such that the map from $\phi : IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r) \to \mathbb{C}^r$ defined by

$$\phi(\sigma) = (<\sigma, C_1>, \ldots, <\sigma, C_r>)$$

is bijective. Define

$$\phi_\ell : IB(\sigma_1, \ldots, \sigma_r, U_1, \ldots, U_r) \to \mathbb{C}$$

by $\phi_\ell(\sigma) = <\sigma, C_\ell>$. It now suffices to show that $\tau_0^{-1}(\phi_\ell^{-1})(U_\ell)$ is open for every open set U_ℓ of $\mathbb{C}$.

By subdividing C_ℓ into small enough pieces, we can assume that the image of C_ℓ is a subset of an open set V of M such that V is homeomorphic, via the natural covering map, to an open set $\tilde{V}$ of $\tilde{M}$. We may then regard C_ℓ as a map from the standard k-simplex Δ_k to $\tilde{M}$.

Let $\rho_0 \in \tau_0^{-1}(\phi_\ell^{-1})(U_\ell)$. Endow G^n with a Riemannian metric. We now show that for sufficiently small δ, the ball of radius δ centered at ρ_0 lies entirely within $\tau_0^{-1}(\phi_\ell^{-1})(U_\ell)$; this will conclude our proof.

Let $\rho_1 \in G^n$ such that the distance from ρ_0 to ρ_1 is less than δ. Let $c(t)$ be a geodesic in G^n with $c(0) = \rho_0$ and $c(1) = \rho_1$. Let $h = a \circ \iota_{c(t)} \circ b$, where a and b are piecewise smooth functions homotopic to $\pi_B \circ \tilde{s}$ and $C_\ell \times \mathrm{id}$, respectively. Note that

$$\phi_\ell(\tau(\rho_j)) = \int_{\Delta_k \times \{j\}} h^*\omega$$

for $j = 0, 1$. Stokes' Theorem then implies that $\phi_\ell(\tau(\rho_1)) - \phi_\ell(\tau(\rho_0))$ can be made arbitrarily small by taking δ to be sufficiently small. $\square$

Question. To what extent does Lemma 2.1.3 generalize?

Remark. If the image of $\tau^{B,G,\omega}$ is totally disconnected (e.g., if it is homeomorphic to a subset of the rational numbers), then Lemma 2.1.3 shows that $\tau^{B,G,\omega}$ is constant on connected components of $\frac{\mathrm{Hom}(\pi_1(M),G)}{G}$. This is the case in the papers cited in the introduction, and it will be the case in our main theorem (Thm. 8.1.1); the number of distinct values of $\tau^{B,G,\omega}$ therefore provides, in these cases, a lower bound for the number of connected components in $\frac{\mathrm{Hom}(\pi_1(M),G)}{G}$.

Example. A simple example shows that $\tau^{B,G,\omega}$ is not always constant on connected components of $\mathrm{Hom}(\pi_1(M), G)$. Let M be the unit circle S^1, let $G = B = \mathbb{R}$ (where G acts on B by translation), and let $\omega = dx$. Let t be the standard generator of $\pi_1(M)$, and identify $\mathrm{Hom}(\pi_1(M), G)$ with $\mathbb{R}$ by $\rho \mapsto \rho(t)$. Since $\mathrm{Hom}(\pi_1(M), G)$ has a single connected component, it suffices to show that the Toledo invariant is not a constant function. Identifying $\tilde{M}$ with $\mathbb{R}$ in the usual way, a ρ-equivariant section of $\tilde{M} \times B$ is given by $x \mapsto (x, \rho(t)x)$. We can then compute that the Toledo invariant $\tau^{B,G,\omega}(\rho)$ is the cohomology class defined by $\rho(t)d\theta$. $\square$

2.2 The $\mathrm{U}(2,1)$ Toledo invariant

We now turn our attention to the special case of this construction that will be the focus of the remainder of this paper. Define $g : \mathbb{C}^3 \to \mathbb{C}$ by $g(z_0, z_1, z_2) = |z_0|^2 - |z_1|^2 - |z_2|^2$. Let $\mathrm{U}(2,1) = \{A \in \mathrm{GL}(3,\mathbb{C}) \mid g(Az) = g(z) \text{ for all } z \in \mathbb{C}^3\}$.

Let $G = \mathrm{U}(2,1)$. Define B by:

$$B = \mathbf{H}^2_{\mathbb{C}} = \{(1, z_1, z_2) \in \mathbb{C}^3 : 1 - |z_1|^2 - |z_2|^2 < 1\}.$$

(B is the ball model of 2-dimensional complex hyperbolic space [15].) Note that B is homeomorphic to $\mathbb{R}^4$, hence solid. G acts on B as follows. Let $z \in B$ and $A \in G$. Define the action of A on z by $A \cdot z = \lambda \cdot (Az)$, where the Az on the right hand side is given by ordinary matrix multiplication (regarding z as a column vector), and λ is the unique complex number such that $\lambda \cdot (Az) \in B$. (We know that λ exists since U(2,1) preserves the indefinite form $|z_0|^2 - |z_1|^2 - |z_2|^2$.) Let $\omega = \frac{i}{2\pi}\partial\overline{\partial}\log g$.

Note that ω is invariant under multiplication by elements of U(2,1) (since g is) and is invariant under multiplication by scalars. By the definition of the action of G on B, then, the restriction of ω to B is G-invariant. The center $Z(\mathrm{U}(2,1))$ of $\mathrm{U}(2,1)$ equals $\{\lambda I | \lambda \in \mathrm{U}(1)\}$. Let $\mathrm{PU}(2,1) = \frac{\mathrm{U}(2,1)}{Z(\mathrm{U}(2,1))}$. Then there is an action of $\mathrm{PU}(2,1)$ on B, inherited from the $\mathrm{U}(2,1)$ action. It follows that ω is $\mathrm{PU}(2,1)$-invariant. From now on, all Toledo invariants will have B and ω as in this paragraph and $G = \mathrm{U}(2,1)$ or $G = \mathrm{PU}(2,1)$.

Remark. ω is the Kähler form associated to the Bergman metric of the Hermitian symmetric space $\mathbf{H}^2_{\mathbb{C}} \cong \frac{\mathrm{U}(2,1)}{\mathrm{U}(2)\times\mathrm{U}(1)}$.

Chapter 3

PU(2,1) representations of fundamental groups of Seifert fibered homology 3-spheres

The goal of this chapter is to note the relationship between PU(2,1) representations of the fundamental group of a Seifert fibered homology 3-sphere and PU(2,1) representations of the fundamental group of a certain elliptic surface called a Dolgachev surface.

3.1 Preliminaries

Let Y be a Seifert fibered homology 3-sphere. (For the definition of Seifert fibered spaces and basic facts about them, we refer to [29].) A $2n+1$-tuple

$$(-b_0; (m_1, b_1), \dots, (m_n, b_n))$$

of integers, with m_k positive for all k, is associated to Y. (These integers are called the Seifert invariants of Y; we may think of m_k as the degree of twisting of the kth singular fibre of Y.) To be a homology 3-sphere, we must have that $\gcd(m_j, m_k) = 1$ whenever $j \neq k$ [14]. The notations Y, n, and $m_1, \ldots, m_n$ will be fixed throughout the rest of this paper.

The fundamental group of Y has the following presentation [29, §5.3]:

$$\pi_1(Y) = < t_1, \ldots, t_n, h \,|\, t_k^{m_k} h^{b_k} = t_1 \ldots t_n h^{b_0} = [h, t_k] = 1 >$$

If G is any group, then let $Z(G)$ denote its center. We have that $Z(\pi_1(Y))$ is generated by h [29, §5.3], so

$$\frac{\pi_1(Y)}{Z(\pi_1(Y))} = < t_1, \ldots, t_n \,|\, t_k^{m_k} = t_1 \ldots t_n = 1 > .$$

Throughout this paper, X will denote a Dolgachev surface whose invariants are $(m_1, \ldots, m_n)$. (See Chapter 4 for a construction of X.)

Lemma 3.1.1. $\pi_1(X) = \frac{\pi_1(Y)}{Z(\pi_1(Y))}$. *If* $n \leq 2$, *then* $\pi_1(X)$ *is trivial. If* $n = 3$ *and* $\{m_1, m_2, m_3\} = \{2, 3, 5\}$, *then* $\pi_1(X)$ *is the alternating group* A_5.

Proof. [9, Chapter II, §3] and [4, Prop. 1.2 and subsequent discussion] □

Because of Lemma 3.1.1, we will impose the restrictions that $n \geq 3$ and that if $n = 3$, then $\{m_1, m_2, m_3\} \neq \{2, 3, 5\}$.

Remark. For Seifert fibered homology 3-spheres Y, the following are equivalent:

(i) Y is large [29].

(ii) $n \geq 3$, and if $n = 3$, then $\{m_1, m_2, m_3\} \neq \{2, 3, 5\}$.

(iii) $\pi_1^{\mathrm{orb}}(O)$ is infinite, where O is the base orbifold of Y and $\pi_1^{\mathrm{orb}}(O)$ is its orbifold fundamental group.

Definition 3.1.2. *The Lie group* PU(2,1) *acts on its Lie algebra* $\mathfrak{g}$ *via the adjoint representation. Consequently, if* H *is a group and* $\rho : H \to \mathrm{PU}(2,1)$ *is a representation, then* ρ *induces an action of* H *on* $\mathfrak{g}$*. We say that* ρ *is* irreducible *(resp.* reducible*) if this induced action is irreducible (resp. reducible). (Recall that an action on a vector space* V *is irreducible if no proper subspace of* V *is invariant under the action, and that the action is reducible otherwise.) We denote the set of irreducible representations* $\rho : H \to \mathrm{PU}(2,1)$ *by* $\mathrm{Hom}^*(H, \mathrm{PU}(2,1))$*.*

Lemma 3.1.3. *Let* H *be a group, and let* $\rho \in \mathrm{Hom}^*(H, \mathrm{PU}(2,1))$*. Then no points and no complex geodesics in* $\mathbf{H}^2_{\mathbb{C}}$ *are invariant under the action of* H *on* $\mathbf{H}^2_{\mathbb{C}}$ *induced by* ρ*.*

Proof. First, suppose that there exists $x \in \mathbf{H}^2_{\mathbb{C}}$ such that $\rho(h) \cdot x = x$ for all $h \in H$. Let $K = \{\phi \in \mathrm{PU}(2,1) | \phi(x) = x\}$. Then K is a Lie subgroup of PU(2,1); in fact, K is conjugate to P(U(2)×U(1)). Let $\mathfrak{g}$ be the Lie algebra of PU(2,1), and let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to K. Since $\rho(H) \subset K$, we have that $\mathfrak{k}$ is invariant under the action of H on $\mathfrak{g}$—but this is a contradiction, since ρ is irreducible.

Similarly, suppose that P is a complex geodesic in $\mathbf{H}^2_{\mathbb{C}}$ such that $\rho(h) \cdot x \in P$ for all $h \in H$ and $x \in P$. In this case, we take K to be the set of all elements in PU(2,1) that preserve P. Again, K is a Lie subgroup of PU(2,1); this time, K is conjugate to P(U(1)×U(1,1)). Again, we find that $\mathfrak{k}$ is invariant under H, contradicting ρ's irreducibility. □

3.2 A bijection

Lemma 3.2.1. *There exists a bijection*

$$\varphi : \mathrm{Hom}^*(\pi_1(\mathrm{Y}), \mathrm{PU}(2,1)) \to \mathrm{Hom}^*(\pi_1(\mathrm{X}), \mathrm{PU}(2,1)).$$

Proof. Since $\pi_1(X) = \frac{\pi_1(Y)}{Z(\pi_1(Y))}$, we have a surjection $\sigma : \pi_1(Y) \to \pi_1(X)$, which in turn induces an injection

$$\overline{\varphi} : \mathrm{Hom}(\pi_1(X), \mathrm{PU}(2,1)) \to \mathrm{Hom}(\pi_1(Y), \mathrm{PU}(2,1)).$$

Now, ρ and $\overline{\varphi}(\rho) = \sigma \circ \rho$ have the same image, so ρ is irreducible if and only if $\overline{\varphi}(\rho)$ is irreducible. Restricting $\overline{\varphi}$ to the irreducible representations then gives us an injection φ from $\mathrm{Hom}^*(\pi_1(X), \mathrm{PU}(2,1))$ to $\mathrm{Hom}^*(\pi_1(Y), \mathrm{PU}(2,1))$. We must now show that φ surjects onto $\mathrm{Hom}^*(\pi_1(Y), \mathrm{PU}(2,1))$.

Let $\tilde{\rho} : \pi_1(Y) \to \mathrm{PU}(2,1)$ be an irreducible representation. We must find

$$\rho : \frac{\pi_1(Y)}{Z(\pi_1(Y))} \to \mathrm{PU}(2,1)$$

such that $\tilde{\rho} = \sigma \circ \rho$. Recalling that the center of $\pi_1(Y)$ is generated by the single element h, we see that it suffices to prove that $\tilde{\rho}$ maps h to the identity element in PU(2,1).

Regard PU(2,1) as the group of isometries of $\mathbf{H}^2_{\mathbb{C}}$. Our first goal is to show that $\tilde{\rho}(h)$ has three linearly independent fixed points x_1, x_2, x_3. [15, p. 203] shows that $\tilde{\rho}(h)$ has a fixed point $x_1 \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$. There must exist $f \in \tilde{\rho}(\pi_1(Y))$ such that $x_2 = f(x_1) \neq x_1$, else $\tilde{\rho}$ would not be irreducible, by Lemma 3.1.3. Since h is central, $\tilde{\rho}(h)$ commutes with f. Thus:

$$\tilde{\rho}(h)(x_2) = f(\tilde{\rho}(h)(x_1)) = x_2.$$

That is, x_2 is another fixed point of $\tilde{\rho}(h)$. Let P be the complex geodesic spanned by x_1 and x_2. By linearity, P is invariant under $\tilde{\rho}(h)$. So, there must exist $g \in \tilde{\rho}(\pi_1(Y))$ and $x \in \{x_1, x_2\}$ such that $x_3 = g(x) \notin P$, else $\tilde{\rho}$ would not be irreducible, again by Lemma 3.1.3. As before, we find that x_3 is a fixed point of $\tilde{\rho}(h)$. By construction, x_1, x_2, and x_3 are linearly independent.

Choose a lift of $\tilde{\rho}(h)$ to U(2,1); denote the lift by $\tilde{h}$. The three linearly independent fixed points x_1, x_2, x_3 yield three linearly independent eigenvectors of $\tilde{h}$. We now prove by contradiction that $\tilde{h}$ has exactly one eigenvalue.

First, suppose that $\tilde{h}$ has 3 distinct eigenvalues. In this case, we have that x_1, x_2, and x_3 are exactly the three one-dimensional eigenspaces of $\tilde{h}$. For each $k \in \{1, \ldots, n\}$, lift $\tilde{\rho}(t_k)$ to U(2,1), and denote the lift by $\tilde{t}_k$. Now, as before, we find that $\tilde{\rho}(t_k)$ maps fixed points of $\tilde{\rho}(h)$ to fixed points of $\tilde{\rho}(h)$. In other words, $\tilde{t}_k$ permutes x_1, x_2, and x_3. Let η_k be this permutation, regarded as an element of the symmetric group S_3. The relation $t_k^{m_k} h^{b_k} = 1$ in $\pi_1(Y)$ implies that $\eta_k^{m_k} = 1$. Consequently, the order $\mathrm{ord}(\eta_k)$ of η_k divides m_k. Now, $\mathrm{ord}(\eta_k) \in \{1, 2, 3\}$, and the m_k's are pairwise coprime. Therefore, there are at most 2 k's such that $\mathrm{ord}(\eta_k) \neq 1$. Moreover, $\mathrm{ord}(\eta_k)$ is relatively prime to $\mathrm{ord}(\eta_{k'})$ whenever $k \neq k'$. The relation $t_1 \ldots t_n h^{b_0} = 1$ in $\pi_1(Y)$ implies that $\eta_1 \ldots \eta_n = 1$. Therefore no η_k has order 2; for if one did, then $\eta_1 \ldots \eta_n$ would be an odd permutation. We must then have that $\eta_k = 1$ for each k, for otherwise $\mathrm{ord}(\eta_1 \ldots \eta_n) = 3$. However, $\eta_k = 1$ if and only if $\tilde{\rho}(t_k)$ fixes x_1, x_2, and x_3. So every element in the image of $\tilde{\rho}$ fixes, say, x_1. By Lemma 3.1.3, this contradicts irreducibility of $\tilde{\rho}$.

Suppose now that $\tilde{h}$ has exactly 2 distinct eigenvalues. Without loss of generality, suppose that x_1 and x_2 belong to the same 2-dimensional eigenspace P and that x_3 is the 1-dimensional eigenspace of $\tilde{h}$. Let f be in the image of $\tilde{\rho}$, and let $\tilde{f}$ be a lift of f to U(2,1). We claim that P is invariant under f. As before, $\tilde{f}$ maps eigenvectors of $\tilde{h}$ to eigenvectors of $\tilde{h}$. In particular, if P is not invariant under $\tilde{f}$, then $\tilde{f}$ maps either x_1 or x_2 to x_3. Let e_1, e_2, and e_3 be nonzero vectors in x_1, x_2, and x_3, respectively. Without loss of generality, assume that $\tilde{f}(e_1) \in x_3$. Since $\tilde{f}$ is nondegenerate, we must then have that $\tilde{f}(e_2) \in P$ and $\tilde{f}(e_3) \in P$. But then $e_2 + e_3$ is an eigenvector of $\tilde{h}$ which is neither in P nor in x_3—a contradiction. So P is invariant under an arbitrary element in the image of $\tilde{\rho}$, once again violating irreducibility.

So, $\tilde{h}$ has three linearly independent eigenvectors and exactly one eigenvalue. Consequently, $\tilde{h}$ is of the form λI, which implies that $\tilde{\rho}(h)$ is the identity in PU(2,1). $\square$

Chapter 4

Dolgachev surfaces

In this chapter, we collect facts about our Dolgachev surface X that will be useful later.

4.1 Construction and definitions

We first show how to construct X. The following description of this construction is taken from [4]. A generic cubic pencil in $\mathbb{CP}^2$ has nine base points. Blowing up at these nine points, we obtain an algebraic surface X_0 along with an elliptic fibration $\pi_0 : X_0 \to \mathbb{CP}^1$. Apply logarithmic transformations [17] along n disjoint nonsingular fibres of X_0 with multiplicities $m_1, \ldots, m_n$. (Recall that $n, m_1, \ldots, m_n$ were fixed in Chapter 3.) The result is an elliptic fibration $\pi : X \to \mathbb{CP}^1$, where X is our Dolgachev surface.

We choose our pencil of curves such that each singular fibre is a rational curve with an ordinary double point. There are, then, 12 such singular fibres in this fibration [12, p. 192]. Denote these 12 fibres by $E_1, \ldots, E_{12}$. Denote the generic fibre of X by F and the multiple fibres of X by $F_1, \ldots, F_n$, where F_k has multiplicity m_k. For all

j, k, we have that E_j is linearly equivalent to F is linearly equivalent to $m_k F_k$.

We say an irreducible curve C is *vertical* if it is set-theoretically equal to $\pi^{-1}(p)$ for some point $p \in \mathbb{CP}^1$. (Note that a multiple fibre F_k is vertical, but it is not the pullback of a divisor on $\mathbb{CP}^1$.) We say that an irreducible curve C is *horizontal* if it is not vertical. We say that a divisor D is *vertical* (respectively *horizontal*) if it is linearly equivalent to a linear combination of vertical (resp. horizontal) irreducible curves, and we say that a line bundle L is *vertical* (resp. *horizontal*) if $L \approx \mathcal{O}_X(D)$ for some vertical (resp. horizontal) divisor D. (Note: this definition of a vertical divisor D is not equivalent to the condition $D \cdot F = 0$, contrary to what one sees occasionally in the literature.) If $V_1 = \mathcal{O}_X(D_1)$ and $V_2 = \mathcal{O}_X(D_2)$ are vertical line bundles, then $c_1(V_1)c_1(V_2) = 0$, since $D_1 \cdot D_2 = 0$. We have that $M \cdot F > 0$ for any effective nonzero horizontal divisor M, since M intersects F transversely. Every divisor on X can be written as $G + M$ where G is vertical and M is horizontal. A divisor D is vertical if and only if it is linearly equivalent to $aF + \sum a_k F_k$ for some integers $a, a_1, \ldots, a_n$. If we write a vertical divisor in this form, we will always assume that $0 \leq a_j < m_j$ for all $j = 1, \ldots, n$, unless otherwise noted.

4.2 Numerical invariants

Lemma 4.2.1 (I. Dolgachev). *The surface X has the following numerical invariants: the topological Euler characteristic $e_X = 12$; the irregularity $q = 0$; the geometric genus $p_g = 0$. Also, the canonical bundle $K_X = \mathcal{O}_X(-F + \sum_k (m_k - 1)F_k)$.*

Proof. As before, let X_0 be the surface obtained by blowing up $\mathbb{CP}^2$ in nine points. The topological Euler characteristic e_{X_0} of X_0 is

$$\sum_{k=0}^{4} (-1)^k \dim_{\mathbb{R}}(H^k(X_0, \mathbb{R})) = 1 - 0 + 10 - 0 + 1 = 12.$$

Excision in a neighborhood of the union of the multiple fibres of an elliptic fibration shows that logarithmic transformations do not change the topological Euler characteristic; therefore, $e_X = e_{X_0} = 12$. Hence, we have that $c_2(X) = 12$, where $c_2(X)$ is the second Chern class of X. The canonical bundle formula [17, p. 572] implies that the canonical bundle K_X of X is vertical, so $K_X \cdot K_X = 0$. By Nöther's formula, we find that $\chi(\mathcal{O}_X) = 1$, where $\chi(\mathcal{O}_X)$ is the holomorphic Euler characteristic of X. Again by the canonical bundle formula, we then have that:

$$K_X = \mathcal{O}_X(-F + \sum_k (m_k - 1)F_k).$$

Since $-F+\sum_k(m_k-1)F_k$ is not linearly equivalent to an effective divisor, we conclude that:

$$p_g = \dim_{\mathbb{C}}(H^0(K_X)) = 0.$$

Then

$$q = \chi(\mathcal{O}_X) + p_g - 1 = 0. \quad \square$$

Lemma 4.2.2 (I. Dolgachev). *X is projective.*

Proof. We have an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X).$$

Since $q = p_g = 0$ by Lemma 4.2.1, we find that

$$H^1(X, \mathcal{O}_X^*) \approx H^2(X, \mathbb{Z}).$$

Dolgachev [9] shows that the first Betti number $b_1(X) = 0$. Poincaré duality implies that $b_3(X) = 0$. Since $b_1(X)$ is even, we have that $b^+(X) = 1$ [2, Theorem 2.6(ii)]. Kodaira's ampleness criterion [2, Theorem 5.2] then implies that X is projective.

Remark. All of the cohomology groups (with integer coefficients) of X can be computed by excising neighborhoods U_k of the multiple fibres F_k such that the U_k's deformation retract onto the F_k's. See [13] for more details on the topology of elliptic surfaces.

4.3 Local description of cohomology groups of vertical line bundles

Lemma 4.3.1.

(i) $h^0(\mathcal{O}_X(\ell F + \sum \ell_k F_k)) = \max(\ell + 1, 0)$, *and*

(ii) $h^1(\mathcal{O}_X(\ell F + \sum \ell_k F_k)) = \max(\ell, -\ell - 1)$.

Proof. [4, Lemma 1.1]

Lemma 4.3.2. *If s is a global section of the locally free sheaf $\mathcal{O}_X(aF + \sum a_k F_k)$, then s is constant on fibres.*

Proof Let $z_0 = \pi(F) \in \mathbb{CP}^1$. Choose a local coordinate z on $\mathbb{CP}^1$ centered at z_0. In order that $H^0(\mathcal{O}_X(aF + \sum a_k F_k)) \neq 0$, we must have $a \geq 0$, by Lemma 4.3.1. Let $f_j = z^{-j}$, for $j = 0, \ldots, a$. The f_j's are linearly independent, so $\{f_j \circ \pi\}$ is a set of $a+1$ linearly independent elements in $H^0(\mathcal{O}_X(aF + \sum a_k F_k)) \neq 0$. By Lemma 4.3.1, s must be a linearly combination of $f_j \circ \pi$'s. Since each $f_j \circ \pi$ is constant on fibres, so is s. □

Lemma 4.3.3. *If $G_1 + M_1$ is linearly equivalent to $G_2 + M_2$ with the G's vertical and the M's horizontal, then G_1 is linearly equivalent to G_2 and M_1 is linearly equivalent to M_2.*

Lemma 4.3.4. *Let F_k be a multiple fibre. Then there exists a collection $\{U_\alpha\}$ of open sets of X such that the U_α's cover F_k; each U_α is a coordinate neighborhood on X; each U_α is disjoint from the singular fibres and from the other multiple fibres;*

and, denoting the coordinates on U_α by (w_α, z_α) and those on U_β by (w_β, z_β), we have that $w_\alpha = \zeta_{\alpha\beta} w_\beta$ and $z_\alpha = z_\beta + t_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ for some complex numbers $\zeta_{\alpha\beta}$ with $\zeta_{\alpha\beta}^{m_k} = 1$ and some holomorphic functions $t_{\alpha\beta}$; the fibration map π locally takes the form $(w_\alpha, z_\alpha) \overset{\pi}{\mapsto} w = w_\alpha^{m_k}$, where w is the local coordinate on $\mathbb{CP}^1$; and $\{w_\alpha = 0\}$ is a set of local defining equations for the divisor F_k.

Proof. As before, let X_0 be the surface obtained by blowing up $\mathbb{CP}^2$ in nine points, and let $\pi_0 : X_0 \to \mathbb{CP}^1$ be the map which expresses X_0 as an elliptic fibration. Let V be a small coordinate disc in $\mathbb{CP}^1$, with coordinate w centered at 0, such that $\pi_0(E_j) \notin V$ for $j = 1, \ldots, 12$. Then we may cover $\pi_0^{-1}(V)$ with coordinate neighborhoods U_γ such that there are coordinates (w, z_γ) on U_γ; the map π_0 is given by $\pi_0(w, z_\gamma) = w$ on U_γ; and the coordinate changes are given by

$$(w, z_{\gamma_1}) = (w, z_{\gamma_2} + s_{\gamma_1\gamma_2})$$

for some $s_{\gamma_1\gamma_2}$. (Here addition is given by fixing a group structure on each fibre.)

We now perform a logarithmic transformation [17] on $\pi_0^{-1}(V)$ with multiplicity m_k. Let

$$\Sigma = \{(u, r) \in V \times \pi_0^{-1}(V) \mid u^{m_k} = w(r)\}.$$

The map $\pi(u, r) = w(r) = u^{m_k}$ expresses Σ as an elliptic fibration over V. We have that $\frac{\mathbb{Z}}{m_k\mathbb{Z}}$ acts on Σ by

$$\ell \cdot (u, r) = (e^{\frac{2\pi i \ell}{m_k}} u, r + \frac{\ell}{m_k}),$$

where $\frac{1}{m_k}$ is an point of order m_k with respect to the group structure on the fibre. Let Σ_0 equal Σ modulo the action of $\frac{\mathbb{Z}}{m_k\mathbb{Z}}$. Then Σ_0 is an open set in X. Moreover, the map $\pi : \Sigma \to V$ is well-defined on Σ_0, and it coincides with our earlier map $\pi : X \to \mathbb{CP}^1$.

Local coordinates on Σ are given by (u, z_γ). If we restrict z_γ to a small disc, we obtain an open set $\tilde{U}_\alpha \subset \Sigma$ such that no two points in $\tilde{U}_\alpha$ are in the same $\frac{\mathbb{Z}}{m_k\mathbb{Z}}$-orbit.

The local coordinates (u, z_γ) on $\tilde{U}_\alpha$ therefore descend to local coordinates (w_α, z_α) on an open set $U_\alpha \subset \Sigma_0$.

We see that these coordinates possess the properties stated in the lemma, as follows. The coordinate change

$$(w_\alpha, z_\alpha) = (\zeta_{\alpha\beta} w_\beta, z_\beta + t_{\alpha\beta})$$

is inherited from the $\frac{\mathbb{Z}}{m_k\mathbb{Z}}$ action. The map π is given by $(u, r) \mapsto u^{m_k}$ in Σ and so descends to $(w_\alpha, z_\alpha) \mapsto w_\alpha^{m_k}$ in Σ_0. Finally, since F_k is the quotient of $u^{-1}(0)$ by $\frac{\mathbb{Z}}{m_k\mathbb{Z}}$, we have that $w_\alpha = u$ is a local defining function for F_k. $\Box$

Let V be as in the proof of Lemma 4.3.4. Without loss of generality, assume that V contains the points $0, \infty$, and $\pi(F_k)$ for each multiple fibre F_k; that $\pi(F_k) \notin \{0, \infty\}$ for all k; and that $F = \pi^{-1}(0)$. As in the proof of Lemma 4.3.4, cover $\pi^{-1}(V - \infty) - \bigcup F_k$ by coordinate neighborhoods V_γ so that there are coordinates (w_γ, z_γ) on V_γ, and the map π is given by $\pi(w_\gamma, z_\gamma) = w$ on V_γ, where w is the coordinate on $\mathbb{CP}^1$ centered at 0. For each multiple fibre F_k, let $\{U_{\alpha,k}\}$ be a system of coordinate neighborhoods covering F_k, where $U_{\alpha,k}$ has coordinates $(w_{\alpha,k}, z_{\alpha,k})$, as in Lemma 4.3.4. Cover $\pi^{-1}(V - 0) - \bigcup_{\alpha,k} \overline{U_{\alpha,k}}$ by coordinate neighborhoods W_ξ so that there are coordinates (w_ξ, z_ξ) on W_ξ, and the map π is given by $\pi(w_\xi, z_\xi) = \frac{1}{w}$ on W_ξ. (Note that $\frac{1}{w}$ is the coordinate on $\mathbb{CP}^1$ centered at ∞.) The relationships between the w's are as follows:

$$\text{On } U_{\alpha_1,k} \cap U_{\alpha_2,k}, \text{we have } w_{\alpha_1,k} = \zeta_{\alpha_1\alpha_2,k} w_{\alpha_2,k} \text{ for some } m_k\text{th root of unity } \zeta_{\alpha_1\alpha_2,k}.$$

$$\text{On } U_{\alpha,k} \cap V_\gamma, \text{we have } w_\gamma = w_{\alpha,k}^{m_k} + t_{\alpha,k;\gamma} \text{ for some } t_{\alpha,k;\gamma}.$$

$$\text{On } V_\gamma \cap W_\xi, \text{we have } w_\xi = \frac{1}{w_\gamma}.$$

Let $L = \mathcal{O}_X(aF + \sum a_k F_k)$ be a vertical line bundle. Local trivializations for L are given by the maps $f \cdot w_{\alpha,k}^{-a_k} \mapsto f$ on $U_{\alpha,k}$; $f \cdot w_\gamma^{-a} \mapsto f$ on V_γ; and $f \mapsto f$ on W_ξ.

From now on, the notations $U_{\alpha,k}, V_\gamma, W_\xi, w_{\alpha,k}, w_\gamma, w_\xi$ will be fixed. Moreover, sections of a vertical line bundle L will be written locally on $U_{\alpha,k}, V_\gamma$, and W_ξ with respect to these trivializations.

Lemma 4.3.5. *Let $L = \mathcal{O}_X(aF + \sum a_k F_k)$ be a vertical line bundle.*

(i) Suppose $a \geq 0$. If $0 \leq j \leq a$, then there exists a section $s_j \in H^0(L)$ such that with respect to the local trivializations, s_j is given by $s_\xi = w_\xi^{a-j}$ on W_ξ; $s_\gamma = w_\gamma^j$ on V_γ; and $s_{\alpha,k} = (w_{\alpha,k}^{m_k} + t_{\alpha,k;\gamma})^j w_{\alpha,k}^{a_k}$ on $U_{\alpha,k}$. Moreover, $\{s_j \,|\, 0 \leq j \leq a\}$ is a basis for $H^0(L)$.

(ii) Suppose $a \leq -2$. If $a < j < 0$, then there exists a Čech 1-cocycle $\sigma_j \in C^1(L)$ such that σ_j is given by $\sigma_{\gamma\xi} = w_\gamma^j$ on $V_\gamma \cap W_\xi$ with respect to the trivialization on V_γ, and $\sigma_{\xi_1\xi_2}, \sigma_{\gamma_1\gamma_2}, \sigma_{\alpha,k;\gamma}$, and $\sigma_{\alpha_1,k;\alpha_2,k}$ vanish on $W_{\xi_1} \cap W_{\xi_2}, V_{\gamma_1} \cap V_{\gamma_2}, U_{\alpha,k} \cap V_\gamma$, and $U_{\alpha_1,k} \cap U_{\alpha_2,k}$, respectively. Moreover, identifying σ_j with its image in $H^1(L)$, we have that $\{\sigma_j \,|\, a < j < 0\}$ is a basis for $H^1(L)$.

Proof. Let $f_j \circ \pi$ be as in the proof of Lemma 4.3.2. Let $s_j = f_j \circ \pi$. From Lemma 4.3.2, we know that the $\{s_j\}$ is a basis for $H^0(L)$. In local coordinates, s_j has the form required in (i). The σ_j's in (ii) are obtained by pulling back a basis for $H^1(\mathcal{O}_{\mathbb{CP}^1}(a))$ via π. Lemma 4.3.1, together with the Leray spectral sequence, guarantees that the σ_j's form a basis for $H^1(L)$. $\square$

4.4 An ample divisor

Definition 4.4.1. *Let H_0 be a fixed ample divisor on X.*

Let $r = min\{M \cdot F \mid M$ is a nonzero effective horizontal divisor$\}$.

Let $s = min\{M \cdot H_0 \mid M$ is a nonzero effective horizontal divisor$\}$.

Let $t = max\{H_0 \cdot (-2F + \sum_k (m_k - 1)F_k), H_0 \cdot F\}$.

Let C be a constant such that $H_0 \cdot D \leq C$ for any divisor D such that $H^0(\mathcal{O}_X(-D) \otimes \Omega^1_X) \neq 0$. (See [12] for a proof that C exists.)

Let k be an integer such that $k > max\{0, \frac{(n+1)t-s}{r}, \frac{t-2s}{2r}, \frac{C}{r}\}$.

Let $H = H_0 + kF$.

Note that F is a fibre of a morphism, so it moves in a linear system without base points; it follows that H is ample, by an exercise in [21]. Throughout this document, the degree of a coherent sheaf—and all related concepts (e.g., stability)—will be with respect to H.

Chapter 5

Toledo invariants on 2-orbifolds and Dolgachev surfaces

In this chapter, we associate to our Seifert fibered space Y a 2-orbifold O. The goal of this chapter will be to show how Toledo invariants on the Dolgachev surface X are related to "orbifold" Toledo invariants which arise from representations of the orbifold fundamental group of O.

5.1 Orbifold Toledo invariants

Let O be the hyperbolic 2-orbifold such that the underlying space $|O|$ of O is the sphere S^2 and O has n elliptic points $p_1, \ldots, p_n$ (also known as cone points) of orders $m_1, \ldots, m_n$, respectively. (We refer to [5], [14], [24], [30], and [35] for details of this construction and for basic facts about orbifolds.) The orbifold fundamental group of O has the following presentation:

$$\pi_1^{\text{orb}}(O) =< u_1, \ldots, u_n \,|\, u_k^{m_k} = u_1 \ldots u_n = 1 >$$

We may think of u_j as a small loop that travels once around the cone point p_j.

In our elliptic fibration $\pi : X \to \mathbb{CP}^1$, we identify $\mathbb{CP}^1$ with $|O|$, and we assume that $p_j = \pi(F_j)$ for each multiple fibre F_j. Let $\tilde{X}$ be the universal cover of our Dolgachev surface X. The orbifold universal cover $\tilde{O}$ of O is the upper half-plane H^2 [35]. Fix a base point x_0 in X and a base point y_0 in O such that $y_0 = \pi(x_0)$ and $x_0 \notin \{E_1, \ldots, E_{12}, F_1, \ldots, F_n\}$. We may regard the elements of $\tilde{X}$ (resp. $\tilde{O}$) as equivalences classes of paths in X (resp. O) beginning at x_0 (resp. y_0). (Caution: One must be careful as to what is meant by a path in O. See [14, §2].) Pushing forward paths in X to paths in O, we obtain a map $\tilde{\pi} : \tilde{X} \to \tilde{O}$ that covers π. If γ is an element of $\pi_1(X)$, then denote the action of γ on $\tilde{X}$ by L_γ. (Similarly for $\tilde{O}$.) Recall that $t_1, \ldots, t_n$ are the generators of $\pi_1(X)$. Then $\tilde{\pi} \circ L_{t_j} = L_{u_j} \circ \tilde{\pi}$. It follows that $\pi_*(t_j) = u_j$, and so π_* is an isomorphism from $\pi_1(X)$ to $\pi_1^{\mathrm{orb}}(O)$.

Lemma 5.1.1. *Let $\rho \in \mathrm{Hom}(\pi_1(\mathrm{X}), \mathrm{PU}(2,1))$, and let $E_\rho = (\tilde{X} \times \mathbf{H}^2_{\mathbb{C}})/\pi_1(X)$. Let $q : \tilde{X} \times \mathbf{H}^2_{\mathbb{C}} \to \mathbf{H}^2_{\mathbb{C}}$ be projection onto the second factor. Then there exists a section s_0 of the fibre bundle E_ρ (as in Definition 2.1.1) and a lift $\tilde{s}_0 : \tilde{X} \to \tilde{X} \times \mathbf{H}^2_{\mathbb{C}}$ of s_0 such that for each point $x \in \tilde{O}$, we have that $q \circ \tilde{s}_0$ is constant on $\tilde{\pi}^{-1}(x)$.*

Definition 5.1.2. *Let $\rho \in \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\mathrm{O}), \mathrm{PU}(2,1))$. We say that a map*

$$s : \tilde{O} \to \tilde{O} \times \mathbf{H}^2_{\mathbb{C}}$$

is ρ-equivariant *if $s(\gamma \cdot x) = \rho(\gamma) \cdot s(x)$ for all $\gamma \in \pi_1^{\mathrm{orb}}(O)$ and $x \in \tilde{O}$. (Here* $\mathrm{PU}(2,1)$ *acts on $\tilde{O} \times \mathbf{H}^2_{\mathbb{C}}$ by acting on its second factor.) If s_1 and s_2 are two ρ-equivariant maps, then we say that s_1 and s_2 are* ρ-equivariantly homotopic *if there exists a homotopy $F : [0,1] \times \tilde{O} \to [0,1] \times \tilde{O} \times \mathbf{H}^2_{\mathbb{C}}$ from s_1 to s_2 such that $F(t, \cdot)$ is ρ-equivariant for all $t \in [0,1]$.*

Lemma 5.1.3. *Let $\rho \in \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\mathrm{O}), \mathrm{PU}(2,1))$. Then there exists a ρ-equivariant* *map* $s : \tilde{O} \to \tilde{O} \times \mathbf{H}^2_{\mathbb{C}}$*. Moreover, if* s_1 *and* s_2 *are any two such equivariant maps, then* s_1 *and* s_2 *are* ρ*-equivariantly homotopic.*

Proof. To construct such a map s, push forward $\tilde{s}_0$ from Lemma 5.1.1. Similarly, to demonstrate that any two such maps are ρ-equivariantly homotopic, push forward a homotopy. □

Definition 5.1.4. *Let* $\rho \in \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\mathrm{O}), \mathrm{PU}(2,1))$*. Let* Σ *be a fundamental domain for the action of* $\pi_1^{\mathrm{orb}}(O)$ *on* $\tilde{O}$*. Take* q *and* s *as in Lemma 5.1.3. Then we define the* orbifold Toledo invariant $\tau_{\mathrm{orb}}(\rho)$ *by*

$$\tau_{\mathrm{orb}}(\rho) = \int_\Sigma s^* q^* \omega$$

where $s : \tilde{O} \to \tilde{O} \times \mathbf{H}^2_{\mathbb{C}}$ *is any* ρ*-equivariant map.*

Lemma 5.1.3 implies that $\tau_{\mathrm{orb}}(\rho)$ is defined and that it is independent of the choice of s. The ρ-equivariance of s implies that $\tau_{\mathrm{orb}}(\rho)$ is independent of the choice of Σ. We now fix $\tilde{s}_0$ as in Lemma 5.1.1, and let s be its ρ-equivariant push-forward, as in Lemma 5.1.2.

5.2 Toledo invariants on X's and O's

Let $H^2_{\mathrm{orb}}(O, \mathbb{Z})$ be the orbifold second cohomology group of O with integer coefficients [14]. (Note that [14] uses the notation "V" in place of "orb," since they use the older terminology "V-manifold" in place of "orbifold.") Let $H^1_{\mathrm{vert}}(X, \mathcal{O}^*_X)$ be the subset of $H^1(X, \mathcal{O}^*_X)$ consisting of vertical line bundles on X. Let

$$H^2_{\mathrm{vert}}(X, \mathbb{Z}) = c_1(H^1_{\mathrm{vert}}(X, \mathcal{O}^*_X))$$

be the set of first Chern classes of vertical line bundles on X.

Lemma 5.2.1. *The map* π *induces an isomorphism* $\pi^* : H^2_{\mathrm{orb}}(O, \mathbb{Z}) \to H^2_{\mathrm{vert}}(X, \mathbb{Z})$*.*

Let $\mathrm{Pic}^t_{\mathrm{orb}}(O)$ be the set of topological isomorphism classes of orbifold line bundles on O. We have that $\mathrm{Pic}^t_{\mathrm{orb}}(O)$ is a group, where the group law is given by the tensor product.

Lemma 5.2.2. $\mathrm{Pic}^{\mathrm{t}}_{\mathrm{orb}}(\mathrm{O}) \cong \mathrm{H}^2_{\mathrm{orb}}(\mathrm{O}, \mathbb{Z})$

Proof. [14, Theorem 2.2(ii)] □

The following lemma spells out the relationship between Toledo invariants on a Dolgachev surface X and those on its underlying orbifold O.

Lemma 5.2.3. *Let* $\rho \in \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\mathrm{O}), \mathrm{PU}(2,1))$*. If*

$$\tau(\rho \circ \pi_*) = c_1(\mathcal{O}_X(aF + \sum a_k F_k)),$$

then $\tau_{\mathrm{orb}}(\rho) = a + \sum \frac{a_k}{m_k}$.

Proof. Let $p \in |O| \backslash \{p_1, \ldots, p_n\}$. Let L_p be the holomorphic point bundle determined by p. L_p is an orbifold line bundle on O with $c_1(L_p) = 1$ [14]. Let $\sigma_k : \frac{\mathbb{Z}}{m_k \mathbb{Z}} \to U(1)$ be the standard representation. Let L_{p_k} be the orbifold line bundle on O with first Chern class $c_1(L_{p_k}) = \frac{1}{m_k}$ and trivial isotropy except at p_k, where it is σ_k. Then $\pi^* L_p = \mathcal{O}_X(F)$ and $\pi^* L_{p_k} = \mathcal{O}_X(F_k)$. Let $L = L_p^{\otimes a} \otimes \left(\bigotimes_k L_{p_k}^{\otimes a_k}\right)$. The following diagrams commute:

$$\begin{array}{ccccc}
\tilde{X} & \xrightarrow{\tilde{s}_0} & \tilde{X} \times \mathbf{H}^2_{\mathbb{C}} & \xrightarrow{q_X} & \mathbf{H}^2_{\mathbb{C}} \\
\tilde{\pi} \downarrow & & \downarrow & & \downarrow \\
\tilde{O} & \xrightarrow{s} & \tilde{O} \times \mathbf{H}^2_{\mathbb{C}} & \xrightarrow{q_O} & \mathbf{H}^2_{\mathbb{C}}
\end{array}$$

and

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{O} \\
\varphi_X \downarrow & & \downarrow \varphi_O \\
X & \xrightarrow{\pi} & O
\end{array}$$

and

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{s}_0} & \tilde{X} \times \mathbf{H}^2_{\mathbb{C}} \\ \varphi_X \downarrow & & \downarrow \varphi \\ X & \xrightarrow{s_0} & E_{\rho\circ\pi_*} \end{array}$$

From these diagrams, we find that

$$\begin{aligned} \tilde{\pi}^* c_1(\varphi_O^* L) &= c_1(\tilde{\pi}^* \varphi_O^* L) \\ &= c_1(\varphi_X^* \pi^* L) \\ &= \varphi_X^* c_1(\mathcal{O}_X(aF + \sum a_k F_k)) \\ &= \varphi_X^* \tau(\rho \circ \pi_*) \\ &= \varphi_X^* s_0^* \varphi_* q_X^* \omega \\ &= \tilde{s}_0^* q_X^* \omega \\ &= \tilde{\pi}^* s^* q_O^* \omega \end{aligned}$$

Let us identify $H^2_{\mathrm{orb}}(O, \mathbb{Z})$ with the set of all $\pi_1^{\mathrm{orb}}(O)$-invariant elements of $H^2(\tilde{O}, \mathbb{Z})$. The equation above then implies that $\pi^* c_1(\varphi_O^* L) = \pi^* s^* q_O^* \omega$. By Lemma 5.2.1, then, we have that

$$\tau_{\mathrm{orb}}(\rho) = \int_\Sigma s^* q_O^* \omega = \int_\Sigma c_1(\varphi_O^* L) = c_1(L) = a + \sum \frac{a_k}{m_k}$$

as desired. □

Chapter 6

U(2,1) Higgs bundles

Hitchin, Simpson, et al. ([22], [31], [33]) have shown that representations of the fundamental group of a Kähler manifold are closely related to holomorphic objects called Higgs bundles. The goal of this chapter is to describe the Higgs bundles that arise from U(2,1) representations of the fundamental group of a Dolgachev surface (actually, any compact Kähler manifold), then describe the Toledo invariant of such a representation in terms of the Chern classes of the associated Higgs bundle.

6.1 Definitions

Definition 6.1.1. *Let M be a complex algebraic manifold, and let H be a fixed ample line bundle on M. A* Higgs bundle *on M is a pair (V, θ), where:*

- *V is a holomorphic vector bundle on M, and*
- *$\theta \in H^0(\mathrm{End}(V) \otimes \Omega^1_M)$, and*
- *$\theta \wedge \theta = 0$.*

(Here End(V) *is the endomorphism bundle of V, and Ω^1_M is the locally free sheaf of holomorphic 1-forms on M. Note that we blur the distinction between a vector*

bundle and a locally free sheaf, if no confusion is likely.)

θ is called the Higgs field*.*

*A subsheaf $\mathcal{S}$ of V is said to be θ-*invariant *if $\theta(\mathcal{S}) \subset \mathcal{S} \otimes \Omega^1_M$.*

The slope *$\mu(\mathcal{S})$ of a coherent sheaf $\mathcal{S}$ on M with* $\text{rank}(\mathcal{S}) > 0$ *is defined by* $\mu(\mathcal{S}) = \frac{\deg(\mathcal{S})}{\text{rank}(\mathcal{S})}$*, where* $\deg(\mathcal{S})$ *is the degree of $\mathcal{S}$ with respect to H.*

A Higgs bundle (V, θ) is stable *if $\mu(\mathcal{S}) < \mu(V)$ for all coherent θ-invariant subsheaves $\mathcal{S}$ of V with* $\text{rank}(\mathcal{S}) > 0$*.*

A Higgs bundle (V, θ) is polystable *if it is a direct sum of stable Higgs bundles, each with the same slope. (One forms the direct sum in the obvious way.)*

A Higgs bundle (V, θ) is reducible *if it is a direct sum of Higgs bundles and is* irreducible *otherwise.*

We say that a Higgs bundle (V, θ) is a U(2,1)-Higgs bundle *if:*

- *$V = V_P \oplus V_Q$ (where V_P and V_Q are vector bundles of rank 2 and 1, respectively), and*
- *θ maps V_P to $V_Q \otimes \Omega^1_M$, and*
- *θ maps V_Q to $V_P \otimes \Omega^1_M$.*

6.2 Higgs bundle Toledo invariants

If H is any group, then let $\text{Hom}^+(\text{H}, \text{U}(2,1))$ denote the space of semisimple representations from H into $\text{U}(2,1)$.

Lemma 6.2.1. *There exists a surjective function* $\phi : \mathcal{H} \to \mathrm{Hom}^{+}(\pi_1(X), \mathrm{U}(2,1))$*, where* $\mathcal{H}$ *is the set of all polystable* U(2,1) *Higgs bundles* (V, θ) *on* X *whose summands have vanishing Chern classes.*

Proof. Let $\mathcal{H}'$ be the set of all polystable rank 3 Higgs bundles (V, θ) on X whose summands have vanishing Chern classes. By Lemma 4.2.2, X is algebraic, hence compact Kähler. In [31], Simpson shows that there is a surjective function

$$\phi : \mathcal{H}' \to \mathrm{Hom}^{+}(\pi_1(X), \mathrm{GL}(3, \mathbb{C})).$$

In [39, Proposition 3.1], Xia shows that

$$\mathcal{H} = \phi^{-1}(\mathrm{Hom}^{+}(\pi_1(X), \mathrm{U}(2,1))).$$

(Xia's proof is for Riemann surfaces, but it goes through for any compact Kähler manifold.) □

Lemma 6.2.2 (Xia). *Let* $\mathcal{H}$ *and* ϕ *be as in Lemma 6.2.1, and let* $(V, \theta) \in \mathcal{H}$*. Write* $V = V_P \oplus V_Q$ *as in Def. 6.1.1. Then* $\tau(\phi(V, \theta)) = c_1(V_P)$*.*

Sketch of proof. We briefly describe here how to make this computation. Let P be the natural principal $\mathrm{U}(2) \times \mathrm{U}(1)$ bundle over $\mathbf{H}^2_{\mathbb{C}} \cong \frac{\mathrm{U}(2,1)}{\mathrm{U}(2)\times\mathrm{U}(1)}$. The Cartan decomposition of $\mathrm{U}(2,1)$ gives us P a natural connection D, which splits into a "U(2)" part and a "U(1)" part. The trace of the curvature of the "U(2)" part of this connection, times $\frac{i}{2\pi}$, is precisely ω. Pulling P back via π_B, then pushing it forward via φ, then pulling it back via s (as in the definition of the Toledo invariant) gives us a principal $\mathrm{U}(2) \times \mathrm{U}(1)$ bundle over X. The associated vector bundle splits into a rank 2 bundle and a rank 1 bundle—these are precisely V_P and V_Q. The first Chern class $c_1(V_P)$ is given by $\frac{i}{2\pi}$ times the trace of the curvature of any connection on V_P. To obtain a connection on V_P, pull D back via π_B, then push it forward via φ, then pull it back via s. We then find that $\frac{i}{2\pi}$ times the trace of the curvature of this connection is $s^*\varphi_*\pi_B^*\omega = \tau(\phi(V,\theta))$. $\square$

Remark: Lemmas 6.2.1 and 6.2.2 serve as a bridge from the world of semisimple representations and Toledo invariants to the world of polystable Higgs bundles and Chern classes. Consequently, while the definition of the Toledo invariant is topological in nature, these two lemmas enable us to employ algebro-geometric techniques in order to compute which Toledo invariants actually occur.

Definition 6.2.3. *If* $(V,\theta) = (V_P \oplus V_Q, \theta)$ *is a* $\mathrm{U}(2,1)$ *Higgs bundle as in Def. 6.1.1, then we define the* Higgs bundle Toledo invariant $\tau_{(V,\theta)}$ *by* $\tau_{(V,\theta)} = \frac{1}{3}(c_1(V_P) - 2c_1(V_Q))$.

Note that if $(V,\theta) \in \mathcal{H}$, then V is flat, in which case Def. 6.2.3 is consistent with Lemma 6.2.2.

Remark: Consider a semisimple representation $\rho : \pi_1(X) \to \mathrm{PU}(2,1)$. The corresponding principal $\mathrm{PU}(2,1)$ bundle lifts to a principal $\mathrm{U}(2,1)$ bundle with an associated vector bundle $V = V_P \oplus V_Q$. One can show that $\tau(\rho) = \frac{1}{3}(c_1(V_P) - 2c_1(V_Q))$; the proof is similar to that of Lemma 6.2.2. This is the motivation for Def. 6.2.3.

Lemma 6.2.4. *Let* $(V,\theta) \in \mathcal{H}$*, and let* L *be a line bundle. Then:*

(i) $(V \otimes L, \theta \otimes 1)$ *is a polystable* $\mathrm{U}(2,1)$ *Higgs bundle with* $\tau_{(V\otimes L,\theta\otimes 1)} = \tau_{(V,\theta)}$.

(ii) $(V^*, \theta) \in \mathcal{H}$*, and* $\tau_{(V^*,\theta)} = -\tau_{(V,\theta)}$.

Chapter 7

Systems of Hodge bundles on Dolgachev surfaces

The results of Chapter 6 imply that to compute Toledo invariants of semisimple U(2,1) representations of the fundamental group of a Dolgachev surface, it suffices to compute Chern classes of the summands of certain polystable U(2,1) Higgs bundles. The goal of this chapter is to show that we may restrict our attention to a special class of these Higgs bundles, namely systems of Hodge bundles. The method is due to Simpson. (See [31] and [32]). Following Xia [39], we then divide these systems of Hodge bundles into two types, binary and ternary.

7.1 Binary and ternary Higgs bundles

Definition 7.1.1 ([32]). *Let M be a complex algebraic manifold. A* system of Hodge bundles *on M is a Higgs bundle (V, θ) such that $V = \bigoplus V^{r,s}$ and*

$$\theta : V^{r,s} \to V^{r-1,s+1} \otimes \Omega^1_M.$$

Lemma 7.1.2. *(a) There exists a quasiprojective variety $\mathcal{M}_{\mathrm{Dol}}$ whose points parametrize* *direct sums of stable Higgs bundles with vanishing Chern classes.*

(b) Let f be the map from $\mathcal{M}_{\mathrm{Dol}}$ to the space of polynomials with coefficients in symmetric powers of the cotangent bundle which takes (V, θ) to the characteristic polynomial of θ. Then f is proper.

(c) Let $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ denote the subspace of $\mathcal{M}_{\mathrm{Dol}}$ whose points parametrize polystable U(2,1) *bundles. Then every connected component of $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ contains a system of Hodge bundles.*

Proof. (a) [32, Prop. 1.4]

(b) [32, Prop. 1.4]

(c) In [32, Theorem 3], Simpson proves that every component of $\mathcal{M}_{\mathrm{Dol}}$ contains a system of Hodge bundles, as follows. Let $\mathbb{C}^*$ act on $\mathcal{M}_{\mathrm{Dol}}$ by

$$t \cdot (V, \theta) = (V, t\theta).$$

As $t \to 0$, we have $f(t \cdot (V, \theta)) \to 0$. Since f is proper, $t \cdot (V, \theta)$ converges to a limit Higgs bundle (V_0, θ_0). Since $\mathcal{M}_{\mathrm{Dol}}$ is Hausdorff, the limit is unique. Consequently, (V_0, θ_0) is fixed under the action of $\mathbb{C}^*$ and is therefore a system of Hodge bundles [32, Lemma 4.1].

Since U(2,1) is closed in $\mathrm{GL}(3, \mathbb{C})$, we have that $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ is closed in $\mathcal{M}_{\mathrm{Dol}}$. Therefore, f restricted to $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ is still proper, and the above proof goes through unchanged. □

Definition 7.1.3 ([39]). *We say a Higgs bundle (V, θ) is* binary *if:*

(i) $V = V_P \oplus V_Q$ where V_P and V_Q are vector bundles of rank 2 and 1, respectively,

and

(ii) θ maps V_P to $V_Q \otimes \Omega^1_X$ and V_Q to 0.

In this situation, denote (V, θ) by $V_P \xrightarrow{\theta \oplus} V_Q$ (omitting θ if it's clear from the context).

We say a Higgs bundle (V, θ) is ternary *if:*

(i) $V = V_2 \oplus V_3 \oplus V_1$ where V_1, V_2, and V_3 are line bundles, and

(ii) θ maps V_2 to $V_3 \otimes \Omega^1_X$, maps V_3 to $V_1 \otimes \Omega^1_X$, and maps V_1 to 0.

In this situation, denote (V, θ) by $V_2 \overset{\oplus}{\to} V_3 \overset{\oplus}{\to} V_1$.

It follows from Def. 7.1.3 and 6.2.4 that if a polystable U(2,1) Higgs bundle is a system of Hodge bundles, then it is either ternary, binary, or dual to a binary bundle. Also, every polystable Higgs bundle is either stable or reducible. We therefore investigate the following four types of polystable U(2,1) Higgs bundles: (1) stable ternary, (2) stable binary, (3) reducible ternary, and (4) reducible binary.

7.2 The case of the stable ternary Higgs bundle

Proposition 7.2.1. *Let $V_2 = \mathcal{O}_X(bF + \sum b_k F_k)$ and $V_1 = \mathcal{O}_X(aF + \sum a_k F_k)$. Then there exists a Higgs field θ such that $(V, \theta) = V_2 \overset{\oplus}{\to} \mathcal{O}_X \overset{\oplus}{\to} V_1$ is a stable ternary Higgs* *bundle if and only if:*

(i) $b \leq -2$, *and*

(ii) $a + \#\{k \,|\, a_k \neq 0\} \geq 2$, *and*

(iii) $2A < B$, *and*

(iv) $A < 2B$.

Here we have used the notations $A = a + \sum \frac{a_k}{m_k}$ *and* $B = b + \sum \frac{b_k}{m_k}$.

Before proving this proposition, we will first prove several preparatory lemmas.

Lemma 7.2.2. *There exists a short exact sequence*

$$0 \to \mathcal{O}_X(-2F + \sum_k (m_k - 1)F_k) \to \Omega^1_X \to I_Z \otimes \mathcal{O}_X(F) \to 0$$

where Z is the reduced subscheme associated to the set of singular points of singular fibres of X, and I_Z is its ideal sheaf.

Proof. Pullback of holomorphic 1-forms via π gives rise to an injection of sheaves

$$0 \to \pi^*\Omega^1_{\mathbb{CP}^1} \to \Omega^1_X$$

[2, p. 98]. Let $\Omega^1_{X/\mathbb{CP}^1}$ denote the sheaf of relative differentials (i.e., the cokernel of this map). We know that $\Omega^1_{\mathbb{CP}^1} = \mathcal{O}_{\mathbb{CP}^1}(-2)$, and so we have that $\pi^*\Omega^1_{\mathbb{CP}^1} = \mathcal{O}_X(-2F)$. From the short exact sequence

$$0 \to \mathcal{O}_X(-2F) \to \Omega^1_X \to \Omega^1_{X/\mathbb{CP}^1} \to 0,$$

we have that

$$\mathcal{O}_X(-F + \sum_k (m_k - 1)F_k) = K_X = \det(\Omega^1_X) = \mathcal{O}_X(-2F) \otimes \det\left(\Omega^1_{X/\mathbb{CP}^1}\right);$$

tensoring with $\mathcal{O}_X(2F)$ shows that

$$\det\left(\Omega^1_{X/\mathbb{CP}^1}\right) = \mathcal{O}_X(F + \sum (m_k - 1)F_k).$$

Let $T = \mathrm{Tor}\left(\Omega^1_{X/\mathbb{CP}^1}\right)$, where $\mathrm{Tor}(\mathcal{S})$ denotes the torsion part of a sheaf $\mathcal{S}$. We claim that T is isomorphic to

$$\bigoplus_{k=1}^{n} \mathcal{O}_{(m_k-1)F_k}((m_k-1)F_k),$$

where $\mathcal{O}_{(m_k-1)F_k}((m_k-1)F_k)$ is the cokernel of the natural inclusion

$$\mathcal{O}_X \to \mathcal{O}_X((m_k-1)F_k).$$

To prove this claim, we first observe that the support of T is contained in the union of the multiple fibres of X [2, p. 98]. Let F_k be a multiple fibre, and let $\{U_\alpha\}$ be a collection of coordinate neighborhoods as in Lemma 4.3.4. It suffices to show that $T|_{\cup U_\alpha}$ is isomorphic to $\mathcal{O}_{(m_k-1)F_k}((m_k-1)F_k)$.

Let V be an open subset of $\cup U_\alpha$. A section s of $\Omega^1_{X/\mathbb{CP}^1}(V)$ is given by a collection $\{(V_\alpha, s_\alpha)\}$ where $\cup V_\alpha = V$, $s_\alpha \in \Omega^1_X(V_\alpha)$, and $s_\beta - s_\alpha \in \pi^*\Omega^1_{\mathbb{CP}^1}(V_\alpha \cap V_\beta)$. Without loss of generality, we assume that $V_\alpha \subset U_\alpha$ for each α. For coordinates on V_α, we take the coordinates (w_α, z_α) from U_α, as in Lemma 4.3.4. Now, $\Omega^1_X(V_\alpha)$ is free; its generators are dw_α and dz_α. Also, $\pi^*\Omega^1_{\mathbb{CP}^1}(V_\alpha)$ is free, with generator

$$\pi^*(du) = d(w_\alpha^{m_k}) = (m_k-1)w_\alpha^{m_k-1}dw_\alpha,$$

where u is the local coordinate on $\mathbb{CP}^1$. We see then that locally, $\Omega^1_{X/\mathbb{CP}^1}$ has two generators, dw_α and dz_α, subject to the relation $w_\alpha^{m_k-1}dw_\alpha = 0$. Therefore, T is given locally by the one generator dw_α subject to the relation $w_\alpha^{m_k-1}dw_\alpha = 0$.

Similarly, we find that $\mathcal{O}_{(m_k-1)F_k}((m_k-1)F_k)$ is given locally by one generator, $w_\alpha^{1-m_k}$, subject to the rather odd-looking relation $w_\alpha^{m_k-1} \cdot w_\alpha^{1-m_k} = 0$. Consequently, the map from T to $\mathcal{O}_{(m_k-1)F_k}((m_k-1)F_k)$ that sends dw_α to $w_\alpha^{1-m_k}$ is a well-defined isomorphism of sheaves.

From the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\sum_k (m_k - 1)F_k) \to T \to 0$$

we find that $\det(T) = \mathcal{O}_X(\sum(m_k - 1)F_k)$. Let $Q = \frac{\Omega^1_{X/\mathbb{CP}^1}}{T}$. From the short exact sequence

$$0 \to T \to \Omega^1_{X/\mathbb{CP}^1} \to Q \to 0$$

we calculate that

$$\det(Q) = \det(T)^* \otimes \det\left(\Omega^1_{X/\mathbb{CP}^1}\right) = \mathcal{O}_X(F).$$

We have a natural map $\Omega^1_X \to Q$, which is surjective. Let N be the kernel of this map. We then have a short exact sequence

$$0 \to N \to \Omega^1_X \to Q \to 0. \tag{7.1}$$

Because Ω^1_X is reflexive and Q is torsion-free, we conclude that N is reflexive [26], hence locally free (since $\dim(X) = 2$). In fact, N is a line bundle, since

$$\operatorname{rank}(N) = \operatorname{rank}\left(\Omega^1_X\right) - \operatorname{rank}(Q) = 2 - 1 = 1.$$

From (7.1) above we compute that

$$N = \mathcal{O}_X(-2F + \sum(m_k - 1)F_k).$$

Since Q is torsion-free, we have that

$$Q = I_Z \otimes \det(Q) = I_Z \otimes \mathcal{O}_X(F)$$

for some codimension 2 subscheme Z [12, p. 33]. The total space of Z is the set of points where Q is not locally free. Now, $\Omega^1_{X/\mathbb{CP}^1}$ fails to be locally free precisely where

π is singular. Since T is supported on the union of the multiple fibres, $Q = \frac{\Omega^1_{X/\mathbb{CP}^1}}{T}$ will fail to be locally free at every singular point of π outside of the multiple fibres. In particular, Z contains the set of singular points of the 12 singular fibres. From (7.1) and the equation [12, p. 29]

$$c_2\left(\Omega^1_X\right) = c_1(N) \cdot c_1\left(\mathcal{O}_X(F)\right) + \ell(Z)$$

(where $\ell(Z)$ is the length of Z), we find that

$$\ell(Z) = c_2\left(\Omega^1_X\right) = 12.$$

We conclude that Z is the subscheme of X associated to the set of singular points of the singular fibres, each point taken with multiplicity one. The exact sequence (7.1) then has the desired form. □

Remark. It will follow from the results of this chapter that Lemma 7.2.2 computes the Harder-Narasimhan filtration of Ω^1_X. □

Throughout this chapter, let N, Q, and Z be as in Lemma 7.2.2.

Lemma 7.2.3. *Let $A = aF + \sum a_k F_k$ be a vertical divisor. If*

$$H^0(\mathcal{O}_X(-A) \otimes Q) \neq 0,$$

then $H^0(\mathcal{O}_X(-A) \otimes N) \neq 0$ and $\deg(\mathcal{O}_X(A)) < 0$.

Proof. A nonzero global section s of $\mathcal{O}_X(-A) \otimes Q$ is a nonzero global section of $\mathcal{O}_X(-A + F)$ that vanishes on the total space of Z. Since $-A + F$ is vertical, s is constant on fibres, by Lemma 4.3.2. Thus s vanishes identically on each singular fibre of X, and hence can be regarded as a nonzero global section of

$$\mathcal{O}_X(-A + F - \sum_{j=1}^{12}(E_j)).$$

Now, $-A + F - \sum_{j=1}^{12}(E_j)$ is linearly equivalent to

$$(-11 - a - \#\{k|a_k \neq 0\})F + \sum_{a_k \neq 0}(m_k - a_k)F_k,$$

so by Lemma 4.3.1,

$$a \leq -11 - \#\{k|a_k \neq 0\} \leq -2.$$

Again by Lemma 4.3.1,

$$h^0(\mathcal{O}_X(-A) \otimes N) = (-2 - a) + 1 > 0,$$

as desired. Moreover,

$$\begin{aligned} \deg(\mathcal{O}_X(A)) &= \left(a + \sum \frac{a_k}{m_k}\right) \deg(F) \\ &\leq \big(a + \#\{k|a_k \neq 0\}\big) \deg(F) \leq -11 \deg(F) < 0. \quad \square \end{aligned}$$

Lemma 7.2.4. *Let M be a horizontal divisor. Let $A = \mathcal{O}_X(bF + \sum b_k F_k + M)$. If $H^0(A^* \otimes \Omega_X^1) \neq 0$, then $-M$ is linearly equivalent to an effective divisor and $b \leq 1$.*

Proof. Tensoring the exact sequence (7.1) from Lemma 7.2.2 with A^* yields an exact sequence

$$0 \to A^* \otimes N \to A^* \otimes \Omega_X^1 \to A^* \otimes Q \to 0. \tag{7.2}$$

From the long exact sequence associated to (7.2), we see that $H^0(A^* \otimes N) \neq 0$ or $H^0(A^* \otimes Q) \neq 0$. Let $B = bF + \sum b_k F_k$.

If $H^0(A^* \otimes N) \neq 0$, then $-B - M - 2F + \sum(m_k - 1)F_k$ is linearly equivalent to an effective divisor, which implies that $b \leq -2$ and $-M$ is linearly equivalent to an effective divisor.

If $H^0(A^* \otimes Q) \neq 0$, then the inclusion $0 \to A^* \otimes Q \to A^* \otimes \mathcal{O}_X(F)$ shows that $H^0(A^* \otimes \mathcal{O}_X(F)) \neq 0$. Therefore $-B - M + F$ is linearly equivalent to an effective divisor, which implies that $b \leq 1$ and $-M$ is linearly equivalent to an effective divisor. $\square$

Lemma 7.2.5. *Let $B = bF + \sum b_k F_k$. Then $H^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) \neq 0$ if and only if $b \leq -2$.*

Proof. We first assume that $b \leq -2$ and show that $H^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) \neq 0$. Tensoring the exact sequence (7.1) from Lemma 7.2.2 with $\mathcal{O}_X(-B)$, we get a short exact sequence

$$0 \to \mathcal{O}_X(-B) \otimes N \to \mathcal{O}_X(-B) \otimes \Omega^1_X \to \mathcal{O}_X(-B) \otimes Q \to 0 \tag{7.3}$$

Consequently, it suffices to show that $H^0(\mathcal{O}_X(-B) \otimes N) \neq 0$. The nonvanishing of $H^0(\mathcal{O}_X(-B) \otimes N)$ follows from the effectiveness of

$$-B + (-2F + \sum_k (m_k - 1)F_k) = (-2 - b)F + \sum (m_k - 1 - b_k)F_k.$$

(Recall the convention that $b_k < m_k$ for all k.)

We now assume that $H^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) \neq 0$ and show that $b \leq -2$. From the exact sequence (7.3) above, we see that

$$H^0(\mathcal{O}_X(-B) \otimes Q) \neq 0$$

or

$$H^0(\mathcal{O}_X(-B) \otimes N) \neq 0.$$

Either way, $H^0(\mathcal{O}_X(-B) \otimes N) \neq 0$, by Lemma 7.2.3. But then

$$(-2 - b)F + \sum (m_k - 1 - b_k)F_k$$

is linearly equivalent to an effective divisor. Therefore $b \leq -2$. □

Remark: By taking B to be 0 in Lemma 7.2.5, we recover the fact that the irregularity $q = H^0(\Omega^1_X)$ of X vanishes.

Remark: Lemma 7.2.5 says that for a vertical line bundle $\mathcal{L}$, the vector space $\mathrm{Hom}(\mathcal{L}, \Omega^1_X)$ of maps from $\mathcal{L}$ to Ω^1_X is nontrivial if and only if $\mathrm{Hom}(\mathcal{L}, N)$ is nontrivial. In fact, there is an isomorphism $\mathrm{Hom}(\mathcal{L}, N) \cong \mathrm{Hom}(\mathcal{L}, \Omega^1_X)$. To see that this isomorphism exists, consider the exact sequence

$$0 \to \pi_*(\mathcal{L}^* \otimes N) \to \pi_*(\mathcal{L}^* \otimes \Omega^1_X) \to \pi_*(\mathcal{L}^* \otimes Q).$$

Then compute that $\pi_*(\mathcal{L}^* \otimes N)$ is a line bundle on $\mathbb{CP}^1$, that $\pi_*(\mathcal{L}^* \otimes \Omega^1_X)$ is a coherent sheaf of rank 1 on $\mathbb{CP}^1$, and that $\pi_*(\mathcal{L}^* \otimes Q)$ is torsion-free. It follows that

$$h^0(\mathcal{L}^* \otimes N) = h^0(\pi_*(\mathcal{L}^* \otimes N)) = h^0(\pi_*(\mathcal{L}^* \otimes \Omega^1_X)) = h^0(\mathcal{L}^* \otimes \Omega^1_X).$$

The exact sequence associated to the global Hom functor then yields the desired isomorphism.

Lemma 7.2.6. *Let $(V, \theta) = V_2 \overset{\oplus}{\to} \mathcal{O}_X \overset{\oplus}{\to} V_1$ be a ternary Higgs bundle. Then (V, θ) is stable if and only if:*

(ST1) Neither $\theta|_{V_2}$ nor $\theta|_{\mathcal{O}_X}$ is the zero map (here $|$ denotes restriction), and

(ST2) $deg(V_1) < \frac{1}{3} \cdot (deg(V_1) + deg(V_2))$, and

(ST3) $\frac{1}{2} \cdot deg(V_1) < \frac{1}{3} \cdot (deg(V_1) + deg(V_2))$

Remark: "ST" stands for "stable ternary."

Proof. We first assume that (V, θ) is stable, and show that (ST1)–(ST3) hold. If $\theta|_{V_2}$ is the zero map, then both V_2 and $\mathcal{O}_X \oplus V_1$ are θ-invariant. Stability then implies that

$$\mathrm{deg(V_2)} = \mu(\mathrm{V_2}) < \mu(\mathrm{V}) = \frac{1}{3} \cdot (\mathrm{deg\,(V_1) + deg\,(V_2)})$$

and

$$\frac{1}{2} \cdot \deg(V_1) = \mu(V_1) < \mu(V) = \frac{1}{3} \cdot (\deg(V_1) + \deg(V_2)).$$

Multiplying the second of these inequalities by 2 and then adding it to the first, we find that

$$\deg(V_1) + \deg(V_2) < \deg(V_1) + \deg(V_2),$$

a contradiction. One obtains a similar contradiction if $\theta|_{\mathcal{O}_X}$. Consequently, (ST1) holds.

The Higgs field θ maps V_1 to 0. Consequently, V_1 is θ-invariant. Stability then implies that

$$\deg(V_1) < \frac{1}{3} \cdot (\deg(V_1) + \deg(V_2)).$$

This proves (ST2). Similarly, the inequality (ST3) follows from the θ-invariance of $\mathcal{O}_X \oplus V_1$.

We now assume (ST1)–(ST3), and show that (V, θ) is stable. Let $\mathcal{S}$ be a θ-invariant subsheaf of V with $0 < \mathrm{rank}(\mathcal{S}) < 3$. We must show that $\mu(\mathcal{S}) < \mu(V)$.

We have that $\mathrm{rank}(\mathcal{S}) = 1$ or $\mathrm{rank}(\mathcal{S}) = 2$. Let us first take the case where $\mathcal{S}$ has rank 2. Observe that $\mathcal{S}$ is torsion-free, since it is a subsheaf of the torsion-free sheaf V. We claim that we have an inclusion of $\mathcal{S}$ into $\mathcal{O}_X \oplus V_1$. If this claim holds, then we have an injection from $\det(\mathcal{S})$ into $\det(\mathcal{O}_\mathrm{X} \oplus \mathrm{V}_1) = \mathrm{V}_1$ [26], which implies that

$$\mu(\mathcal{S}) = \frac{1}{2} \cdot \deg(\mathcal{S}) \leq \frac{1}{2} \cdot \deg(V_1) < \frac{1}{3} \cdot (\deg(V_1) + \deg(V_2)) = \mu(V),$$

with the next-to-last inequality coming from (ST3). Hence, it suffices to prove our claim.

If L is a summand in the direct sum decomposition $V = V_2 \oplus \mathcal{O}_X \oplus V_1$, then let $\pi_L : V \to V$ denote projection onto L. It is enough to show that $\pi_{V_2}(\mathcal{S}) = 0$. Suppose to the contrary that there is an open set U and $s_1 \in \mathcal{S}(U)$ such that $\pi_{V_2}(s_1) \neq 0$.

We observe that $\theta|_{V_2}$ is injective. To prove this statement, suppose that

$$R_1 = \ker(\theta|_{\mathrm{V}_2}) \neq 0.$$

Now, V_2 is torsion-free, so R_1 is also torsion-free. It follows that R_1 has rank 1; for if R_1 has rank 0, then R_1 is torsion (as well as torsion-free) and therefore vanishes. But if R_1 has rank 1, then $R_2 = \mathrm{im}(\theta|_{V_2})$ has rank 0. So by the same argument, $R_2 = 0$, contradicting (ST1).

The injectivity of $\theta|_{V_2}$ implies that $\theta(\pi_{V_2}(s_1)) \neq 0$. Since $\mathcal{S}$ is θ-invariant, we have that

$$\begin{aligned}\theta(s_1) &= \theta(\pi_{V_2}(s_1)) + \theta(\pi_{\mathcal{O}_X}(s_1)) + \theta(\pi_{V_1}(s_1)) \\ &= \theta(\pi_{V_2}(s_1)) + \theta(\pi_{\mathcal{O}_X}(s_1)) \in \mathcal{S} \otimes \Omega^1_X(U).\end{aligned}$$

We also have that

$$\theta(\pi_{V_2}(s_1)) + \theta(\pi_{\mathcal{O}_X}(s_1)) \in (\mathcal{O}_X \oplus V_1) \otimes \Omega^1_X(U).$$

Consequently, there exists $s_2 \in \mathcal{S}(U) \cap (\mathcal{O}_X \oplus V_1)(U)$ such that $\pi_{\mathcal{O}_X}(s_1) \neq 0$. (One may need to shrink the open set U in order to guarantee s_2's existence.)

Similarly, we find that there exists a nonzero element $s_3 \in \mathcal{S}(U) \cap V_1(U)$. But s_1, s_2, and s_3 are linearly independent over $\mathcal{O}_X(U)$, contradicting the fact that $\mathcal{S}$ has rank 2. Therefore $\pi_{V_2}(\mathcal{S}) = 0$, as desired.

The case where $\mathcal{S}$ has rank 1 is quite similar. In this case we find (using the same sort of argument) that there is an inclusion of $\mathcal{S}$ into V_1. The inequality in (ST3) then shows that $\mu(\mathcal{S}) < \mu(V)$, as desired. □

Lemma 7.2.7. *Suppose that* $(V, \theta) = V_2 \overset{\oplus}{\to} \mathcal{O}_X \overset{\oplus}{\to} V_1$ *is a stable ternary Higgs bundle. Then* V_2 *and* V_1 *are vertical.*

Proof. Let G_1, G_2, M_1, and M_2 be divisors such that G_1 and G_2 are vertical, M_1 and M_2 are horizontal, $V_1 = \mathcal{O}_X(G_1 + M_1)$, and $V_2 = \mathcal{O}_X(G_2 + M_2)$. We wish to show that $M_1 = M_2 = 0$. First we show that M_1 vanishes; suppose to the contrary that $M_1 \neq 0$.

Lemma 7.2.6 implies that there are nonzero maps

$$V_2 \to \mathcal{O}_X \otimes \Omega^1_X$$

and

$$\mathcal{O}_X \to V_1 \otimes \Omega^1_X.$$

That is, $H^0(V_2^* \otimes \Omega^1_X) \neq 0$ and $H^0(V_1 \otimes \Omega^1_X) \neq 0$. It follows from Lemma 7.2.4 that $-M_2$ and M_1 are linearly equivalent to effective divisors. Consequently, we have that $-M_2 \cdot F \geq 0$ and $M_1 \cdot F \geq r$. We also have that $-M_2 \cdot H_0 \geq 0$ and $M_1 \cdot H_0 \geq s$. (Here r, s, and H_0 are as in the Definition 4.4.1, as is t below.)

From the long exact sequence in cohomology associated to the short exact sequence of sheaves obtained by tensoring the exact sequence from Lemma 7.2.2 with V_2^*, we find that either $-G_2 - 2F + \sum(m_k - 1)F_k$ or $-G_2 + F$ is linearly equivalent to an effective divisor. Similarly (this time tensoring with V_1), we have that either $G_1 - 2F + \sum(m_k - 1)F_k$ or $G_1 + F$ is linearly equivalent to an effective divisor. Consequently $G_2 \cdot H \leq t$ and $-G_1 \cdot H \leq t$.

From condition (ST3) in Lemma 7.2.6, we have that $\deg(\mathrm{V}_1) < 2\deg(\mathrm{V}_2)$. On the other hand,

$$\begin{aligned}
\deg(\mathrm{V}_1) &= H \cdot (G_1 + M_1) \\
&\geq -t + s + kr \\
&\geq 2t \qquad \text{(Here we use that } k \geq \tfrac{3t-s}{r} \text{ by our choice of } k\text{.)} \\
&\geq 2H \cdot G_2 \\
&\geq 2H \cdot (G_2 + M_2) \\
&= 2\deg(\mathrm{V}_2),
\end{aligned}$$

yielding the desired contradiction. A similar argument shows that $M_2 = 0$. □

Proof of Prop. 7.2.1. For all k, we have that $F = m_k F$. Consequently,

$$\deg(V_1) = \left(a + \sum \frac{a_k}{m_k}\right)(H \cdot F)$$

and

$$\deg(V_2) = \left(b + \sum \frac{b_k}{m_k}\right)(H \cdot F).$$

It follows that (iii) is equivalent to (ST2) from Lemma 7.2.6 and (iv) is equivalent to (ST3) from Lemma 7.2.6.

If (V, θ) is stable, then (ST1) from Lemma 7.2.6 implies that $H^0(V_2^* \otimes \Omega_X^1) \neq 0$ and $H^0(V_1 \otimes \Omega_X^1) \neq 0$. Now,

$$V_1^* = \mathcal{O}_X \left((-a - \#\{k \mid a_k \neq 0\})F + \sum_{a_k \neq 0} (m_k - a_k)F_k \right).$$

Lemma 7.2.5 then implies that (i) is equivalent to the nonvanishing of $H^0(V_2^* \otimes \Omega_X^1)$ and (ii) is equivalent to the nonvanishing of $H^0(V_1 \otimes \Omega_X^1)$.

On the other hand, if (i) and (ii) hold, then let θ_2' be a nonzero global map from V_2^* to N and θ_1' a nonzero global map from V_1 to N. (The proof of Lemma 7.2.5 shows that θ_2' and θ_1' exist.) Let ι be the inclusion $N \hookrightarrow \Omega_X^1$ as in Lemma 7.2.2. Let $\theta_1 = \iota \circ \theta_1'$ and $\theta_2 = \iota \circ \theta_2'$. Let (w_γ, z_γ) be coordinates on V_γ, as in the discussion following Lemma 4.3.4. On V_γ, then, θ_1 has the form $g_1 dw_\gamma$ for some meromorphic function g_1. Likewise, $\theta_2 = g_2 dw_\gamma$ on V_γ for some meromorphic g_2. Define θ by $\theta|V_2 = \theta_2$, $\theta|\mathcal{O}_X = \theta_1$, and $\theta|V_1 = 0$. Then $\theta \wedge \theta = \theta_1 \wedge \theta_2 = 0$ on V_γ. Similarly, we find that $\theta \wedge \theta$ vanishes outside the union of the singular fibres and the multiple fibres. Hence $\theta \wedge \theta = 0$ everywhere, making (V, θ) a $\mathrm{U}(2,1)$ Higgs bundle such that (ST1) from Lemma 7.2.6 holds. □

7.3 The case of the stable binary Higgs bundle with rank(im(θ))=1

Let $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ be a stable projectively flat binary Higgs bundle. When restricted to V_P, the Higgs field $\theta|V_P$ is a map from V_P to Ω_X^1. The image $\mathrm{im}(\theta|V_P)$

of this map is a subsheaf of Ω^1_X. Arguing as in (ST1) of Lemma 7.2.6, one sees that $\theta|V_P$ cannot be the zero map. It follows that $\mathrm{im}(\theta|V_P)$ has rank 1 or rank 2. We shall take these cases separately, beginning with the rank 1 case.

Proposition 7.3.1. *If $(V,\theta) = V_P \stackrel{\oplus}{\longrightarrow} \mathcal{O}_X$ is a stable projectively flat binary Higgs bundle with* $\mathrm{rank}(\mathrm{im}(\theta|V_P)) = 1$*, then V_P can be written as an extension of the form*

$$0 \to V_1 \to V_P \stackrel{\beta}{\to} V_2 \to 0, \tag{7.4}$$

where $V_1 = \mathcal{O}_X(aF + \sum a_k F_k)$ and $V_2 = \mathcal{O}_X(bF + \sum b_k F_k)$ with the a's and b's subject to the following numerical conditions:

(i) $-B < A < \frac{1}{2}B$*, and*

(ii) $d_2 \leq -2$*, and*

(iii) $b \leq -2$*, and*

(iv) *If $(c, c_1, \ldots, c_n)$ is an $(n+1)$-tuple of integers such that $0 \leq c_k < m_k$ for all k and $d_1 \geq 0$ and $C \geq \frac{2}{3}(A+B)$, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.*

Here we have used the notations $A = a + \sum \frac{a_k}{m_k}$; $B = b + \sum \frac{b_k}{m_k}$; $C = c + \sum \frac{c_k}{m_k}$; $d_1 = b - c - \#\{b_k < c_k\}$; $d_2 = a - b - \#\{a_k < b_k\}$; and $d_3 = a - c - \#\{a_k < c_k\}$.

Conversely, given a's and b's satisfying (i)–(iv)*, let $V_1 = \mathcal{O}_X(aF + \sum a_k F_k)$ and $V_2 = \mathcal{O}_X(bF + \sum b_k F_k)$. Then there exists a stable projectively flat binary Higgs bundle $V_P \stackrel{\oplus}{\longrightarrow} \mathcal{O}_X$ with V_P given as an extension of the form* (7.4).

Before proving this proposition, we will first prove several preparatory lemmas.

Lemma 7.3.2. *Let $(V,\theta) = V_P \stackrel{\oplus}{\longrightarrow} \mathcal{O}_X$ be a binary Higgs bundle such that im$(\theta|V_P)$*

has rank 1. Let $V_1 = ker(\theta|V_P)$. Then (V, θ) is stable if and only if:

(SB1) $deg(V_1) < \frac{1}{3}\, deg(V_P)$, and

(SB2) $deg(\mathcal{S}) < \frac{2}{3}\, deg(V_P)$ for every rank 1 subsheaf $\mathcal{S}$ of V_P, and

(SB3) $deg(V_P) > 0$.

Proof. Let us first assume that (V, θ) is stable. Then (SB1)–(SB3) follow directly from the fact that the θ-invariant subsheaves $V_1, \mathcal{S} \oplus \mathcal{O}_X$, and $\mathcal{O}_X$, respectively, do not destabilize V.

Let us now assume that (SB1)–(SB3) hold, and show that (V, θ) is stable. Let $\mathcal{S}'$ be a proper θ-invariant subsheaf of V. We must show that $\mathcal{S}'$ is not destabilizing. Arguing as in the proof of Lemma 7.2.6, we find that one of the following three possibilities must hold:

(i) $\mathcal{S}'$ is a rank 1 subsheaf of V_1, or

(ii) $\mathcal{S}' = \mathcal{S} \oplus \mathcal{O}_X$, where $\mathcal{S}$ is a rank 1 subsheaf of V_P, or

(iii) $\mathcal{S}'$ is a rank 1 subsheaf of $\mathcal{O}_X$.

The inequalities in (SB1)–(SB3) then imply that $\mathcal{S}'$ is not destabilizing. □

Lemma 7.3.3. *Let $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ be a binary Higgs bundle. Then V is projectively flat if and only if $c_1^2(V_P) = 3c_2(V_P)$.*

Proof. [26]

Lemma 7.3.4. *Let $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ be a stable projectively flat binary Higgs bundle such that $im(\theta|V_P)$ has rank 1. Let $V_1 = ker(\theta|V_P)$ and $V_2 = im(\theta|V_P)$. Then V_1 and V_2 are vertical line bundles.*

Proof. We have an exact sequence $0 \to V_1 \to V_P \to V_2 \to 0$. Now, V_2 is a subsheaf of Ω^1_X, hence torsion-free. V_P is a vector bundle, hence reflexive. It follows that V_1

is reflexive [26], hence locally free (since dim X=2). Therefore there exist vertical divisors G_1 and G_2, horizontal divisors M_1 and M_2, and a dimension 0 subscheme $\tilde{Z}$ such that $V_1 = \mathcal{O}_X(G_1 + M_1)$ and $V_2 = I_{\tilde{Z}} \otimes \mathcal{O}_X(G_2 + M_2)$, where $I_{\tilde{Z}}$ is the ideal sheaf associated to $\tilde{Z}$ [12]. Let $\ell(\tilde{Z})$ denote the length of $\tilde{Z}$. Then there is a natural inclusion

$$0 \to \mathcal{O}_X(-\ell(\tilde{Z})F + G_2 + M_2) \to V_2.$$

Since V_2 is the image of $\theta|V_P$, which maps to Ω^1_X, we have that

$$h^0(\mathcal{O}_X(\ell(\tilde{Z})F - G_2 - M_2) \otimes \Omega^1_X) \geq h^0(V_2^* \otimes \Omega^1_X) > 0.$$

So by Lemma 7.2.4, $-M_2$ is effective.

We claim that $M_2 = 0$. Suppose to the contrary that M_2 is nonzero. Then

$$\deg(V_2) = H \cdot (G_2 + M_2) = H_0 \cdot (G_2 + M_2) + kF \cdot M_2 \leq C - kr < 0.$$

(Here H_0, k, C, and r are as in Definition 4.4.1.) Now since

$$\deg(V_P) = \deg(V_1) + \deg(V_2),$$

we have that (SB1) from Lemma 7.3.2 is equivalent to the inequality

$$\deg(V_1) < \frac{1}{2}\deg(V_2).$$

Consequently, $\deg(V_1) < 0$. But then $\deg(V_P) < 0$, which contradicts (SB3) from Lemma 7.3.2.

We now prove that V_1 is vertical. Write G_2 as $\mathcal{O}_X(bF + \sum b_k F_k)$. Lemma 7.2.5 implies that $b \leq -2$. Conditions (SB1) and (SB3) from Lemma 7.3.2, together with Def. 4.4.1, show that $(G_1 + M_1) \cdot F = 0$. From the exact sequence

$$0 \to \mathcal{O}_X(G_1 + M_1) \to V_P \to I_{\tilde{Z}} \otimes \mathcal{O}_X(G_2) \to 0,$$

we find that

$$c_2(V_P) = M_1 \cdot G_2 + \ell(\tilde{Z})$$

and

$$c_1^2(V_P) = M_1 \cdot G_1 + M_1 \cdot G_2 + M_1^2.$$

By Lemma 7.3.3, we have

$$(G_1 - M_1)^2 = 3\ell(\tilde{Z}) \geq 0.$$

The Hodge index theorem, applied to

$$(H \cdot F)(G_1 - M_1) - [H \cdot (G_1 - M_1)]F,$$

then shows that $G_1 - M_1$ is vertical. Therefore, $M_1 = 0$, so V_1 is vertical.

From the equation

$$0 = c_1^2(V_P) = 3c_2(V_P) = 3M_1 \cdot G_2 + 3\ell(\tilde{Z}) = 3\ell(\tilde{Z}),$$

we then find that V_2 is a vertical line bundle, as desired. □

Lemma 7.3.5. *Let $V_1 = \mathcal{O}_X(aF + \sum a_k F_k)$ and $V_2 = \mathcal{O}_X(bF + \sum b_k F_k)$ be vertical line bundles such that $d_2 \leq -2$, where $d_2 = a - b - \#\{a_k < b_k\}$.*

If there is a nonsplit extension of the form

$$0 \to V_1 \to V_P \xrightarrow{\beta} V_2 \to 0, \tag{7.5}$$

and $L = \mathcal{O}_X(cF + \sum c_k F_k)$ is a vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$, where $d_1 = b - c - \#\{b_k < c_k\}$ and $d_3 = a - c - \#\{a_k < c_k\}$, such that $H^0(L^* \otimes V_P) = 0$, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.

Conversely, there exists a nonsplit extension (7.5) such that if

$$L = \mathcal{O}_X(cF + \sum c_k F_k)$$

is any vertical line bundle with $d_1 \geq 0$ *and* $d_3 \leq -2$ *such that*

$$d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1),$$

then $H^0(L^* \otimes V_P) = 0$.

Proof. First, we show that if V_P and L are subject to the given conditions, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.

Observe that $L^* \otimes V_1 = \mathcal{O}_X(d_3 F + \sum r_k F_k)$ for some r_k with $r_k \geq 0$. The condition $d_3 \leq -2$ then implies that $H^0(L^* \otimes V_1) = 0$. Similarly, $L^* \otimes V_2 = \mathcal{O}_X(d_1 F + \sum r_k F_k)$ for some r_k with $r_k \geq 0$, so the condition $d_1 \geq 0$ implies that $H^0(L^* \otimes V_2) \neq 0$. Consider the short exact sequence

$$0 \to L^* \otimes V_1 \to L^* \otimes V_P \to L^* \otimes V_2 \to 0. \tag{7.6}$$

The associated long exact sequence in cohomology then implies that the coboundary map $\delta : H^0(L^* \otimes V_2) \to H^1(L^* \otimes V_1)$ is injective. Consequently,

$$h^0(L^* \otimes V_2) \leq h^1(L^* \otimes V_1),$$

and so by Lemma 4.3.1, we have that $d_1 + 1 \leq -d_3 - 1$.

We now show that $d_1 + 1 \leq -d_2 - 1$. Suppose that $d_1 + 1 > -d_2 - 1$. Let σ be an element of $H^1(V_2^* \otimes V_1)$ which defines the extension (7.5), as in [18]. Note that $V_2^* \otimes V_1 = \mathcal{O}_X(d_2 F + \sum r_k F_k)$ for some r_k with $r_k \geq 0$. Taking notation from Lemma 4.3.5(ii), we have that σ equals $\sigma_{_{-1}} w_\gamma^{-1} + \cdots + \sigma_{_{d_2+1}} w_\gamma^{d_2+1}$ on $V_\gamma \cap W_\xi$ and 0 elsewhere for some $\sigma_{_{-1}}, \ldots, \sigma_{_{d_2+1}}$. Let $\{\phi'_{\alpha\beta}\}$ be a system of transition functions for the line bundle $L^* \otimes V_1$, and let $\{\phi''_{\alpha\beta}\}$ be a system of transition functions for the line bundle $L^* \otimes V_2$. We may regard σ as the extension class of (7.6). Transition matrices for $L^* \otimes V_P$ are then given by $\begin{pmatrix} \phi'_{\alpha\beta} & \phi''_{\alpha\beta}\sigma_{\alpha\beta} \\ 0 & \phi''_{\alpha\beta} \end{pmatrix}$. If $s \in H^0(L^* \otimes V_2)$, one can then compute that $\delta(s)$ equals $s_\gamma \sigma_{\gamma\xi}$ on $V_\gamma \cap W_\xi$ and 0 elsewhere.

Let $s \in H^0(L^* \otimes V_2)$ be the nonzero section such that with respect to the trivialization on V_γ, we have $s_\gamma = w_\gamma^{d_1}$, as in Lemma 4.3.5(i). Then

$$\delta(s) = \sigma_{_{-1}} w_\gamma^{d_1-1} + \cdots + \sigma_{_{d_2+1}} w_\gamma^{d_1+d_2+1}$$

on $V_\gamma \cap W_\xi$ and equals 0 elsewhere. So, by Lemma 4.3.5(ii) and the inequality $d_1+1 > -d_2-1$, we have that $\delta(s) = 0 \in H^1(L^* \otimes V_1)$. But since δ is injective, this yields the desired contradiction.

We now show that there exists a nonsplit extension (7.5) such that if

$$L = \mathcal{O}_X(cF + \sum c_k F_k)$$

is any vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$ such that

$$d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1),$$

then $H^0(L^* \otimes V_P) = 0$. Let $(\sigma_{_{-1}}, \sigma_{_{-2}}, \ldots, \sigma_{_{d_2+1}})$ be a $(-d_2-2)$-tuple of complex numbers such that for any ℓ_1, ℓ_3 with $\ell_1 \geq 0$ and $\ell_3 \leq -2$ such that

$$\ell_1 + 1 \leq \min(-d_2 - 1, -\ell_3 - 1),$$

the matrix

$$\Theta_{\ell_1,\ell_3} = \begin{pmatrix} \sigma_{_{\ell_3+1}} & \sigma_{_{\ell_3}} & \cdots & \sigma_{_{d_2+1}} & 0 & 0 & \cdots & 0 \\ \sigma_{_{\ell_3+2}} & \sigma_{_{\ell_3+1}} & \cdots & \sigma_{_{d_2+2}} & \sigma_{_{d_2+1}} & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & \vdots \\ \sigma_{_{\ell_3+d_2+1}} & \sigma_{_{\ell_3+d_2}} & \cdots & & \cdots & & \cdots & \sigma_{_{d_2+1}} \\ \vdots & & & & & & & \vdots \\ \sigma_{_{-1}} & \sigma_{_{-2}} & \cdots & & \cdots & & \cdots & \sigma_{_{-\ell_1-1}} \end{pmatrix}$$

has maximal rank. (One may construct such a sequence of σ's by induction on $-d_2-1$; given $\sigma_{_{-1}}, \sigma_{_{-2}}, \ldots, \sigma_{_{d_2+2}}$, choose $\sigma_{_{d_2+1}}$ so that every *square* matrix of the above form has nonzero determinant. This is possible because there are only finitely many such matrices, and for each such matrix, the determinant is zero for only finitely many values of σ_{d_2+1}.)

Let σ be the element in $H^1(V_2^* \otimes V_1)$ represented by a 1-cocycle which equals $\sigma_{-1}w_\gamma^{-1} + \cdots + \sigma_{d_2+1}w_\gamma^{d_2+1}$ on $V_\gamma \cap W_\xi$ and 0 elsewhere. Let V_P be the rank 2 bundle given as an extension as in (7.5) whose extension class is determined by σ. Since σ is nonzero, (7.5) does not split. Let $L = \mathcal{O}_X(cF + \sum c_k F_k)$ be a vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$ such that $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$. We must show that $H^0(L^* \otimes V_P) = 0$.

The condition $d_3 \leq -2$ guarantees that $H^0(L^* \otimes V_1) = 0$. It therefore suffices to show that the coboundary map $\delta : H^0(L^* \otimes V_2) \to H^1(L^* \otimes V_1)$ is injective. We now show that if $\delta(s) = 0$, then $s = 0$.

Let $s \in H^0(L^* \otimes V_2)$. From Lemma 4.3.5(i), we know that on V_γ, the section s is of the form $s_\gamma = s_0 + s_1 w_\gamma + \cdots + s_{d_1} w_\gamma^{d_1}$ with respect to the trivialization on V_γ. From Lemma 4.3.5(ii), we know that if c is the 1-cocycle given by w^j on $V_\gamma \cap W_\xi$ and 0 elsewhere, then $[c] = 0 \in H^1(L^* \otimes V_1)$ if and only if $j \geq 0$ or $j \leq -d_3$. Recall that $\delta(s)$ equals $s_\gamma \sigma_{\gamma\xi}$ on $V_\gamma \cap W_\xi$ and 0 elsewhere. Therefore, $\delta(s) = 0$ if and only if the following equalities hold:

$$\sigma_{d_3+1}s_0 + \sigma_{d_3}s_1 + \cdots + \sigma_{d_2+1}s_{d_3-d_2} = 0$$

$$\sigma_{d_3+2}s_0 + \sigma_{d_3+1}s_1 + \cdots + \sigma_{d_2+1}s_{d_3-d_2+1} = 0$$

$$\cdots$$

$$\sigma_{d_3+d_2+1}s_0 + \sigma_{d_3+d_2}s_1 + \cdots + \sigma_{d_2+1}s_{d_1} = 0$$

$$\cdots$$

$$\sigma_{-1}s_0 + \sigma_{-2}s_1 + \cdots + \sigma_{-d_1-1}s_{d_1} = 0$$

Since Θ_{d_1,d_3} has maximal rank and $d_1+1 \leq -d_3-1$ (which is to say, regarding the s's as variables, that there are at least as many equations as variables), we conclude that $s = 0$. □

Proof of Prop. 7.3.1. We first show that if $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ is a stable projectively flat binary Higgs bundle with $\mathrm{rank}(\mathrm{im}(\theta|\mathrm{V_P})) = 1$, then V_P has the stated form.

Lemma 7.3.4 implies that $V_1 = \ker(\theta|\mathrm{V_P})$ and $V_2 = \mathrm{im}(\theta|\mathrm{V_P})$ are vertical line bundles; we therefore obtain the extension (7.4). The inequality $-B < A$ then follows from (SB3) of Lemma 7.3.2. The inequality $A < \frac{1}{2}B$ follows from (SB1) of Lemma 7.3.2. Hence, condition (i) holds.

If the extension (7.4) splits, then both V_1 and $V_2 \oplus \mathcal{O}_X$ are θ-invariant, and so $\mu(V_1) < \mu(V)$ and $\mu(V_2 \oplus \mathcal{O}_X) < \mu(V)$, which implies the contradictory inequality $\mu(V) < \mu(V)$. Therefore, (7.4) does not split. The nonsplitting of (7.4) implies that $h^1(V_2^* \otimes V_1) > 0$. We have that

$$V_2^* \otimes V_1 = \mathcal{O}_X(d_2F + \sum r_kF_k),$$

where $0 \leq r_k < m_k$ for all k. Condition (i) implies that $A < B$, so

$$(d_2 + \sum \frac{r_k}{m_k})H \cdot F = \deg(V_2^* \otimes V_1) = (A - B)H \cdot F < 0.$$

Consequently, $d_2 < 0$. Condition (ii) then follows from Lemma 4.3.1(ii) and the inequalities $h^1(V_2^* \otimes V_1) > 0$ and $d_2 < 0$.

Since $\mathrm{im}(\theta|\mathrm{V_P})$ is a subsheaf of Ω^1_X, we must have that $H^0(V_2^* \otimes \Omega^1_X) \neq 0$. Condition (iii) then follows from Lemma 7.2.5.

Let $(c, c_1, \ldots, c_n)$ be an $(n+1)$-tuple of integers such that $0 \leq c_k < m_k$ for all k and $d_1 \geq 0$ and $C \geq \frac{2}{3}(A + B)$. Let $L = \mathcal{O}_X(cF + \sum c_kF_k)$. From (SB2) of Lemma

7.3.2, we know that $H^0(L^* \otimes V_P) = 0$. Note that

$$L^* \otimes V_1 = \mathcal{O}_X(d_3 F + \sum r_k F_k)$$

for some r_k with $0 \leq r_k < m_k$ for all k. Arguing as in the proof that condition (ii) holds, we see that $d_3 < 0$. From the long exact sequence in cohomology associated to (7.6), we find that $H^1(L^* \otimes V_1) \neq 0$. Lemma 4.3.1 then implies that $d_3 \leq -2$. Condition (iv) then follows from Lemma 7.3.5.

Conversely, suppose that we are given a's and b's satisfying conditions (i)–(iv), and let $V_1 = \mathcal{O}_X(aF + \sum a_k F_k)$ and $V_2 = \mathcal{O}_X(bF + \sum b_k F_k)$. We will show that there exists a stable projectively flat binary Higgs bundle $V_P \xrightarrow{\oplus} \mathcal{O}_X$ with $\mathrm{rank}(\mathrm{im}(\theta|\mathrm{V_P})) = 1$ and V_P as in (7.4).

Lemma 7.3.5 and condition (ii) guarantee the existence of a rank 2 bundle V_P and a nonsplit extension (7.4) such that if $L = \mathcal{O}_X(cF + \sum c_k F_k)$ is any vertical line bundle with $d_1 \geq 0$ and $d_3 < 0$ such that $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$, then $H^0(L^* \otimes V_P) = 0$. Let V_P be such a bundle. By Lemma 7.2.5 and condition (iii), there exists a nonzero map $\alpha : V_2 \to \Omega^1_X$. Let $V = V_P \oplus \mathcal{O}_X$. Define a Higgs field θ by $\theta|V_P = \alpha \circ \beta$ and $\theta|\mathcal{O}_X = 0$. Note that $\theta \wedge \theta = 0$. Then (V, θ) is a binary Higgs bundle with $\mathrm{rank}(\mathrm{im}(\theta|\mathrm{V_P})) = 1$. Moreover, V is projectively flat by Lemma 7.3.3.

It remains to be shown that (V, θ) is stable. (SB1) and (SB3) from Lemma 7.3.2 follow from condition (i). Let us now verify that (SB2) holds. Suppose to the contrary that there exists a rank 1 subsheaf $\mathcal{S}$ of V_P such that $\deg(\mathcal{S}) \geq \frac{2}{3}\deg(V_P)$. Let L be the kernel of the natural map $V_P \to \frac{\frac{V_P}{\mathcal{S}}}{\mathrm{Tor}(\frac{V_P}{\mathcal{S}})}$. Then L is a line bundle, $\deg(L) \geq \deg(\mathcal{S})$, and $H^0(L^* \otimes V_P) \neq 0$. (See [26].)

We claim that L is vertical. To see this, consider the short exact sequence (7.6) from Lemma 7.3.5. Since

$$\deg(L) \geq \deg(\mathcal{S}) \geq \frac{2}{3}\deg(V_P) > \deg(V_1),$$

we must have $H^0(L^* \otimes V_1) = 0$. From the long exact sequence in cohomology associated to (7.6), we then find that $H^0(L^* \otimes V_2) \neq 0$. Writing $L = \mathcal{O}_X(G + M)$ with G vertical and M horizontal, we see that

$$-G - M + bF + \sum b_k F_k$$

must be linearly equivalent to an effective divisor. In particular, $-M$ must be linearly equivalent to an effective divisor, and

$$\deg(G) \leq \deg(V_2) \leq \deg(-2F + \sum (m_k - 1)F_k),$$

where the second inequality comes from the condition $b \leq -2$. The condition

$$k \geq \frac{t-s}{r}$$

from Definition 4.4.1 implies that if M is not linearly equivalent to zero, then

$$\deg(M) \leq -\deg(-2F + \sum (m_k - 1)F_k),$$

which would yield the contradictory inequality

$$0 < \frac{2}{3}\deg(V_P) \leq \deg(L) = \deg(M) + \deg(G) \leq 0.$$

Therefore L is vertical.

Write $L = \mathcal{O}_X(cF + \sum c_k F_k)$. Dividing both sides of $\deg(L) \geq \frac{2}{3}\deg(V_P)$ by $H \cdot F$, we find that $C \geq \frac{2}{3}(A + B)$. Note that $L^* \otimes V_2 = \mathcal{O}_X(d_1 F + \sum r_k F_k)$, and so $H^0(L^* \otimes V_2) \neq 0$ implies that $d_1 \geq 0$. It now follows from condition (iv) that $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$. Moreover, $d_3 < 0$ since $H^0(L^* \otimes V_1) = 0$. Our choice of V_P then implies that $H^0(L^* \otimes V_P) = 0$, contradicting our earlier assertion that $H^0(L^* \otimes V_P) \neq 0$. Therefore $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ is stable, as desired. □

7.4 The case of the stable binary Higgs bundle with rank(im(θ))=2

In this subsection, we show that there does not exist a stable binary Higgs bundle (V,θ) on X with rank(im(θ))=2.

Throughout this section, let $N = \mathcal{O}_X(-2F+\sum_k(m_k-1)F_k)$ and $Q = I_Z\otimes\mathcal{O}_X(F)$, as in Lemma 7.2.2.

Lemma 7.4.1. *Suppose that* $(V,\theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ *is a stable projectively flat binary Higgs bundle with* $\mathrm{rank}(\mathrm{im}(\theta)) = 2$*. Then there exists an exact sequence*

$$0 \to V_1 \to V_P \to V_2 \to 0,$$

where V_1 *and* V_2 *are vertical line bundles and* $H^0(V_2^* \otimes Q) \neq 0$*.*

Proof. Let β be the map in the exact sequence of Lemma 7.2.2 from Ω^1_X to Q. Let $V_2 = \mathrm{im}(\beta\circ(\theta|V_P))$, and let $V_1 = \ker(\beta\circ(\theta|V_P))$. This gives us an exact sequence

$$0 \to V_1 \to V_P \to V_2 \to 0.$$

Since $\mathrm{rank}(\mathrm{im}(\theta)) = 2$, we see that $1 = \mathrm{rank}(\mathrm{V}_2) = \mathrm{rank}(\mathrm{V}_1)$. The proof of Lemma 7.3.4 shows that V_1 and V_2 are vertical line bundles. Moreover, the inclusion map $\iota : V_2 \hookrightarrow Q$ yields a nonzero element of $H^0(V_2^* \otimes Q)$. □

Proposition 7.4.2. *If* (V,θ) *is a stable projectively flat binary Higgs bundle, then* $\mathrm{im}(\theta)$ *has rank* 1*.*

Proof. By tensoring with a line bundle, as in Lemma 6.2.4, we may assume that (V,θ) is of the form $V_P \xrightarrow{\oplus} \mathcal{O}_X$. Then $\mathrm{im}(\theta)$ is a subsheaf of Ω^1_X and so has rank 0, 1, or 2. As noted in the introduction to §7.3, $\mathrm{im}(\theta)$ cannot have rank 0.

Suppose $\mathrm{im}(\theta)$ has rank 2. By Lemma 7.4.1, we have an exact sequence

$$0 \to V_1 \to V_P \to V_2 \to 0,$$

where V_1 and V_2 are vertical line bundles and $H^0(V_2^* \otimes Q) \neq 0$. By Lemma 7.2.3, we have that $\deg(V_2) < 0$. As in Lemma 7.3.2, stability implies that $\deg(V_P) > 0$, whence we see that

$$0 < \deg(V_P) = \deg(V_1) + \deg(V_2) < \deg(V_1).$$

The proof of Lemma 7.3.2 also shows that $\deg(V_1) < \frac{2}{3}\deg(V_P)$, whereby one obtains the contradictory inequality

$$0 < \deg(V_1) < 2\deg(V_2) < 0. \quad \square$$

Remark. If we drop the projective flatness condition, then Prop. 7.4.2 is no longer true. For example, if $\{m_k\} = \{2, 3, 7\}$ and $V_P = \Omega^1_X$ and $\theta|V_P$ is given by the identity map, then $V_P \xrightarrow{\oplus} \mathcal{O}_X$ is stable and $\mathrm{im}(\theta)$ has rank 2.

7.5 The case of the reducible ternary Higgs bundle

We now consider reducible, polystable, ternary Higgs bundles of the form

$$(V, \theta) = V_2 \overset{\oplus}{\to} V_3 \overset{\oplus}{\to} V_1.$$

In this case, either $\theta|V_2$ or $\theta|V_3$ must be the zero map. (For if not, then V is not reducible.) We divide into three cases accordingly, depending whether the first map only is zero, the second map only is zero, or both are.

Case 1: $\theta|V_2 = 0$ and $\theta|V_3 \neq 0$

Proposition 7.5.1. *There exists a polystable ternary Higgs bundle*

$$(V, \theta) = V_2 \overset{\oplus}{\to} V_3 \overset{\oplus}{\to} V_1$$

with $\theta|V_2 = 0$ *and* $\theta|V_3 \neq 0$ *and* $c_1(V_2) = c_1(V_3 \oplus V_1) = c_2(V_3 \oplus V_1) = 0$ *if and only if* $V_2 = \mathcal{O}_X$ *and* $V_3 = \mathcal{O}_X(bF + \sum b_k F_k)$ *and* $V_1 = V_3^*$*, where the b's are subject to the following numerical conditions:*

(i) $b + \sum \frac{b_k}{m_k} > 0$*, and*

(ii) $2b + \#\{b_k \geq \frac{m_k}{2}\} \leq -2$.

Proof. First, let $V_2 = \mathcal{O}_X$ and $V_3 = \mathcal{O}_X(bF + \sum b_k F_k)$ and $V_1 = V_3^*$, where the b's satisfy (i) and (ii). Note that

$$V_3 \otimes V_3 = \mathcal{O}_X((2b + \#\{b_k \geq \frac{m_k}{2}\})F + \sum r_k F_k)$$

for some r_k with $0 \leq r_k < m_k$. Condition (ii) guarantees that there exists a nonzero map $\theta : V_3 \to V_1 \otimes \Omega^1_X$, by Lemma 7.2.5. Extend θ to V by letting $\theta|V_2 = \theta|V_1 = 0$; then $\theta \wedge \theta = 0$. Condition (i) guarantees that $V_3 \xrightarrow{\oplus} V_1$ is stable, since every coherent rank 1 θ-invariant subsheaf $\mathcal{S}$ of $V_3 \oplus V_1$ is a subsheaf of V_1, in which case

$$\mu(\mathcal{S}) \leq \mu(V_1) = -\mu(V_3) < 0 = \mu(V_3 \oplus V_1).$$

We have $c_1(V_3 \oplus V_1) = 0$ since $V_1 = V_3^*$. Also, $c_2(V_3 \oplus V_1) = c_1(V_3)c_1(V_1) = 0$ since V_3 and V_1 are vertical.

Now let $(V, \theta) = V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$ be a polystable ternary Higgs bundle with $\theta|V_2 = 0$ and $\theta|V_3 \neq 0$ and $c_1(V_2) = c_1(V_3 \oplus V_1) = c_2(V_3 \oplus V_1) = 0$. Then $V_2 = \mathcal{O}_X$ and $V_1 = V_3^*$, since $c_1(V_2) = 0$ and $c_1(V_3 \oplus V_1) = 0$. Write $V_3 = \mathcal{O}_X(bF + \sum b_k F_k + M)$, where M is a horizontal divisor. Since $\theta|V_3 \neq 0$, we have $H^0((V_3 \otimes V_3)^* \otimes \Omega^1_X) \neq 0$, so Lemma 7.2.4 then implies that $-2M$ is linearly equivalent to an effective divisor and that $2b + \#\{b_k \geq \frac{m_k}{2}\} \leq 1$. Stability of $V_3 \xrightarrow{\oplus} V_1$ and θ-invariance of V_1 show that $\deg(V_3) > 0$. The condition $k > \frac{(n+1)t-s}{r}$ from Definition 4.4.1 then implies that $M \equiv 0$—in other words, V_3 is a vertical line bundle. We obtain condition (i) by diving both sides of the inequality $\deg(V_3) > 0$ by $H \cdot F$. Lemma 7.2.5 yields condition (ii). $\square$

Case 2: $\theta | V_2 \neq 0$ and $\theta | V_3 = 0$.

This case is the same as Case 1, with the V's relabeled.

Case 3: $\theta | V_2 = \theta | V_3 = 0$

This case is trivial; there exists a polystable Higgs bundle $V_2 \overset{\oplus}{\to} V_3 \overset{\oplus}{\to} V_1$ with $c_1(V_2) = c_1(V_3) = c_1(V_1) = 0$ and $\theta | V_2 = \theta | V_3 = 0$ if and only if $V_2 = V_3 = V_1 = \mathcal{O}_X$.

7.6 The case of the reducible binary Higgs bundle

Let $(V, \theta) = V_P \overset{\oplus}{\longrightarrow} V_Q$ be a reducible polystable binary Higgs bundle whose summands have vanishing Chern classes, where rank(V_P)=2 and rank(V_Q)=1. The rank R of the image of θ in $V_Q \otimes \Omega^1_X$ is either 2, 1, or 0. If $R = 2$, then (V, θ) can not be reducible. If $R = 1$, then we must have $V_P = V_1 \oplus V_2$, where $V_1 = \ker(\theta | V_P)$; this case was discussed in section §7.5. If $R = 0$, then θ is the zero map. In this case, we must have $V_Q = \mathcal{O}_X$ and V_P stable. An explicit description of all stable rank 2 bundles on X with vanishing Chern classes can be found in [4, Proposition 4.1]. (The method of proof of Prop. 7.3.1 also yields such a description.)

Chapter 8

Main Theorem and Examples

Putting together the pieces from the previous sections, we have the following explicit description of all orbifold Toledo invariants that arise from semisimple U(2,1) representations of the orbifold fundamental group of the 2-orbifold associated to a Seifert fibered homology 3-sphere.

8.1 Main theorem

Theorem 8.1.1. *Let O be the 2-orbifold of genus zero with n cone points of orders $m_1, \ldots, m_n$ and no other singular points. Assume that $n \geq 3$ and that if $n = 3$, then $\{m_1, m_2, m_3\} \neq \{2, 3, 5\}$. Let $\tau \in \mathbb{R}$. Then there exists a semisimple representation $\rho : \pi_1^{\mathrm{orb}}(O) \to \mathrm{U}(2,1)$ such that $\tau = \tau_{\mathrm{orb}}(\rho)$ if and only if $\tau = \pm(y + \sum \frac{y_k}{m_k})$ for some integers $(y, y_1, \ldots, y_n)$ with $0 \leq y_k < m_k$ such that at least one of the numerical conditions* (i)–(iv) *holds:*

(i) *There exist integers $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$ with $0 \leq a_k, b_k < m_k$ such that $b \leq -2$, and $a + \#\{k \mid a_k \neq 0\} \geq 2$, and $2A < B$, and $A < 2B$, and $(\star)$ below holds.*

(ii) *There exist integers* $a, a_1, \ldots, a_n, b, b_1, \ldots, b_n$ *with* $0 \leq a_k, b_k < m_k$ *such that* $-B < A < \frac{1}{2}B$, *and* $d_2 \leq -2$, *and* $b \leq -2$, *and* $(\star)$ *below holds, and* $d_1+1 \leq \min(-d_2-1, -d_3-1)$ *for every* $(n+1)$*-tuple of integers* $(c, c_1, \ldots, c_n)$ *such that* $0 \leq c_k < m_k$ *for all* k *and* $d_1 \geq 0$ *and* $C \geq \frac{2}{3}(A+B)$.

(iii) $y + \sum \frac{y_k}{m_k} > 0$, *and* $2y + \#\{y_k \geq \frac{m_k}{2}\} \leq -2$.

(iv) $y = y_k = 0$ *for all* k.

$(\star)$ $3y + \sum \lfloor \frac{3y_k}{m_k} \rfloor = a + b$, *and* $3y_k - \lfloor \frac{3y_k}{m_k} \rfloor m_k = a_k + b_k$ *for* $k = 1, \ldots, n$.

Here we have used the notations $A = a + \sum \frac{a_k}{m_k}$; $B = b + \sum \frac{b_k}{m_k}$; $C = c + \sum \frac{c_k}{m_k}$; $d_1 = b - c - \#\{b_k < c_k\}$; $d_2 = a - b - \#\{a_k < b_k\}$; *and* $d_3 = a - c - \#\{a_k < c_k\}$.

Proof. By Lemmas 6.2.2, 6.2.4, 7.1.2(c), and 5.2.3 it suffices to show that

$$c_1(\mathcal{O}_X(yF + \sum y_k F_k))$$

equals the Higgs bundle Toledo invariant of a stable ternary, stable binary, reducible ternary, or reducible binary Higgs bundle whose summands have vanishing Chern classes if and only if the y's satisfy one of (i)–(iv).

Suppose $(V, \theta) = V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$ is a stable ternary Higgs bundle with vanishing Chern classes. By Lemma 6.2.4, tensoring with V_3^* yields a stable ternary Higgs bundle

$$(V',\, \theta') = (V \otimes V_3^*,\, \theta \otimes 1) = (V_2 \otimes V_3^*) \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} (V_1 \otimes V_3^*)$$

with $\tau_{(V,\theta)} = \tau_{(V',\theta')}$. By Lemma 7.2.7, Prop. 7.2.1, and Definition 6.2.3, we then have that

$$\tau_{(V',\theta')} = \frac{1}{3} c_1(\mathcal{O}_X((a+b)F + \sum (a_k + b_k) F_k)),$$

where the a's and b's satisfy (i)–(iv) from Prop. 7.2.1. Moreover,

$$\mathcal{O}_X((a+b)F + \sum(a_k+b_k)F_k) = \det(V') = V_3^* \otimes V_3^* \otimes V_3^*$$

is vertical. Thus V_3^* is of the form $\mathcal{O}_X(yF + \sum y_k F_k)$ with

$$3(yF + \sum y_k F_k) \equiv (a+b)F + \sum(a_k+b_k)F_k.$$

Condition ($\star$), which is equivalent to the condition that $(a+b)F + \sum(a_k+b_k)F_k$ is "divisible by 3," therefore holds.

Conversely, given a's and b's satisfying (i), Prop. 7.2.1 and Def. 6.2.3 guarantee the existence of a stable projectively flat ternary Higgs bundle (V', θ') with

$$\tau_{(V',\theta')} = \mathcal{O}_X((a+b)F + \sum(a_k+b_k)F_k).$$

Condition ($\star$) is then equivalent to the existence of a vertical line bundle

$$V_3 = \mathcal{O}_X(yF + \sum y_k F_k)$$

such that $c_1(V' \otimes V_3) = c_2(V' \otimes V_3) = 0$. By Lemma 6.2.4, $V' \otimes V_3$ is a stable ternary Higgs bundle with $\tau_{(V,\theta)} = \tau_{(V',\theta')}$. To summarize: $c_1(\mathcal{O}_X(yF + \sum y_k F_k))$ equals the Higgs bundle Toledo invariant of a stable flat ternary Higgs bundle on X if and only if the y's satisfy (i).

A similar argument, using Prop. 7.3.1 instead of Prop. 7.2.1, shows that

$$c_1(\mathcal{O}_X(yF + \sum y_k F_k))$$

equals the Higgs bundle Toledo invariant of a stable binary Higgs bundle (V, θ) with $c_1(V) = c_2(V) = 0$ and rank(im(θ))=1 if and only if the a's and b's satisfy (ii). Prop. 7.4.2 shows that there are no stable binary Higgs bundles with rank(im(θ))=2. (iii) covers Cases 1 and 2 from §7.5, and (iv) covers Case 3 from §7.5 as well as the reducible binary case with $\theta = 0$ (as discussed in §7.6), since the Toledo invariant vanishes in both of these cases. □

Remark: One can check whether ($\star$) holds by the method of [38, Exercise 12.3].

Corollary 8.1.2. (a) *A lower bound for the number of distinct connected components in the representation space* $\mathcal{R}^{+}_{\mathrm{U}(2,1)}(O) = \frac{\mathrm{Hom}^{+}(\pi_1^{\mathrm{orb}}(O), \mathrm{U}(2,1))}{\mathrm{U}(2,1)}$ *is given by the number of distinct values* $\pm(y + \sum \frac{y_k}{m_k})$, *where the y's satisfy one of* (i)–(iv) *from Thm.* 8.1.1.

(b) *A lower bound for the number of distinct connected components in the representation space* $\mathcal{R}^{*}_{\mathrm{PU}(2,1)}(Y) = \frac{\mathrm{Hom}^{*}(\pi_1(Y), \mathrm{PU}(2,1))}{\mathrm{PU}(2,1)}$ *is given by the number of distinct values* $\pm(y + \sum \frac{y_k}{m_k})$, *where the y's satisfy* (i) *or* (ii) *from Thm.* 8.1.1.

Proof. We prove (b) only; the proof of (a) is similar. Lemma 3.2.1 shows that we may replace Y by X in the statement of this theorem. Lemma 2.1.3 shows that (equivalence classes of) PU(2,1) representations with distinct Toledo invariants lie in distinct components of $\frac{\mathrm{Hom}^{*}(\pi_1(X), \mathrm{PU}(2,1))}{\mathrm{PU}(2,1)}$. If $\rho \in \mathrm{Hom}^{*}(\pi_1(X),\, \mathrm{U}(2,1))$, then $\varphi \circ \rho \in \mathrm{Hom}^{*}(\pi_1(X),\, \mathrm{PU}(2,1))$, where $\varphi : \mathrm{U}(2,1) \to \mathrm{PU}(2,1)$ is the canonical homomorphism. Lemmas 6.2.1 and 6.2.2 show that the number of distinct Toledo invariants arising from irreducible U(2,1) representations of $\pi_1(X)$ exactly equals the number of distinct Higgs bundle Toledo invariants of stable U(2,1) Higgs bundles on X with vanishing Chern classes. There exist y's satisfying (i) or (ii) from Thm. 8.1.1 if and only if $\pm c_1(\mathcal{O}_X(yF + \sum y_k F_k))$ equals the Higgs bundle Toledo invariant of a stable U(2,1) Higgs bundle on X with vanishing Chern classes—in which case, by Lemma 5.2.3, the corresponding orbifold Toledo invariant is $\pm(y + \sum \frac{y_k}{m_k})$. □

8.2 An example

Example. Let $n = 3$, and let $(m_1, m_2, m_3) = (2, 3, 11)$. Departing from our previous notations, let F_{m_k} (instead of F_k) denote the multiple fibre on X of multiplicity m_k.

Let $(V_1, \theta_1) = \mathcal{O}_X(-2F + F_2 + 2F_3 + 10F_{11}) \overset{\oplus}{\to} \mathcal{O}_X \overset{\oplus}{\to} \mathcal{O}_X(-F + F_2 + F_3 + F_{11})$ be a stable ternary Higgs bundle.

Let $(V_2, \theta_2) = \mathcal{O}_X \overset{\oplus}{\to} \mathcal{O}_X \overset{\oplus}{\to} \mathcal{O}_X$, where θ_2 is the zero map.

Let (V_3, θ_3) be a stable binary Higgs bundle of the form $V_P \overset{\oplus}{\to} \mathcal{O}_X$, where V_P is given by a nontrivial extension

$$0 \to \mathcal{O}_X(-F + F_3 + 7F_{11}) \to V_P \to \mathcal{O}_X(-2F + F_2 + 2F_3 + 10F_{11}) \to 0.$$

Let $(V_4, \theta_4) = \mathcal{O}_X(-2F + F_2 + 2F_3 + 10F_{11}) \overset{\oplus}{\to} \mathcal{O}_X \overset{\oplus}{\to} \mathcal{O}_X(-F + F_2 + F_3 + 2F_{11})$ be a stable ternary Higgs bundle.

The proof of Thm. 8.1.1 guarantees that all orbifold Toledo invariants arise from these four Higgs bundles and their duals. Let τ_k be the orbifold Toledo invariant corresponding to (V_k, θ_k). Then $0 = \tau_1 = \tau_2$, $0.0152 \approx \tau_3$, and $0.0303 \approx \tau_4$. We conclude that in this case, $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$ contains at least 5 distinct connected components.

Though (V_1, θ_1) and (V_2, θ_2) have the same Higgs bundle Toledo invariant, we conjecture that they lie in different components of $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$. If this conjecture is true, then $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$ has more than 5 components. We plan to study the matter further, with the goal of precisely determining the number of components in this representation space.

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Appendix: Some computations of orbifold Toledo invariants

We provide in this appendix several explicit examples of orbifold Toledo invariants. In each case, for ease of computation, we take $n = 3$. Because of this restriction, condition (iii) from Thm. 8.1.1 never occurs. We list only a's and b's satisfying (i) or (ii) from Thm. 8.1.1, since (iv) is trivial. Moreover, we list only those a's and b's that yield positive values of the orbifold Toledo invariant. For each given triple (m_1, m_2, m_3), the list of a's and b's satisfying (i) or (ii) and yielding a positive orbifold Toledo invariant is complete. The column labelled "Cond" indicates whether condition (i) or (ii) is satisfied, and the column labelled "τ" gives the corresponding orbifold Toledo invariant.

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	3	7	(i)	-1	1	1	1	-2	1	2	6	0.0000
2	3	11	(i)	-1	1	1	1	-2	1	2	10	0.0000
2	3	11	(ii)	-1	0	1	7	-2	1	2	10	0.0152
2	3	11	(i)	-1	1	1	2	-2	1	2	10	0.0303
2	3	13	(i)	-1	1	1	1	-2	1	2	12	0.0000
2	3	13	(i)	-1	1	1	2	-2	1	2	11	0.0000
2	3	13	(ii)	-1	0	1	8	-2	1	2	12	0.0128
2	3	13	(i)	-1	1	1	2	-2	1	2	12	0.0256
2	3	13	(ii)	-1	0	1	9	-2	1	2	12	0.0385
2	3	17	(i)	-1	1	1	1	-2	1	2	16	0.0000
2	3	17	(i)	-1	1	1	2	-2	1	2	15	0.0000
2	3	17	(ii)	-1	0	1	10	-2	1	2	16	0.0098
2	3	17	(ii)	-1	0	1	11	-2	1	2	15	0.0098
2	3	17	(i)	-1	1	1	2	-2	1	2	16	0.0196
2	3	17	(i)	-1	1	1	3	-2	1	2	15	0.0196
2	3	17	(ii)	-1	0	1	11	-2	1	2	16	0.0294
2	3	17	(i)	-1	1	1	3	-2	1	2	16	0.0392
2	3	17	(ii)	-1	0	1	12	-2	1	2	16	0.0490
2	3	19	(i)	-1	1	1	1	-2	1	2	18	0.0000
2	3	19	(i)	-1	1	1	2	-2	1	2	17	0.0000
2	3	19	(i)	-1	1	1	3	-2	1	2	16	0.0000
2	3	19	(ii)	-1	0	1	11	-2	1	2	18	0.0088
2	3	19	(ii)	-1	0	1	12	-2	1	2	17	0.0088
2	3	19	(i)	-1	1	1	2	-2	1	2	18	0.0175
2	3	19	(i)	-1	1	1	3	-2	1	2	17	0.0175
2	3	19	(ii)	-1	0	1	12	-2	1	2	18	0.0263
2	3	19	(ii)	-1	0	1	13	-2	1	2	17	0.0263
2	3	19	(i)	-1	1	1	3	-2	1	2	18	0.0351
2	3	19	(ii)	-1	0	1	13	-2	1	2	18	0.0439
2	3	19	(i)	-1	1	1	4	-2	1	2	18	0.0526
2	5	7	(i)	-1	1	1	1	-2	1	4	6	0.0000
2	5	7	(i)	-1	1	1	2	-2	1	4	5	0.0000
2	5	7	(ii)	-1	0	3	2	-2	1	4	6	0.0143
2	5	7	(ii)	-1	0	2	4	-2	1	4	6	0.0429

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	5	9	(i)	-1	1	1	1	-2	1	4	8	0.0000
2	5	9	(i)	-1	1	1	2	-2	1	4	7	0.0000
2	5	9	(ii)	-1	0	2	4	-2	1	4	8	0.0111
2	5	9	(ii)	-1	0	2	5	-2	1	4	7	0.0111
2	5	9	(ii)	-1	0	1	7	-2	1	4	8	0.0556
2	5	9	(i)	-1	1	2	1	-2	1	4	8	0.0667
2	5	9	(ii)	-1	0	3	4	-2	1	4	8	0.0778
2	5	11	(i)	-1	1	1	1	-2	1	4	10	0.0000
2	5	11	(i)	-1	1	1	2	-2	1	4	9	0.0000
2	5	11	(i)	-1	1	1	3	-2	1	4	8	0.0000
2	5	11	(i)	-1	1	2	1	-2	1	3	10	0.0000
2	5	11	(ii)	-1	0	3	3	-2	1	4	10	0.0273
2	5	11	(ii)	-1	0	3	4	-2	1	4	9	0.0273
2	5	11	(i)	-1	1	2	1	-2	1	4	9	0.0364
2	5	11	(ii)	-1	0	1	8	-2	1	4	10	0.0455
2	5	11	(ii)	-1	0	2	7	-2	1	4	10	0.0818
2	5	11	(i)	-1	1	1	4	-2	1	4	10	0.0909
2	5	13	(i)	-1	1	1	1	-2	1	4	12	0.0000
2	5	13	(i)	-1	1	1	2	-2	1	4	11	0.0000
2	5	13	(i)	-1	1	1	3	-2	1	4	10	0.0000
2	5	13	(i)	-1	1	2	1	-2	1	3	12	0.0000
2	5	13	(i)	-1	1	2	1	-2	1	4	10	0.0154
2	5	13	(ii)	-1	0	1	9	-2	1	4	12	0.0385
2	5	13	(ii)	-1	0	1	10	-2	1	4	11	0.0385
2	5	13	(ii)	-1	0	2	7	-2	1	4	12	0.0538
2	5	13	(ii)	-1	0	2	8	-2	1	4	11	0.0538
2	5	13	(ii)	-1	0	3	5	-2	1	4	12	0.0692
2	5	13	(ii)	-1	0	3	6	-2	1	4	11	0.0692
2	5	13	(i)	-1	1	1	4	-2	1	4	12	0.0769
2	5	13	(i)	-1	1	2	2	-2	1	4	12	0.0923
2	5	17	(i)	-1	1	1	1	-2	1	4	16	0.0000
2	5	17	(i)	-1	1	1	2	-2	1	4	15	0.0000
2	5	17	(i)	-1	1	1	3	-2	1	4	14	0.0000
2	5	17	(i)	-1	1	1	4	-2	1	4	13	0.0000

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	5	17	(i)	-1	1	1	5	-2	1	4	12	0.0000
2	5	17	(i)	-1	1	2	1	-2	1	3	16	0.0000
2	5	17	(ii)	-1	0	3	4	-2	1	4	15	0.0059
2	5	17	(ii)	-1	0	3	5	-2	1	4	14	0.0059
2	5	17	(ii)	-1	0	3	6	-2	1	4	13	0.0059
2	5	17	(i)	-1	1	1	5	-2	1	3	16	0.0118
2	5	17	(ii)	-1	0	2	7	-2	1	4	16	0.0176
2	5	17	(ii)	-1	0	2	8	-2	1	4	15	0.0176
2	5	17	(ii)	-1	0	2	9	-2	1	4	14	0.0176
2	5	17	(ii)	-1	0	2	10	-2	1	4	13	0.0176
2	5	17	(ii)	-1	0	1	11	-2	1	4	16	0.0294
2	5	17	(ii)	-1	0	1	12	-2	1	4	15	0.0294
2	5	17	(ii)	-2	1	3	13	-2	1	4	16	0.0353
2	5	17	(i)	-1	1	2	1	-2	1	4	15	0.0471
2	5	17	(i)	-1	1	2	2	-2	1	4	14	0.0471
2	5	17	(i)	-1	1	1	4	-2	1	4	16	0.0588
2	5	17	(i)	-1	1	1	5	-2	1	4	15	0.0588
2	5	17	(i)	-1	1	1	6	-2	1	4	14	0.0588
2	5	17	(ii)	-1	0	3	6	-2	1	4	16	0.0647
2	5	17	(ii)	-1	0	3	7	-2	1	4	15	0.0647
2	5	17	(ii)	-1	0	2	10	-2	1	4	16	0.0765
2	5	17	(ii)	-1	0	2	11	-2	1	4	15	0.0765
2	5	17	(ii)	-1	0	1	14	-2	1	4	16	0.0882
2	5	17	(i)	-1	1	2	3	-2	1	4	16	0.1059
2	5	17	(i)	-1	1	1	7	-2	1	4	16	0.1176
2	5	19	(i)	-1	1	1	1	-2	1	4	18	0.0000
2	5	19	(i)	-1	1	1	2	-2	1	4	17	0.0000
2	5	19	(i)	-1	1	1	3	-2	1	4	16	0.0000
2	5	19	(i)	-1	1	1	4	-2	1	4	15	0.0000
2	5	19	(i)	-1	1	1	5	-2	1	4	14	0.0000
2	5	19	(i)	-1	1	2	1	-2	1	3	18	0.0000
2	5	19	(ii)	-1	0	2	7	-2	1	4	18	0.0053
2	5	19	(ii)	-1	0	2	8	-2	1	4	17	0.0053
2	5	19	(ii)	-1	0	2	9	-2	1	4	16	0.0053

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	5	19	(ii)	-1	0	2	10	-2	1	4	15	0.0053
2	5	19	(ii)	-1	0	2	11	-2	1	4	14	0.0053
2	5	19	(i)	-1	1	1	6	-2	1	3	18	0.0211
2	5	19	(ii)	-1	0	1	12	-2	1	4	18	0.0263
2	5	19	(ii)	-1	0	1	13	-2	1	4	17	0.0263
2	5	19	(ii)	-1	0	1	14	-2	1	4	16	0.0263
2	5	19	(i)	-1	1	2	1	-2	1	4	16	0.0316
2	5	19	(i)	-1	1	2	2	-2	1	4	15	0.0316
2	5	19	(ii)	-1	0	3	5	-2	1	4	18	0.0368
2	5	19	(ii)	-1	0	3	6	-2	1	4	17	0.0368
2	5	19	(ii)	-1	0	3	7	-2	1	4	16	0.0368
2	5	19	(ii)	-1	0	3	8	-2	1	4	15	0.0368
2	5	19	(i)	-1	1	1	4	-2	1	4	18	0.0526
2	5	19	(i)	-1	1	1	5	-2	1	4	17	0.0526
2	5	19	(i)	-1	1	1	6	-2	1	4	16	0.0526
2	5	19	(ii)	-1	0	2	10	-2	1	4	18	0.0579
2	5	19	(ii)	-1	0	2	11	-2	1	4	17	0.0579
2	5	19	(ii)	-1	0	2	12	-2	1	4	16	0.0579
2	5	19	(ii)	-1	0	1	15	-2	1	4	18	0.0789
2	5	19	(ii)	-1	0	1	16	-2	1	4	17	0.0789
2	5	19	(i)	-1	1	2	2	-2	1	4	18	0.0842
2	5	19	(i)	-1	1	2	3	-2	1	4	17	0.0842
2	5	19	(ii)	-1	0	3	8	-2	1	4	18	0.0895
2	5	19	(ii)	-1	0	3	9	-2	1	4	17	0.0895
2	5	19	(i)	-1	1	1	7	-2	1	4	18	0.1053
2	5	19	(ii)	-1	0	2	13	-2	1	4	18	0.1105
2	7	9	(i)	-1	1	1	1	-2	1	6	8	0.0000
2	7	9	(i)	-1	1	1	2	-2	1	6	7	0.0000
2	7	9	(i)	-1	1	1	3	-2	1	6	6	0.0000
2	7	9	(i)	-1	1	2	1	-2	1	5	8	0.0000
2	7	9	(ii)	-1	0	1	7	-2	1	5	8	0.0079
2	7	9	(ii)	-1	0	5	2	-2	1	6	7	0.0238
2	7	9	(ii)	-1	0	3	4	-2	1	6	8	0.0397
2	7	9	(ii)	-1	0	3	5	-2	1	6	7	0.0397

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	9	(ii)	-1	0	4	4	-2	1	5	8	0.0397
2	7	9	(i)	-1	1	2	1	-2	1	6	8	0.0476
2	7	9	(i)	-1	1	2	2	-2	1	6	7	0.0476
2	7	9	(i)	-1	1	3	1	-2	1	5	8	0.0476
2	7	9	(ii)	-2	1	5	7	-2	1	6	8	0.0794
2	7	9	(ii)	-1	0	4	4	-2	1	6	8	0.0873
2	7	9	(i)	-1	1	3	1	-2	1	6	8	0.0952
2	7	9	(ii)	-1	0	2	7	-2	1	6	8	0.1032
2	7	9	(i)	-1	1	1	4	-2	1	6	8	0.1111
2	7	11	(i)	-1	1	1	1	-2	1	6	10	0.0000
2	7	11	(i)	-1	1	1	2	-2	1	6	9	0.0000
2	7	11	(i)	-1	1	1	3	-2	1	6	8	0.0000
2	7	11	(i)	-1	1	2	1	-2	1	5	10	0.0000
2	7	11	(i)	-1	1	2	2	-2	1	5	9	0.0000
2	7	11	(ii)	-1	0	4	2	-2	1	6	10	0.0065
2	7	11	(ii)	-1	0	4	3	-2	1	6	9	0.0065
2	7	11	(ii)	-1	0	4	4	-2	1	6	8	0.0065
2	7	11	(i)	-1	1	1	3	-2	1	5	10	0.0130
2	7	11	(i)	-1	1	1	4	-2	1	5	9	0.0130
2	7	11	(ii)	-1	0	3	4	-2	1	6	10	0.0195
2	7	11	(ii)	-1	0	3	5	-2	1	6	9	0.0195
2	7	11	(ii)	-1	0	3	6	-2	1	6	8	0.0195
2	7	11	(ii)	-1	0	4	4	-2	1	5	10	0.0195
2	7	11	(ii)	-1	0	2	6	-2	1	6	10	0.0325
2	7	11	(ii)	-1	0	2	7	-2	1	6	9	0.0325
2	7	11	(ii)	-1	0	3	6	-2	1	5	10	0.0325
2	7	11	(ii)	-1	0	1	8	-2	1	6	10	0.0455
2	7	11	(ii)	-1	0	2	8	-2	1	5	10	0.0455
2	7	11	(i)	-1	1	3	1	-2	1	6	9	0.0649
2	7	11	(i)	-1	1	2	2	-2	1	6	10	0.0779
2	7	11	(i)	-1	1	2	3	-2	1	6	9	0.0779
2	7	11	(ii)	-1	0	5	3	-2	1	6	10	0.0844
2	7	11	(ii)	-1	0	5	4	-2	1	6	9	0.0844
2	7	11	(i)	-1	1	1	4	-2	1	6	10	0.0909

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	11	(ii)	-1	0	4	5	-2	1	6	10	0.0974
2	7	11	(ii)	-1	0	3	7	-2	1	6	10	0.1104
2	7	11	(ii)	-1	0	2	9	-2	1	6	10	0.1234
2	7	13	(i)	-1	1	1	1	-2	1	6	12	0.0000
2	7	13	(i)	-1	1	1	2	-2	1	6	11	0.0000
2	7	13	(i)	-1	1	1	3	-2	1	6	10	0.0000
2	7	13	(i)	-1	1	1	4	-2	1	6	9	0.0000
2	7	13	(i)	-1	1	2	1	-2	1	5	12	0.0000
2	7	13	(i)	-1	1	2	2	-2	1	5	11	0.0000
2	7	13	(ii)	-1	0	3	4	-2	1	6	12	0.0055
2	7	13	(ii)	-1	0	3	5	-2	1	6	11	0.0055
2	7	13	(ii)	-1	0	3	6	-2	1	6	10	0.0055
2	7	13	(ii)	-1	0	3	7	-2	1	6	9	0.0055
2	7	13	(ii)	-1	0	4	4	-2	1	5	12	0.0055
2	7	13	(ii)	-1	0	4	5	-2	1	5	11	0.0055
2	7	13	(ii)	-1	0	1	10	-2	1	5	12	0.0165
2	7	13	(i)	-1	1	2	1	-2	1	6	11	0.0220
2	7	13	(i)	-1	1	2	2	-2	1	6	10	0.0220
2	7	13	(i)	-1	1	2	3	-2	1	6	9	0.0220
2	7	13	(i)	-1	1	3	1	-2	1	5	11	0.0220
2	7	13	(ii)	-1	0	4	3	-2	1	6	12	0.0275
2	7	13	(ii)	-1	0	4	4	-2	1	6	11	0.0275
2	7	13	(ii)	-1	0	4	5	-2	1	6	10	0.0275
2	7	13	(ii)	-1	0	1	9	-2	1	6	12	0.0385
2	7	13	(ii)	-1	0	2	9	-2	1	5	12	0.0385
2	7	13	(i)	-1	1	3	1	-2	1	6	10	0.0440
2	7	13	(ii)	-1	0	5	3	-2	1	6	11	0.0495
2	7	13	(ii)	-1	0	5	4	-2	1	6	10	0.0495
2	7	13	(i)	-1	1	1	5	-2	1	5	12	0.0549
2	7	13	(ii)	-1	0	2	8	-2	1	6	12	0.0604
2	7	13	(ii)	-1	0	2	9	-2	1	6	11	0.0604
2	7	13	(ii)	-1	0	3	8	-2	1	5	12	0.0604
2	7	13	(i)	-1	1	1	4	-2	1	6	12	0.0769
2	7	13	(i)	-1	1	1	5	-2	1	6	11	0.0769

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	13	(ii)	-1	0	3	7	-2	1	6	12	0.0824
2	7	13	(ii)	-1	0	3	8	-2	1	6	11	0.0824
2	7	13	(ii)	-2	1	5	10	-2	1	6	12	0.0879
2	7	13	(i)	-1	1	2	3	-2	1	6	12	0.0989
2	7	13	(i)	-1	1	2	4	-2	1	6	11	0.0989
2	7	13	(ii)	-1	0	4	6	-2	1	6	12	0.1044
2	7	13	(i)	-1	1	3	2	-2	1	6	12	0.1209
2	7	13	(ii)	-1	0	5	5	-2	1	6	12	0.1264
2	7	13	(ii)	-1	0	2	11	-2	1	6	12	0.1374
2	7	15	(i)	-1	1	1	1	-2	1	6	14	0.0000
2	7	15	(i)	-1	1	1	2	-2	1	6	13	0.0000
2	7	15	(i)	-1	1	1	3	-2	1	6	12	0.0000
2	7	15	(i)	-1	1	1	4	-2	1	6	11	0.0000
2	7	15	(i)	-1	1	1	5	-2	1	6	10	0.0000
2	7	15	(i)	-1	1	2	1	-2	1	5	14	0.0000
2	7	15	(i)	-1	1	2	2	-2	1	5	13	0.0000
2	7	15	(i)	-1	1	2	3	-2	1	5	12	0.0000
2	7	15	(i)	-1	1	3	1	-2	1	4	14	0.0000
2	7	15	(ii)	-1	0	2	7	-2	1	6	14	0.0143
2	7	15	(ii)	-1	0	2	8	-2	1	6	13	0.0143
2	7	15	(ii)	-1	0	2	9	-2	1	6	12	0.0143
2	7	15	(ii)	-1	0	2	10	-2	1	6	11	0.0143
2	7	15	(ii)	-1	0	3	7	-2	1	5	14	0.0143
2	7	15	(ii)	-1	0	3	8	-2	1	5	13	0.0143
2	7	15	(i)	-1	1	1	4	-2	1	5	14	0.0190
2	7	15	(i)	-1	1	1	5	-2	1	5	13	0.0190
2	7	15	(ii)	-1	0	5	3	-2	1	6	12	0.0238
2	7	15	(ii)	-1	0	5	4	-2	1	6	11	0.0238
2	7	15	(i)	-1	1	3	1	-2	1	6	11	0.0286
2	7	15	(ii)	-1	0	1	10	-2	1	6	14	0.0333
2	7	15	(ii)	-1	0	2	10	-2	1	5	14	0.0333
2	7	15	(ii)	-1	0	2	11	-2	1	5	13	0.0333
2	7	15	(ii)	-1	0	4	4	-2	1	6	14	0.0429
2	7	15	(ii)	-1	0	4	5	-2	1	6	13	0.0429

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	15	(ii)	-1	0	4	6	-2	1	6	12	0.0429
2	7	15	(ii)	-1	0	4	7	-2	1	6	11	0.0429
2	7	15	(i)	-1	1	2	1	-2	1	6	14	0.0476
2	7	15	(i)	-1	1	2	2	-2	1	6	13	0.0476
2	7	15	(i)	-1	1	2	3	-2	1	6	12	0.0476
2	7	15	(i)	-1	1	3	1	-2	1	5	14	0.0476
2	7	15	(ii)	-1	0	1	13	-2	1	5	14	0.0524
2	7	15	(ii)	-2	1	5	11	-2	1	6	13	0.0571
2	7	15	(ii)	-1	0	3	7	-2	1	6	14	0.0619
2	7	15	(ii)	-1	0	3	8	-2	1	6	13	0.0619
2	7	15	(ii)	-1	0	3	9	-2	1	6	12	0.0619
2	7	15	(ii)	-1	0	4	7	-2	1	5	14	0.0619
2	7	15	(i)	-1	1	1	4	-2	1	6	14	0.0667
2	7	15	(i)	-1	1	1	5	-2	1	6	13	0.0667
2	7	15	(i)	-1	1	1	6	-2	1	6	12	0.0667
2	7	15	(i)	-1	1	2	4	-2	1	5	14	0.0667
2	7	15	(ii)	-1	0	2	10	-2	1	6	14	0.0810
2	7	15	(ii)	-1	0	2	11	-2	1	6	13	0.0810
2	7	15	(ii)	-1	0	5	4	-2	1	6	14	0.0905
2	7	15	(ii)	-1	0	5	5	-2	1	6	13	0.0905
2	7	15	(i)	-1	1	3	1	-2	1	6	14	0.0952
2	7	15	(i)	-1	1	3	2	-2	1	6	13	0.0952
2	7	15	(ii)	-1	0	4	7	-2	1	6	14	0.1095
2	7	15	(ii)	-1	0	4	8	-2	1	6	13	0.1095
2	7	15	(i)	-1	1	2	4	-2	1	6	14	0.1143
2	7	15	(ii)	-2	1	5	13	-2	1	6	14	0.1238
2	7	15	(ii)	-1	0	3	10	-2	1	6	14	0.1286
2	7	15	(i)	-1	1	1	7	-2	1	6	14	0.1333
2	7	15	(i)	-1	1	4	1	-2	1	6	14	0.1429
2	7	17	(i)	-1	1	1	1	-2	1	6	16	0.0000
2	7	17	(i)	-1	1	1	2	-2	1	6	15	0.0000
2	7	17	(i)	-1	1	1	3	-2	1	6	14	0.0000
2	7	17	(i)	-1	1	1	4	-2	1	6	13	0.0000
2	7	17	(i)	-1	1	1	5	-2	1	6	12	0.0000

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	17	(i)	-1	1	1	6	-2	1	6	11	0.0000
2	7	17	(i)	-1	1	2	1	-2	1	5	16	0.0000
2	7	17	(i)	-1	1	2	2	-2	1	5	15	0.0000
2	7	17	(i)	-1	1	2	3	-2	1	5	14	0.0000
2	7	17	(i)	-1	1	3	1	-2	1	4	16	0.0000
2	7	17	(ii)	-1	0	5	3	-2	1	6	13	0.0042
2	7	17	(ii)	-1	0	5	4	-2	1	6	12	0.0042
2	7	17	(i)	-1	1	2	1	-2	1	6	14	0.0084
2	7	17	(i)	-1	1	2	2	-2	1	6	13	0.0084
2	7	17	(i)	-1	1	2	3	-2	1	6	12	0.0084
2	7	17	(i)	-1	1	3	1	-2	1	5	14	0.0084
2	7	17	(i)	-1	1	3	1	-2	1	6	12	0.0168
2	7	17	(ii)	-1	0	1	13	-2	1	5	16	0.0210
2	7	17	(ii)	-1	0	1	14	-2	1	5	15	0.0210
2	7	17	(ii)	-1	0	1	11	-2	1	6	16	0.0294
2	7	17	(ii)	-1	0	2	11	-2	1	5	16	0.0294
2	7	17	(ii)	-1	0	2	12	-2	1	5	15	0.0294
2	7	17	(ii)	-2	1	5	12	-2	1	6	14	0.0336
2	7	17	(ii)	-1	0	2	9	-2	1	6	16	0.0378
2	7	17	(ii)	-1	0	2	10	-2	1	6	15	0.0378
2	7	17	(ii)	-1	0	2	11	-2	1	6	14	0.0378
2	7	17	(ii)	-1	0	2	12	-2	1	6	13	0.0378
2	7	17	(ii)	-1	0	3	9	-2	1	5	16	0.0378
2	7	17	(ii)	-1	0	3	10	-2	1	5	15	0.0378
2	7	17	(ii)	-1	0	3	7	-2	1	6	16	0.0462
2	7	17	(ii)	-1	0	3	8	-2	1	6	15	0.0462
2	7	17	(ii)	-1	0	3	9	-2	1	6	14	0.0462
2	7	17	(ii)	-1	0	3	10	-2	1	6	13	0.0462
2	7	17	(ii)	-1	0	4	7	-2	1	5	16	0.0462
2	7	17	(ii)	-1	0	4	8	-2	1	5	15	0.0462
2	7	17	(i)	-1	1	1	6	-2	1	5	16	0.0504
2	7	17	(ii)	-1	0	4	5	-2	1	6	16	0.0546
2	7	17	(ii)	-1	0	4	6	-2	1	6	15	0.0546
2	7	17	(ii)	-1	0	4	7	-2	1	6	14	0.0546

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	17	(ii)	-1	0	4	8	-2	1	6	13	0.0546
2	7	17	(i)	-1	1	1	4	-2	1	6	16	0.0588
2	7	17	(i)	-1	1	1	5	-2	1	6	15	0.0588
2	7	17	(i)	-1	1	1	6	-2	1	6	14	0.0588
2	7	17	(i)	-1	1	1	7	-2	1	6	13	0.0588
2	7	17	(i)	-1	1	2	4	-2	1	5	16	0.0588
2	7	17	(ii)	-1	0	5	4	-2	1	6	15	0.0630
2	7	17	(ii)	-1	0	5	5	-2	1	6	14	0.0630
2	7	17	(i)	-1	1	2	2	-2	1	6	16	0.0672
2	7	17	(i)	-1	1	2	3	-2	1	6	15	0.0672
2	7	17	(i)	-1	1	2	4	-2	1	6	14	0.0672
2	7	17	(i)	-1	1	3	2	-2	1	5	16	0.0672
2	7	17	(i)	-1	1	3	1	-2	1	6	15	0.0756
2	7	17	(i)	-1	1	3	2	-2	1	6	14	0.0756
2	7	17	(ii)	-2	1	5	13	-2	1	6	16	0.0924
2	7	17	(ii)	-2	1	5	14	-2	1	6	15	0.0924
2	7	17	(ii)	-1	0	2	12	-2	1	6	16	0.0966
2	7	17	(ii)	-1	0	2	13	-2	1	6	15	0.0966
2	7	17	(ii)	-1	0	3	10	-2	1	6	16	0.1050
2	7	17	(ii)	-1	0	3	11	-2	1	6	15	0.1050
2	7	17	(ii)	-1	0	4	8	-2	1	6	16	0.1134
2	7	17	(ii)	-1	0	4	9	-2	1	6	15	0.1134
2	7	17	(i)	-1	1	1	7	-2	1	6	16	0.1176
2	7	17	(i)	-1	1	1	8	-2	1	6	15	0.1176
2	7	17	(ii)	-1	0	5	6	-2	1	6	16	0.1218
2	7	17	(i)	-1	1	2	5	-2	1	6	16	0.1261
2	7	17	(i)	-1	1	3	3	-2	1	6	16	0.1345
2	7	17	(i)	-1	1	4	1	-2	1	6	16	0.1429
2	7	19	(i)	-1	1	1	1	-2	1	6	18	0.0000
2	7	19	(i)	-1	1	1	2	-2	1	6	17	0.0000
2	7	19	(i)	-1	1	1	3	-2	1	6	16	0.0000
2	7	19	(i)	-1	1	1	4	-2	1	6	15	0.0000
2	7	19	(i)	-1	1	1	5	-2	1	6	14	0.0000
2	7	19	(i)	-1	1	1	6	-2	1	6	13	0.0000

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	19	(i)	-1	1	2	1	-2	1	5	18	0.0000
2	7	19	(i)	-1	1	2	2	-2	1	5	17	0.0000
2	7	19	(i)	-1	1	2	3	-2	1	5	16	0.0000
2	7	19	(i)	-1	1	2	4	-2	1	5	15	0.0000
2	7	19	(i)	-1	1	3	1	-2	1	4	18	0.0000
2	7	19	(ii)	-1	0	2	8	-2	1	6	18	0.0038
2	7	19	(ii)	-1	0	2	9	-2	1	6	17	0.0038
2	7	19	(ii)	-1	0	2	10	-2	1	6	16	0.0038
2	7	19	(ii)	-1	0	2	11	-2	1	6	15	0.0038
2	7	19	(ii)	-1	0	2	12	-2	1	6	14	0.0038
2	7	19	(ii)	-1	0	3	8	-2	1	5	18	0.0038
2	7	19	(ii)	-1	0	3	9	-2	1	5	17	0.0038
2	7	19	(ii)	-1	0	3	10	-2	1	5	16	0.0038
2	7	19	(i)	-1	1	3	1	-2	1	6	13	0.0075
2	7	19	(ii)	-1	0	4	3	-2	1	6	18	0.0113
2	7	19	(ii)	-1	0	4	4	-2	1	6	17	0.0113
2	7	19	(ii)	-1	0	4	5	-2	1	6	16	0.0113
2	7	19	(ii)	-1	0	4	6	-2	1	6	15	0.0113
2	7	19	(ii)	-1	0	4	7	-2	1	6	14	0.0113
2	7	19	(ii)	-1	0	4	8	-2	1	6	13	0.0113
2	7	19	(i)	-1	1	1	5	-2	1	5	18	0.0226
2	7	19	(i)	-1	1	1	6	-2	1	5	17	0.0226
2	7	19	(i)	-1	1	1	7	-2	1	5	16	0.0226
2	7	19	(ii)	-1	0	1	12	-2	1	6	18	0.0263
2	7	19	(ii)	-1	0	1	13	-2	1	6	17	0.0263
2	7	19	(ii)	-1	0	2	12	-2	1	5	18	0.0263
2	7	19	(ii)	-1	0	2	13	-2	1	5	17	0.0263
2	7	19	(ii)	-1	0	2	14	-2	1	5	16	0.0263
2	7	19	(i)	-1	1	2	1	-2	1	6	17	0.0301
2	7	19	(i)	-1	1	2	2	-2	1	6	16	0.0301
2	7	19	(i)	-1	1	2	3	-2	1	6	15	0.0301
2	7	19	(i)	-1	1	2	4	-2	1	6	14	0.0301
2	7	19	(i)	-1	1	3	1	-2	1	5	17	0.0301
2	7	19	(ii)	-1	0	3	7	-2	1	6	18	0.0338

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	19	(ii)	-1	0	3	8	-2	1	6	17	0.0338
2	7	19	(ii)	-1	0	3	9	-2	1	6	16	0.0338
2	7	19	(ii)	-1	0	3	10	-2	1	6	15	0.0338
2	7	19	(ii)	-1	0	3	11	-2	1	6	14	0.0338
2	7	19	(ii)	-1	0	4	7	-2	1	5	18	0.0338
2	7	19	(ii)	-1	0	4	8	-2	1	5	17	0.0338
2	7	19	(ii)	-1	0	5	4	-2	1	6	16	0.0414
2	7	19	(ii)	-1	0	5	5	-2	1	6	15	0.0414
2	7	19	(ii)	-1	0	5	6	-2	1	6	14	0.0414
2	7	19	(ii)	-1	0	1	16	-2	1	5	18	0.0489
2	7	19	(i)	-1	1	1	4	-2	1	6	18	0.0526
2	7	19	(i)	-1	1	1	5	-2	1	6	17	0.0526
2	7	19	(i)	-1	1	1	6	-2	1	6	16	0.0526
2	7	19	(i)	-1	1	1	7	-2	1	6	15	0.0526
2	7	19	(i)	-1	1	2	4	-2	1	5	18	0.0526
2	7	19	(i)	-1	1	2	5	-2	1	5	17	0.0526
2	7	19	(ii)	-1	0	2	11	-2	1	6	18	0.0564
2	7	19	(ii)	-1	0	2	12	-2	1	6	17	0.0564
2	7	19	(ii)	-1	0	2	13	-2	1	6	16	0.0564
2	7	19	(ii)	-1	0	2	14	-2	1	6	15	0.0564
2	7	19	(ii)	-1	0	3	11	-2	1	5	18	0.0564
2	7	19	(i)	-1	1	3	1	-2	1	6	16	0.0602
2	7	19	(i)	-1	1	3	2	-2	1	6	15	0.0602
2	7	19	(ii)	-1	0	4	6	-2	1	6	18	0.0639
2	7	19	(ii)	-1	0	4	7	-2	1	6	17	0.0639
2	7	19	(ii)	-1	0	4	8	-2	1	6	16	0.0639
2	7	19	(ii)	-1	0	4	9	-2	1	6	15	0.0639
2	7	19	(ii)	-2	1	5	14	-2	1	6	17	0.0677
2	7	19	(i)	-1	1	1	8	-2	1	5	18	0.0752
2	7	19	(ii)	-1	0	1	15	-2	1	6	18	0.0789
2	7	19	(ii)	-1	0	2	15	-2	1	5	18	0.0789
2	7	19	(i)	-1	1	2	3	-2	1	6	18	0.0827
2	7	19	(i)	-1	1	2	4	-2	1	6	17	0.0827
2	7	19	(i)	-1	1	2	5	-2	1	6	16	0.0827

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	7	19	(ii)	-1	0	3	10	-2	1	6	18	0.0865
2	7	19	(ii)	-1	0	3	11	-2	1	6	17	0.0865
2	7	19	(ii)	-1	0	3	12	-2	1	6	16	0.0865
2	7	19	(ii)	-1	0	5	5	-2	1	6	18	0.0940
2	7	19	(ii)	-1	0	5	6	-2	1	6	17	0.0940
2	7	19	(ii)	-1	0	5	7	-2	1	6	16	0.0940
2	7	19	(i)	-1	1	1	7	-2	1	6	18	0.1053
2	7	19	(i)	-1	1	1	8	-2	1	6	17	0.1053
2	7	19	(ii)	-1	0	2	14	-2	1	6	18	0.1090
2	7	19	(ii)	-1	0	2	15	-2	1	6	17	0.1090
2	7	19	(i)	-1	1	3	2	-2	1	6	18	0.1128
2	7	19	(i)	-1	1	3	3	-2	1	6	17	0.1128
2	7	19	(ii)	-1	0	4	9	-2	1	6	18	0.1165
2	7	19	(ii)	-1	0	4	10	-2	1	6	17	0.1165
2	7	19	(ii)	-2	1	5	16	-2	1	6	18	0.1203
2	7	19	(i)	-1	1	2	6	-2	1	6	18	0.1353
2	7	19	(ii)	-1	0	3	13	-2	1	6	18	0.1391
2	7	19	(i)	-1	1	4	1	-2	1	6	18	0.1429
2	7	19	(ii)	-1	0	5	8	-2	1	6	18	0.1466
2	9	11	(i)	-1	1	1	1	-2	1	8	10	0.0000
2	9	11	(i)	-1	1	1	2	-2	1	8	9	0.0000
2	9	11	(i)	-1	1	1	3	-2	1	8	8	0.0000
2	9	11	(i)	-1	1	1	4	-2	1	8	7	0.0000
2	9	11	(i)	-1	1	2	1	-2	1	7	10	0.0000
2	9	11	(i)	-1	1	2	2	-2	1	7	9	0.0000
2	9	11	(i)	-1	1	2	3	-2	1	7	8	0.0000
2	9	11	(i)	-1	1	3	1	-2	1	6	10	0.0000
2	9	11	(ii)	-1	0	4	3	-2	1	8	10	0.0051
2	9	11	(ii)	-1	0	4	4	-2	1	8	9	0.0051
2	9	11	(ii)	-1	0	4	5	-2	1	8	8	0.0051
2	9	11	(ii)	-1	0	4	6	-2	1	8	7	0.0051
2	9	11	(ii)	-1	0	5	3	-2	1	7	10	0.0051
2	9	11	(ii)	-1	0	5	4	-2	1	7	9	0.0051
2	9	11	(ii)	-1	0	1	7	-2	1	8	10	0.0152

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	11	(ii)	-1	0	2	7	-2	1	7	10	0.0152
2	9	11	(ii)	-1	0	2	8	-2	1	7	9	0.0152
2	9	11	(ii)	-1	0	3	7	-2	1	6	10	0.0152
2	9	11	(ii)	-1	0	7	2	-2	1	8	8	0.0253
2	9	11	(i)	-1	1	1	2	-2	1	8	10	0.0303
2	9	11	(i)	-1	1	1	3	-2	1	8	9	0.0303
2	9	11	(i)	-1	1	1	4	-2	1	8	8	0.0303
2	9	11	(i)	-1	1	2	2	-2	1	7	10	0.0303
2	9	11	(i)	-1	1	2	3	-2	1	7	9	0.0303
2	9	11	(i)	-1	1	3	2	-2	1	6	10	0.0303
2	9	11	(ii)	-1	0	4	4	-2	1	8	10	0.0354
2	9	11	(ii)	-1	0	4	5	-2	1	8	9	0.0354
2	9	11	(ii)	-1	0	4	6	-2	1	8	8	0.0354
2	9	11	(ii)	-1	0	5	4	-2	1	7	10	0.0354
2	9	11	(ii)	-1	0	5	5	-2	1	7	9	0.0354
2	9	11	(ii)	-1	0	2	8	-2	1	7	10	0.0455
2	9	11	(i)	-1	1	4	1	-2	1	8	8	0.0505
2	9	11	(ii)	-1	0	7	2	-2	1	8	9	0.0556
2	9	11	(ii)	-1	0	7	3	-2	1	8	8	0.0556
2	9	11	(i)	-1	1	1	3	-2	1	8	10	0.0606
2	9	11	(i)	-1	1	1	4	-2	1	8	9	0.0606
2	9	11	(i)	-1	1	2	3	-2	1	7	10	0.0606
2	9	11	(ii)	-1	0	4	5	-2	1	8	10	0.0657
2	9	11	(ii)	-1	0	4	6	-2	1	8	9	0.0657
2	9	11	(ii)	-1	0	5	5	-2	1	7	10	0.0657
2	9	11	(ii)	-2	1	7	8	-2	1	8	9	0.0707
2	9	11	(ii)	-1	0	2	9	-2	1	7	10	0.0758
2	9	11	(i)	-1	1	4	1	-2	1	8	9	0.0808
2	9	11	(ii)	-1	0	7	3	-2	1	8	9	0.0859
2	9	11	(i)	-1	1	1	4	-2	1	8	10	0.0909
2	9	11	(i)	-1	1	1	5	-2	1	8	9	0.0909
2	9	11	(i)	-1	1	2	4	-2	1	7	10	0.0909
2	9	11	(ii)	-1	0	4	6	-2	1	8	10	0.0960
2	9	11	(ii)	-1	0	4	7	-2	1	8	9	0.0960

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	11	(ii)	-2	1	7	8	-2	1	8	10	0.1010
2	9	11	(i)	-1	1	4	1	-2	1	8	10	0.1111
2	9	11	(ii)	-1	0	7	3	-2	1	8	10	0.1162
2	9	11	(i)	-1	1	1	5	-2	1	8	10	0.1212
2	9	11	(ii)	-1	0	4	7	-2	1	8	10	0.1263
2	9	11	(ii)	-2	1	7	9	-2	1	8	10	0.1313
2	9	11	(i)	-1	1	4	2	-2	1	8	10	0.1414
2	9	11	(ii)	-1	0	7	4	-2	1	8	10	0.1465
2	9	13	(i)	-1	1	1	1	-2	1	8	12	0.0000
2	9	13	(i)	-1	1	1	2	-2	1	8	11	0.0000
2	9	13	(i)	-1	1	1	3	-2	1	8	10	0.0000
2	9	13	(i)	-1	1	1	4	-2	1	8	9	0.0000
2	9	13	(i)	-1	1	1	5	-2	1	8	8	0.0000
2	9	13	(i)	-1	1	2	1	-2	1	7	12	0.0000
2	9	13	(i)	-1	1	2	2	-2	1	7	11	0.0000
2	9	13	(i)	-1	1	2	3	-2	1	7	10	0.0000
2	9	13	(i)	-1	1	3	1	-2	1	6	12	0.0000
2	9	13	(i)	-1	1	3	2	-2	1	6	11	0.0000
2	9	13	(ii)	-1	0	7	2	-2	1	8	9	0.0043
2	9	13	(ii)	-1	0	1	8	-2	1	8	12	0.0128
2	9	13	(ii)	-1	0	2	8	-2	1	7	12	0.0128
2	9	13	(ii)	-1	0	2	9	-2	1	7	11	0.0128
2	9	13	(ii)	-1	0	3	8	-2	1	6	12	0.0128
2	9	13	(ii)	-1	0	4	4	-2	1	8	12	0.0214
2	9	13	(ii)	-1	0	4	5	-2	1	8	11	0.0214
2	9	13	(ii)	-1	0	4	6	-2	1	8	10	0.0214
2	9	13	(ii)	-1	0	4	7	-2	1	8	9	0.0214
2	9	13	(ii)	-1	0	5	4	-2	1	7	12	0.0214
2	9	13	(ii)	-1	0	5	5	-2	1	7	11	0.0214
2	9	13	(ii)	-1	0	5	6	-2	1	7	10	0.0214
2	9	13	(i)	-1	1	1	2	-2	1	8	12	0.0256
2	9	13	(i)	-1	1	1	3	-2	1	8	11	0.0256
2	9	13	(i)	-1	1	1	4	-2	1	8	10	0.0256
2	9	13	(i)	-1	1	1	5	-2	1	8	9	0.0256

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	13	(i)	-1	1	2	2	-2	1	7	12	0.0256
2	9	13	(i)	-1	1	2	3	-2	1	7	11	0.0256
2	9	13	(i)	-1	1	3	2	-2	1	6	12	0.0256
2	9	13	(ii)	-1	0	7	2	-2	1	8	10	0.0299
2	9	13	(ii)	-1	0	7	3	-2	1	8	9	0.0299
2	9	13	(i)	-1	1	4	1	-2	1	8	9	0.0342
2	9	13	(ii)	-1	0	2	9	-2	1	7	12	0.0385
2	9	13	(ii)	-1	0	2	10	-2	1	7	11	0.0385
2	9	13	(ii)	-1	0	3	9	-2	1	6	12	0.0385
2	9	13	(ii)	-2	1	7	9	-2	1	8	10	0.0427
2	9	13	(ii)	-1	0	4	5	-2	1	8	12	0.0470
2	9	13	(ii)	-1	0	4	6	-2	1	8	11	0.0470
2	9	13	(ii)	-1	0	4	7	-2	1	8	10	0.0470
2	9	13	(ii)	-1	0	5	5	-2	1	7	12	0.0470
2	9	13	(ii)	-1	0	5	6	-2	1	7	11	0.0470
2	9	13	(i)	-1	1	1	3	-2	1	8	12	0.0513
2	9	13	(i)	-1	1	1	4	-2	1	8	11	0.0513
2	9	13	(i)	-1	1	1	5	-2	1	8	10	0.0513
2	9	13	(i)	-1	1	2	3	-2	1	7	12	0.0513
2	9	13	(i)	-1	1	2	4	-2	1	7	11	0.0513
2	9	13	(ii)	-1	0	7	3	-2	1	8	10	0.0556
2	9	13	(i)	-1	1	4	1	-2	1	8	10	0.0598
2	9	13	(ii)	-1	0	2	10	-2	1	7	12	0.0641
2	9	13	(ii)	-2	1	7	9	-2	1	8	11	0.0684
2	9	13	(ii)	-1	0	4	6	-2	1	8	12	0.0726
2	9	13	(ii)	-1	0	4	7	-2	1	8	11	0.0726
2	9	13	(ii)	-1	0	4	8	-2	1	8	10	0.0726
2	9	13	(ii)	-1	0	5	6	-2	1	7	12	0.0726
2	9	13	(i)	-1	1	1	4	-2	1	8	12	0.0769
2	9	13	(i)	-1	1	1	5	-2	1	8	11	0.0769
2	9	13	(i)	-1	1	1	6	-2	1	8	10	0.0769
2	9	13	(i)	-1	1	2	4	-2	1	7	12	0.0769
2	9	13	(ii)	-1	0	7	3	-2	1	8	11	0.0812
2	9	13	(i)	-1	1	4	1	-2	1	8	11	0.0855

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	13	(ii)	-1	0	2	11	-2	1	7	12	0.0897
2	9	13	(ii)	-2	1	7	10	-2	1	8	11	0.0940
2	9	13	(ii)	-1	0	4	7	-2	1	8	12	0.0983
2	9	13	(ii)	-1	0	4	8	-2	1	8	11	0.0983
2	9	13	(ii)	-1	0	5	7	-2	1	7	12	0.0983
2	9	13	(i)	-1	1	1	5	-2	1	8	12	0.1026
2	9	13	(i)	-1	1	1	6	-2	1	8	11	0.1026
2	9	13	(ii)	-1	0	7	3	-2	1	8	12	0.1068
2	9	13	(ii)	-1	0	7	4	-2	1	8	11	0.1068
2	9	13	(i)	-1	1	4	1	-2	1	8	12	0.1111
2	9	13	(i)	-1	1	4	2	-2	1	8	11	0.1111
2	9	13	(ii)	-2	1	7	10	-2	1	8	12	0.1197
2	9	13	(ii)	-1	0	4	8	-2	1	8	12	0.1239
2	9	13	(i)	-1	1	1	6	-2	1	8	12	0.1282
2	9	13	(ii)	-1	0	7	4	-2	1	8	12	0.1325
2	9	13	(i)	-1	1	4	2	-2	1	8	12	0.1368
2	9	13	(ii)	-2	1	7	11	-2	1	8	12	0.1453
2	9	13	(ii)	-1	0	4	9	-2	1	8	12	0.1496
2	9	13	(i)	-1	1	1	7	-2	1	8	12	0.1538
2	9	17	(i)	-1	1	1	1	-2	1	8	16	0.0000
2	9	17	(i)	-1	1	1	2	-2	1	8	15	0.0000
2	9	17	(i)	-1	1	1	3	-2	1	8	14	0.0000
2	9	17	(i)	-1	1	1	4	-2	1	8	13	0.0000
2	9	17	(i)	-1	1	1	5	-2	1	8	12	0.0000
2	9	17	(i)	-1	1	1	6	-2	1	8	11	0.0000
2	9	17	(i)	-1	1	2	1	-2	1	7	16	0.0000
2	9	17	(i)	-1	1	2	2	-2	1	7	15	0.0000
2	9	17	(i)	-1	1	2	3	-2	1	7	14	0.0000
2	9	17	(i)	-1	1	2	4	-2	1	7	13	0.0000
2	9	17	(i)	-1	1	3	1	-2	1	6	16	0.0000
2	9	17	(i)	-1	1	3	2	-2	1	6	15	0.0000
2	9	17	(ii)	-1	0	4	4	-2	1	8	16	0.0033
2	9	17	(ii)	-1	0	4	5	-2	1	8	15	0.0033
2	9	17	(ii)	-1	0	4	6	-2	1	8	14	0.0033

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	17	(ii)	-1	0	4	7	-2	1	8	13	0.0033
2	9	17	(ii)	-1	0	4	8	-2	1	8	12	0.0033
2	9	17	(ii)	-1	0	4	9	-2	1	8	11	0.0033
2	9	17	(ii)	-1	0	5	4	-2	1	7	16	0.0033
2	9	17	(ii)	-1	0	5	5	-2	1	7	15	0.0033
2	9	17	(ii)	-1	0	5	6	-2	1	7	14	0.0033
2	9	17	(ii)	-1	0	5	7	-2	1	7	13	0.0033
2	9	17	(ii)	-1	0	1	10	-2	1	8	16	0.0098
2	9	17	(ii)	-1	0	2	10	-2	1	7	16	0.0098
2	9	17	(ii)	-1	0	2	11	-2	1	7	15	0.0098
2	9	17	(ii)	-1	0	2	12	-2	1	7	14	0.0098
2	9	17	(ii)	-1	0	3	10	-2	1	6	16	0.0098
2	9	17	(ii)	-1	0	3	11	-2	1	6	15	0.0098
2	9	17	(i)	-1	1	4	1	-2	1	8	11	0.0131
2	9	17	(ii)	-1	0	7	3	-2	1	8	12	0.0163
2	9	17	(ii)	-1	0	7	4	-2	1	8	11	0.0163
2	9	17	(i)	-1	1	1	2	-2	1	8	16	0.0196
2	9	17	(i)	-1	1	1	3	-2	1	8	15	0.0196
2	9	17	(i)	-1	1	1	4	-2	1	8	14	0.0196
2	9	17	(i)	-1	1	1	5	-2	1	8	13	0.0196
2	9	17	(i)	-1	1	1	6	-2	1	8	12	0.0196
2	9	17	(i)	-1	1	2	2	-2	1	7	16	0.0196
2	9	17	(i)	-1	1	2	3	-2	1	7	15	0.0196
2	9	17	(i)	-1	1	2	4	-2	1	7	14	0.0196
2	9	17	(i)	-1	1	2	5	-2	1	7	13	0.0196
2	9	17	(i)	-1	1	3	2	-2	1	6	16	0.0196
2	9	17	(i)	-1	1	3	3	-2	1	6	15	0.0196
2	9	17	(ii)	-1	0	4	5	-2	1	8	16	0.0229
2	9	17	(ii)	-1	0	4	6	-2	1	8	15	0.0229
2	9	17	(ii)	-1	0	4	7	-2	1	8	14	0.0229
2	9	17	(ii)	-1	0	4	8	-2	1	8	13	0.0229
2	9	17	(ii)	-1	0	4	9	-2	1	8	12	0.0229
2	9	17	(ii)	-1	0	5	5	-2	1	7	16	0.0229
2	9	17	(ii)	-1	0	5	6	-2	1	7	15	0.0229

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	17	(ii)	-1	0	5	7	-2	1	7	14	0.0229
2	9	17	(ii)	-2	1	7	11	-2	1	8	13	0.0261
2	9	17	(ii)	-1	0	1	11	-2	1	8	16	0.0294
2	9	17	(ii)	-1	0	2	11	-2	1	7	16	0.0294
2	9	17	(ii)	-1	0	2	12	-2	1	7	15	0.0294
2	9	17	(ii)	-1	0	2	13	-2	1	7	14	0.0294
2	9	17	(ii)	-1	0	3	11	-2	1	6	16	0.0294
2	9	17	(i)	-1	1	4	1	-2	1	8	12	0.0327
2	9	17	(ii)	-1	0	7	3	-2	1	8	13	0.0359
2	9	17	(ii)	-1	0	7	4	-2	1	8	12	0.0359
2	9	17	(i)	-1	1	1	3	-2	1	8	16	0.0392
2	9	17	(i)	-1	1	1	4	-2	1	8	15	0.0392
2	9	17	(i)	-1	1	1	5	-2	1	8	14	0.0392
2	9	17	(i)	-1	1	1	6	-2	1	8	13	0.0392
2	9	17	(i)	-1	1	1	7	-2	1	8	12	0.0392
2	9	17	(i)	-1	1	2	3	-2	1	7	16	0.0392
2	9	17	(i)	-1	1	2	4	-2	1	7	15	0.0392
2	9	17	(i)	-1	1	2	5	-2	1	7	14	0.0392
2	9	17	(i)	-1	1	3	3	-2	1	6	16	0.0392
2	9	17	(ii)	-1	0	4	6	-2	1	8	16	0.0425
2	9	17	(ii)	-1	0	4	7	-2	1	8	15	0.0425
2	9	17	(ii)	-1	0	4	8	-2	1	8	14	0.0425
2	9	17	(ii)	-1	0	4	9	-2	1	8	13	0.0425
2	9	17	(ii)	-1	0	4	10	-2	1	8	12	0.0425
2	9	17	(ii)	-1	0	5	6	-2	1	7	16	0.0425
2	9	17	(ii)	-1	0	5	7	-2	1	7	15	0.0425
2	9	17	(ii)	-1	0	5	8	-2	1	7	14	0.0425
2	9	17	(ii)	-1	0	1	12	-2	1	8	16	0.0490
2	9	17	(ii)	-1	0	2	12	-2	1	7	16	0.0490
2	9	17	(ii)	-1	0	2	13	-2	1	7	15	0.0490
2	9	17	(ii)	-1	0	3	12	-2	1	6	16	0.0490
2	9	17	(i)	-1	1	4	1	-2	1	8	13	0.0523
2	9	17	(ii)	-1	0	7	3	-2	1	8	14	0.0556
2	9	17	(ii)	-1	0	7	4	-2	1	8	13	0.0556

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	17	(i)	-1	1	1	4	-2	1	8	16	0.0588
2	9	17	(i)	-1	1	1	5	-2	1	8	15	0.0588
2	9	17	(i)	-1	1	1	6	-2	1	8	14	0.0588
2	9	17	(i)	-1	1	1	7	-2	1	8	13	0.0588
2	9	17	(i)	-1	1	2	4	-2	1	7	16	0.0588
2	9	17	(i)	-1	1	2	5	-2	1	7	15	0.0588
2	9	17	(ii)	-1	0	4	7	-2	1	8	16	0.0621
2	9	17	(ii)	-1	0	4	8	-2	1	8	15	0.0621
2	9	17	(ii)	-1	0	4	9	-2	1	8	14	0.0621
2	9	17	(ii)	-1	0	4	10	-2	1	8	13	0.0621
2	9	17	(ii)	-1	0	5	7	-2	1	7	16	0.0621
2	9	17	(ii)	-1	0	5	8	-2	1	7	15	0.0621
2	9	17	(ii)	-2	1	7	12	-2	1	8	14	0.0654
2	9	17	(ii)	-1	0	2	13	-2	1	7	16	0.0686
2	9	17	(ii)	-1	0	2	14	-2	1	7	15	0.0686
2	9	17	(i)	-1	1	4	1	-2	1	8	14	0.0719
2	9	17	(i)	-1	1	4	2	-2	1	8	13	0.0719
2	9	17	(ii)	-1	0	7	4	-2	1	8	14	0.0752
2	9	17	(ii)	-1	0	7	5	-2	1	8	13	0.0752
2	9	17	(i)	-1	1	1	5	-2	1	8	16	0.0784
2	9	17	(i)	-1	1	1	6	-2	1	8	15	0.0784
2	9	17	(i)	-1	1	1	7	-2	1	8	14	0.0784
2	9	17	(i)	-1	1	2	5	-2	1	7	16	0.0784
2	9	17	(i)	-1	1	2	6	-2	1	7	15	0.0784
2	9	17	(ii)	-1	0	4	8	-2	1	8	16	0.0817
2	9	17	(ii)	-1	0	4	9	-2	1	8	15	0.0817
2	9	17	(ii)	-1	0	4	10	-2	1	8	14	0.0817
2	9	17	(ii)	-1	0	5	8	-2	1	7	16	0.0817
2	9	17	(ii)	-2	1	7	12	-2	1	8	15	0.0850
2	9	17	(ii)	-2	1	7	13	-2	1	8	14	0.0850
2	9	17	(ii)	-1	0	2	14	-2	1	7	16	0.0882
2	9	17	(i)	-1	1	4	1	-2	1	8	15	0.0915
2	9	17	(i)	-1	1	4	2	-2	1	8	14	0.0915
2	9	17	(ii)	-1	0	7	4	-2	1	8	15	0.0948

m_1	m_2	m_3	Cond	a	a_1	a_2	a_3	b	b_1	b_2	b_3	τ
2	9	17	(ii)	-1	0	7	5	-2	1	8	14	0.0948
2	9	17	(i)	-1	1	1	6	-2	1	8	16	0.0980
2	9	17	(i)	-1	1	1	7	-2	1	8	15	0.0980
2	9	17	(i)	-1	1	1	8	-2	1	8	14	0.0980
2	9	17	(i)	-1	1	2	6	-2	1	7	16	0.0980
2	9	17	(ii)	-1	0	4	9	-2	1	8	16	0.1013
2	9	17	(ii)	-1	0	4	10	-2	1	8	15	0.1013
2	9	17	(ii)	-1	0	4	11	-2	1	8	14	0.1013
2	9	17	(ii)	-1	0	5	9	-2	1	7	16	0.1013
2	9	17	(ii)	-2	1	7	13	-2	1	8	15	0.1046
2	9	17	(ii)	-1	0	2	15	-2	1	7	16	0.1078
2	9	17	(i)	-1	1	4	1	-2	1	8	16	0.1111
2	9	17	(i)	-1	1	4	2	-2	1	8	15	0.1111
2	9	17	(ii)	-1	0	7	4	-2	1	8	16	0.1144
2	9	17	(ii)	-1	0	7	5	-2	1	8	15	0.1144
2	9	17	(i)	-1	1	1	7	-2	1	8	16	0.1176
2	9	17	(i)	-1	1	1	8	-2	1	8	15	0.1176
2	9	17	(ii)	-1	0	4	10	-2	1	8	16	0.1209
2	9	17	(ii)	-1	0	4	11	-2	1	8	15	0.1209
2	9	17	(ii)	-2	1	7	13	-2	1	8	16	0.1242
2	9	17	(ii)	-2	1	7	14	-2	1	8	15	0.1242
2	9	17	(i)	-1	1	4	2	-2	1	8	16	0.1307
2	9	17	(i)	-1	1	4	3	-2	1	8	15	0.1307
2	9	17	(ii)	-1	0	7	5	-2	1	8	16	0.1340
2	9	17	(ii)	-1	0	7	6	-2	1	8	15	0.1340
2	9	17	(i)	-1	1	1	8	-2	1	8	16	0.1373
2	9	17	(ii)	-1	0	4	11	-2	1	8	16	0.1405
2	9	17	(ii)	-2	1	7	14	-2	1	8	16	0.1438
2	9	17	(i)	-1	1	4	3	-2	1	8	16	0.1503
2	9	17	(ii)	-1	0	7	6	-2	1	8	16	0.1536
2	9	17	(i)	-1	1	1	9	-2	1	8	16	0.1569
2	9	17	(ii)	-1	0	4	12	-2	1	8	16	0.1601
2	9	17	(ii)	-2	1	7	15	-2	1	8	16	0.1634

MIKE KREBS

Curriculum Vitae

3807 Pleasant Place	mkrebs@math.jhu.edu	Phone: 410-243-5179
Baltimore, MD 21211	http://math.jhu.edu/~mkrebs	Citizenship: United States
	Born August 24, 1973 in Princeton, NJ	

EDUCATION

JOHNS HOPKINS UNIVERSITY, Ph.D. candidate, Mathematics
Dissertation: *Toledo invariants on 2-orbifolds* — May 2005 (expected)
Advisor: Dr. Richard Wentworth

POMONA COLLEGE, Bachelor's Degree, Mathematics — May 1994

TEACHING AWARDS

Krieger School of Arts and Sciences Teaching Assistant Award, Johns Hopkins University — 2003–2004

Krieger School of Arts and Sciences Teaching Assistant Award (Finalist), Johns Hopkins University — 2002–2003

Deans' Teaching Fellowship, Krieger School of Arts and Sciences, Johns Hopkins University. Designed and taught a new course: *110.207 Infinity*. — Fall 2002

William Kelso Morrill Award for Excellence in the Teaching of Mathematics, Department of Mathematics, Johns Hopkins University — 2001–2002

Mathematics Teacher of the Year, Brandon Hall School — 1996–1997

RESEARCH

Research Interests: Algebraic geometry, gauge theory, and applications to low dimensional topology

Dissertation abstract: To each component in the space of semisimple representations from the orbifold fundamental group of the base of a Seifert fibered homology 3-sphere into U(2,1), we associate a numerical invariant. Using the theory of Higgs bundles, we compute all values this invariant takes on.

WORK EXPERIENCE AND COURSES TAUGHT

Teaching Assistant, Johns Hopkins University, 1998–present

- Deans' Teaching Fellow. Designed and taught a new course: *110.207 Infinity*. Fall 2002.
- Taught *110.660 Qualifying Exam Problems* for graduate students, covering algebra, analysis, topology, and complex analysis. Spring 2000, Fall 2000, Spring 2001.
- Graduate teaching assistant for:
  - *Calculus I and II (Biological and Social Sciences)*
  - *Calculus I and II (Physical Sciences and Engineering)*
  - *Calculus III—Calculus of Several Variables*
  - *Differential Equations with Applications*
  - *Linear Algebra*
  - *Elementary Number Theory*
  - *Geometry and Relativity*

Johns Hopkins University Center for Talented Youth, Summers since 1994

- Subject Area Coordinator. Acted as *de facto* department chair for Math and Computer Science Department. Summer 1997 and Summer 1998.
- Consultant. Conducted teacher trainings for mathematics teachers at the Institute for Talented Students, Hamilton, Bermuda. Spring 2002 and Spring 2003.
- Keynote speaker. Closing Ceremonies, Franklin & Marshall College site. Summer 2004.
- Instructor. Team-taught *Number Theory*, *Geometry*, and *Individually Paced Math Sequence.*

Houghton Mifflin, 1997–1998
- Editor, McDougal Littell. Edited high school mathematics textbooks.

Fay School, 1996–1997
- Mathematics Teacher. Taught *Pre-Algebra* and *Algebra I*.

Brandon Hall School, 1995–1996
- Mathematics Teacher. Taught *Algebra I*, *Geometry*, *Algebra II*, *Trigonometry*, *Precalculus*, *Calculus*, and *SAT Math*. Wrote curricula for *Precalculus* and *Calculus*.

INVITED TALKS

Geometry and Topology Seminar, University of Maryland at College Park	Fall 2004
Algebraic Geometry Seminar, Johns Hopkins University	Fall 2004
Mathematics Colloquium, Reed College: "When topology meets group representations"	Fall 2004
Graduate Seminar, Johns Hopkins University: "The Kodaira-Enriques Classification"	Spring 2004
Graduate Seminar, Johns Hopkins University: "Vector bundles and moduli spaces"	Spring 2003

PUBLICATIONS

Higgs bundles, orbifold Toledo invariants, and Seifert fibered homology 3-spheres, submitted, 2004

CONFERENCES ATTENDED

Japanese-American Mathematical Institute Conference	Spring 2002

MEMBERSHIPS

American Mathematical Society	1998–present

GRANTS

Kenan Grant Teaching Assistant for *110.407 Geometry and Relativity*, under Dr. Mark Haskins, Johns Hopkins University	Spring 2002
Summer Research Support, Johns Hopkins University	Summer 2002

INSTITUTIONAL SERVICE

Teaching Assitant Training, Johns Hopkins University Department of Mathematics, Fall 2003
- Led training sessions for teaching assistants

Chair, Graduate Representative Organization, Johns Hopkins University, 2002–2003
- Oversaw all aspects of this group, including:
- Running graduate student orientation
- Publishing and distributing the *GRO-Hopkins Guide to Living in Baltimore* (a 200+ page book)
- Advocating for graduate student concerns (e.g., health insurance)
- Organizing social activities
- Managing a $100,000 budget

LaTeX seminar, Johns Hopkins University, Fall 2000
- Initiated and conducted *Introduction to LaTeX* seminar, Johns Hopkins University

Mentor Program, Johns Hopkins University Department of Mathematics
- Instituted a Mentor Program in the Johns Hopkins University Department of Mathematics, wherein returning graduate students counsel incoming graduate students. This program later served as a model for the university-wide graduate student Welcome Wagon program.