Global Strichartz Estimates for Solutions of the Wave Equation Exterior to a Convex Obstacle

by

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ABSTRACT

In this thesis, we show that certain local Strichartz estimates for solutions of the wave equation exterior to a convex obstacle can be extended to estimates that are global in both space and time. This extends the work that was done previously by H. Smith and C. Sogge in odd spatial dimensions. In order to prove the global estimates, we explore weighted Strichartz estimates for solutions of the wave equation when the Cauchy data and forcing term are compactly supported.

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Contents

1 Introduction

The purpose of this paper is to show that certain local Strichartz estimates for solutions to the wave equation exterior to a nontrapping obstacle can be extended to estimates that are global in both space and time. In [23], Smith and Sogge proved this result for odd spatial dimensions $n \geq 3$. Here, we extend this result to all spatial dimensions $n \geq 2$.

If Ω is the exterior domain in $\mathbb{R}^n$ to a compact obstacle and $n \geq 2$ is an even integer, we are looking at solutions to the following wave equation

$$\begin{cases} \Box u(t,x) = \partial_t^2 u(t,x) - \Delta u(t,x) = F(t,x)\,, \quad (t,x) \in \mathbb{R} \times \Omega\,, \\ u(0,x) = f(x) \in \dot{H}^\gamma_D(\Omega)\,, \\ \partial_t u(0,x) = g(x) \in \dot{H}^{\gamma-1}_D(\Omega)\,, \\ u(t,x) = 0\,, \quad x \in \partial\Omega\,. \end{cases} \tag{1.1}$$

Here Ω is the complement in $\mathbb{R}^n$ to a compact set contained in $\{|x| \leq R\}$ with C^∞ boundary. Moreover, Ω is nontrapping in the sense that there is a T_R such that no geodesic of length T_R is completely contained in $\{|x| \leq R\} \cap \Omega$. The case $\Omega = \mathbb{R}^n$ is permitted.

We say that $1 \leq r, s \leq 2 \leq p, q \leq \infty$ and γ are admissible if the following two estimates hold.

Local Strichartz estimates. *For $f, g, F(t,\cdot)$ supported in $\{|x| \leq R\}$, solutions to* (1.1) *satisfy*

$$\|u\|_{L^p_t L^q_x([0,1]\times\Omega)} + \sup_{0\leq t\leq 1} \|u(t,\cdot)\|_{\dot{H}^\gamma_D(\Omega)} + \sup_{0\leq t\leq 1} \|\partial_t u(t,\cdot)\|_{\dot{H}^{\gamma-1}_D(\Omega)} \\ \leq C\left(\|f\|_{\dot{H}^\gamma_D(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)} + \|F\|_{L^r_t L^s_x([0,1]\times\Omega)} \right). \tag{1.2}$$

Global Minkowski Strichartz estimates. *In the case of $\Omega = \mathbb{R}^n$, solutions to* (1.1) *satisfy*

$$\|u\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} + \sup_t \|u(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \sup_t \|\partial_t u(t,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \\ \leq C\left(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L^r_t L^s_x(\mathbb{R}^{1+n})} \right). \tag{1.3}$$

Additionally, for technical reasons we need to assume $2 > r$. If $n = 2$, we must also assume $q > 2$.

The global Minkowski Strichartz estimate (1.3) is a generalization of the work of Strichartz [30, 31]. The local Strichartz estimates (1.2) for solutions to the homogeneous ($F = 0$) wave equation in a domain exterior to a convex obstacle were established by Smith and Sogge in [24]. In [23], Smith and Sogge demonstrated that a lemma of Christ and Kiselev [6] (see also [23] for a proof) could be used to establish local estimates for solutions to the nonhomogeneous problem.

While the arguments that follow are valid in any domain exterior to a nontrapping obstacle, it is not currently known whether the local Strichartz estimates (1.2) hold if the obstacle is not convex. Related eigenfunction estimates are, however, known to fail if $\partial\Omega$ has a point of convexity.

We note here that p, q, r, s, γ are admissible in the above sense if the obstacle is convex, $n \geq 3$,

$$q, s' < \frac{2(n-1)}{n-3}; \quad \frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{r} + \frac{n}{s} - 2$$

$$\frac{1}{p} = \left(\frac{n-1}{2}\right)\left(\frac{1}{2} - \frac{1}{q}\right); \quad \frac{1}{r'} = \left(\frac{n-1}{2}\right)\left(\frac{1}{2} - \frac{1}{s'}\right)$$

where r', s' represent the conjugate exponents to r, s respectively. In particular, notice that we have admissibility in the conformal case

$$p, q = \frac{2(n+1)}{n-1}; \quad r, s = \frac{2(n+1)}{n+3}; \quad \gamma = \frac{1}{2}.$$

Additionally, we note that it is well-known (see, e.g., [13]) that in the homogeneous case ($F = 0$) the Global Minkowski Strichartz estimate (1.3) holds if and only if $n \geq 2$, $2 \leq p \leq \infty$, $2 \leq q < \infty$, $\gamma = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}$, and

$$\frac{2}{p} \leq \frac{n-1}{2}\left(1 - \frac{2}{q}\right) \tag{1.4}$$

Thus, (1.4) provides a necessary condition for admissibility.

The main result of this paper states that for such a set of indices a similar global estimate holds for solutions to the wave equation in the exterior domain.

Theorem 1.1. *If* p, q, r, s, γ *are admissible and* u *is a solution to the Cauchy problem* (1.1), *then*

$$\|u\|_{L_t^p L_x^q(\mathbb{R}\times\Omega)} \leq C\left(\|f\|_{\dot{H}_D^\gamma(\Omega)} + \|g\|_{\dot{H}_D^{\gamma-1}(\Omega)} + \|F\|_{L_t^r L_x^s(\mathbb{R}\times\Omega)} \right).$$

The key differences between this case and the odd dimensional case are the lack of strong Huygens' principle and the fact that the local energy no longer decays exponentially. Local energy decay and the homogeneous Sobolev spaces $\dot{H}_D^\gamma(\Omega)$ will be discussed in more detail in the next section.

At the final stage of preparation, we learned that N. Burq [4] has independently obtained the results from this paper using a slightly different method.

2 Energy Estimates

2.1 Homogeneous Sobolev Spaces

We begin here with a few notes on the homogeneous Sobolev spaces $\dot{H}^\gamma_D(\Omega)$. Fixing a smooth cutoff function $\beta \in C_c^\infty$ such that $\beta(x) \equiv 1$ for $|x| \le R$, for $|\gamma| < n/2$, we are able to define

$$\|f\|_{\dot{H}^\gamma_D(\Omega)} = \|\beta f\|_{\dot{H}^\gamma_D(\tilde{\Omega})} + \|(1-\beta)f\|_{\dot{H}^\gamma(\mathbb{R}^n)}$$

where $\tilde{\Omega}$ is a compact manifold with boundary containing $B_R = \Omega \cap \{|x| \le R\}$. In particular, notice that for functions (or distributions) supported in $\{|x| \le R\}$, we have $\|f\|_{\dot{H}^\gamma_D(\Omega)} = \|f\|_{\dot{H}^\gamma_D(\tilde{\Omega})}$.

The homogeneous Sobolev norms, $\dot{H}^\gamma(\mathbb{R}^n)$, are given by

$$\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \|(\sqrt{-\Delta})^\gamma f\|_{L^2(\mathbb{R}^n)}.$$

For functions supported on a fixed compact set, the homogeneous Sobolev spaces $\dot{H}^\gamma(\mathbb{R}^n)$ are comparable to the inhomogeneous Sobolev spaces $H^\gamma(\mathbb{R}^n)$. In particular, for $r < s$ and for f compactly supported, we have

$$\|f\|_{\dot{H}^r(\mathbb{R}^n)} \le C\|f\|_{\dot{H}^s(\mathbb{R}^n)}.$$

Functions $f \in \dot{H}^\gamma_D(\tilde{\Omega})$ satisfy the Dirichlet condition $f|_{\partial\tilde{\Omega}} = 0$ (when this makes sense). Additionally, when $\gamma \ge 2$, we must require the extra compatibility condition $\Delta^j f|_{\partial\tilde{\Omega}} = 0$ for $2j \le \gamma$. Since in this paper, we always have $\gamma - 1 \ge -1$ the additional compatibility conditions are irrelevant for $\gamma < 0$.

With the Dirichlet condition fixed, we may define the spaces $\dot{H}^\gamma_D(\tilde{\Omega})$ in terms of eigenfunctions of Δ. Since $\tilde{\Omega}$ is compact, we have an orthonormal basis of $L^2(\tilde{\Omega})$, $\{u_j\} \subset H^1_D(M) \cap C^\infty(M)$ with $\Delta u_j = -\lambda_j u_j$ where $0 < \lambda_j \nearrow \infty$. Thus, for $\gamma \ge 0$, it is natural to define

$$\dot{H}^\gamma_D(\tilde{\Omega}) = \left\{ v \in L^2(\tilde{\Omega}) : \sum_{j \ge 0} |\hat{v}(j)|^2 \lambda_j^\gamma < \infty \right\}$$

where $\hat{v}(j) = (v, u_j)$. The $\dot{H}^\gamma_D(\tilde{\Omega})$ norm is given by

$$\|v\|^2_{\dot{H}^\gamma_D(\tilde{\Omega})} = \sum_j |\hat{v}(j)|^2 \lambda_j^\gamma.$$

Defining $\dot{H}^\gamma_D(\tilde{\Omega})$ for $\gamma < 0$ in terms of duality, it is not difficult to see that the above characterization for the norm also holds for negative γ. Additionally, we mention that

$$\|v\|^2_{\dot{H}^1_D(\tilde{\Omega})} = \|v'\|^2_{L^2(\tilde{\Omega})}$$

and for $r < s$,

$$\|v\|^2_{\dot{H}^r_D(\tilde{\Omega})} \leq C\|v\|^2_{\dot{H}^s_D(\tilde{\Omega})}.$$

See, e.g., [33] for further details.

Let $\|f\|_{\dot{H}^\gamma(\Omega)} = \|(\sqrt{-\Delta})^\gamma f\|_{L^2(\Omega)}$ for $\gamma \geq 0$ (as in [24]), and define $\|f\|_{\dot{H}^{-\gamma}(\Omega)}$ via duality. Suppose f is supported in $\{|x| \leq R\}$. Since

$$\|(\sqrt{-\Delta})^\gamma f\|_{L^2(\tilde{\Omega})} \leq \|(\sqrt{-\Delta})^\gamma f\|_{L^2(\Omega)} \leq \|(\sqrt{-\Delta})^\gamma f\|_{L^2(\mathbb{R}^n)},$$

it follows easily that for a distribution g supported in $\{|x| \leq R\}$,

$$\|g\|_{\dot{H}^{-\gamma}(\Omega)} \leq \|g\|_{\dot{H}^{-\gamma}_D(\tilde{\Omega})} = \|g\|_{\dot{H}^{-\gamma}_D(\Omega)}$$

and

$$\|g\|_{\dot{H}^{-\gamma}(\mathbb{R}^n)} \leq \|g\|_{\dot{H}^{-\gamma}_D(\tilde{\Omega})} = \|g\|_{\dot{H}^{-\gamma}_D(\Omega)}$$

for $\gamma \geq 0$.

In order to prove the analogous inequalities for $\dot{H}^\gamma$ spaces with $\gamma \geq 0$, we will need the following proposition.

Proposition 2.1. *For $\Omega, \tilde{\Omega}$ as above, $\gamma \geq 0$, there exist extension operators $\mathfrak{E}_{\tilde{\Omega},\mathbb{R}^n}$, $\mathfrak{E}_{\tilde{\Omega},\Omega}$ such that if $f \in \dot{H}^\gamma_D(\tilde{\Omega})$,*

$$\mathfrak{E}_{\tilde{\Omega},\mathbb{R}^n} f = \mathfrak{E}_{\tilde{\Omega},\Omega} f = f \text{ in } \tilde{\Omega}$$

$$\|\mathfrak{E}_{\tilde{\Omega},\mathbb{R}^n} f\|_{\dot{H}^\gamma(\mathbb{R}^n)} \leq C\|f\|_{\dot{H}^\gamma_D(\tilde{\Omega})}$$

$$\|\mathfrak{E}_{\tilde{\Omega},\Omega} f\|_{\dot{H}^\gamma(\Omega)} \leq C\|f\|_{\dot{H}^\gamma_D(\tilde{\Omega})}$$

Moreover, if f vanishes in a neighborhood of $\partial\tilde{\Omega}$, then $\mathfrak{E}_{\tilde{\Omega},\mathbb{R}^n} f(x) = \mathfrak{E}_{\tilde{\Omega},\Omega} f(x) = 0$ for $x \notin \tilde{\Omega}$.

For γ integral, this result follows from Calderon [5] (Theorem 12). The result for non-integral γ then follows via complex interpolation.

2.2 Local Energy Decay

One of the key results that will allow us to establish the global estimates from the local estimates and the global Minkowski estimates is local energy decay. It is this result that requires the nontrapping assumption on the obstacle. In odd dimensions, we are able to get exponential energy decay: see Taylor [32], Lax-Philips [14], Vainberg [34], Morawetz-Ralston-Strauss [20], Strauss [29], and Morawetz [17, 19]. In even spatial dimensions, the decay is significantly less. The version that we will use in this paper is

Local energy decay. *For $n \geq 2$ even, data f, g supported in $\{|x| \leq R\}$, $0 \leq \gamma < \frac{n}{2}$, and $\beta(x)$ smooth, supported in $\{|x| \leq R\}$, there exist $C < \infty$ such that for solutions to* (1.1) *where $F = 0$ the following holds*

$$\|\beta u(t,\cdot)\|_{\dot{H}^{\gamma}_{D}(\Omega)} + \|\beta \partial_t u(t,\cdot)\|_{\dot{H}^{\gamma-1}_{D}(\Omega)} \leq C\,|t|^{-n/2}\Big(\|f\|_{\dot{H}^{\gamma}_{D}(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_{D}(\Omega)} \Big). \quad (2.1)$$

This is a generalized version of the results of Melrose [16] ($n \geq 4$) and Morawetz [17] ($n = 2$). Before showing how we can derive this generalized version of local energy decay, we would like to mention here the works of Ralston [21] and Strauss [29].

Proof of Equation (2.1). By density, we may, without loss of generality, assume that f, g are C^∞. When $n \geq 2$ is even, Melrose [16] ($n \geq 4$) and Morawetz [17] ($n = 2$) were able to show that a solution to the homogeneous ($F = 0$) Cauchy problem (1.1) outside a nontrapping obstacle with data f, g supported in $\{|x| \leq R\}$ must satisfy

$$\int_{B_R} |\nabla u(t,x)|^2\,dx + \int_{B_R} (\partial_t u(t,x))^2\,dx \leq Ct^{-n}\left(\int |\nabla f|^2\,dx + \int |g|^2\,dx\right) \quad (2.2)$$

where $B_R = \{|x| \leq R\} \cap \Omega$.

Let $\tilde{g}$ be the solution of

$$\begin{cases} \Delta \tilde{g}(x) = g(x) & \text{in } \{|x| \leq R\} \cap \Omega \\ \tilde{g}(x) = 0 & \text{on } \{|x| = R\} \cup \partial\Omega \end{cases}$$

and extend $\tilde{g}$ to all of Ω by setting it to 0 outside $\{|x| \leq R\}$. Let v be the solution to the Cauchy problem

$$\begin{cases} \Box v(t,x) = 0 \\ v(0,x) = \tilde{g}(x) \\ \partial_t v(0,x) = f(x) \\ v(t,x) = 0\,, \quad x \in \partial\Omega\,. \end{cases}$$

By (2.2), we have

$$\|\partial_t v(t,\cdot)\|_{L^2(B_R)} \le Ct^{-n/2}\left(\|\tilde{g}\|_{\dot{H}^1_D(\tilde{\Omega})} + \|f\|_{L^2(\tilde{\Omega})}\right) \tag{2.3}$$

We claim that $u = \partial_t v$. Indeed, since $[\Box, \partial_t] = 0$, we have

$$\begin{cases} \Box \partial_t v(t,x) = 0 \\ \partial_t v(0,x) = f(x) \\ \partial_t\left(\partial_t v(0,x)\right) = \partial_t^2 v(0,x) = \Delta v(0,x) = \Delta\tilde{g}(x) = g(x) \\ \partial_t v(t,x) = 0\,, \quad x \in \partial\Omega\,. \end{cases}$$

Thus, by the uniqueness of solutions to the Cauchy problem, we have that $u = \partial_t v$ and (2.3) yields

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(B_R)} &= \|\partial_t v(t,\cdot)\|_{L^2(B_R)} \\ &\le Ct^{-n/2}\left(\|\tilde{g}\|_{\dot{H}^1_D(\tilde{\Omega})} + \|f\|_{L^2(\tilde{\Omega})}\right) \\ &= Ct^{-n/2}\left(\|\Delta\tilde{g}\|_{\dot{H}^{-1}_D(\tilde{\Omega})} + \|f\|_{L^2(\tilde{\Omega})}\right) \\ &= Ct^{-n/2}\left(\|g\|_{\dot{H}^{-1}_D(\tilde{\Omega})} + \|f\|_{L^2(\tilde{\Omega})}\right) \\ &\le Ct^{-n/2}\left(\|f\|_{\dot{H}^1_D(\tilde{\Omega})} + \|g\|_{L^2(\tilde{\Omega})}\right) \end{aligned}$$

Since $\partial_j(\beta u) = \beta_j u + \beta u_j$, the above calculation and (2.2) yield

$$\|\beta(\cdot)u(t,\cdot)\|_{\dot{H}^1_D(\tilde{\Omega})} + \|\beta(\cdot)\partial_t u(t,\cdot)\|_{L^2(\tilde{\Omega})} \le Ct^{-n/2}\left(\|f\|_{\dot{H}^1_D(\tilde{\Omega})} + \|g\|_{L^2(\tilde{\Omega})}\right) \tag{2.4}$$

Letting v and $\tilde{g}$ be as above, (2.4) gives

$$\begin{aligned} \|\beta(\cdot)v(t,\cdot)\|_{\dot{H}^1_D(\tilde{\Omega})} + \|\beta(\cdot)\partial_t v(t,\cdot)\|_{L^2(\tilde{\Omega})} &\le Ct^{-n/2}\left(\|\tilde{g}\|_{\dot{H}^1_D(\tilde{\Omega})} + \|f\|_{L^2(\tilde{\Omega})}\right) \\ &\le Ct^{-n/2}\left(\|g\|_{\dot{H}^{-1}_D(\tilde{\Omega})} + \|f\|_{L^2(\tilde{\Omega})}\right) \end{aligned} \tag{2.5}$$

Since $u = \partial_t v$, we have

$$
\begin{aligned}
\|\beta\partial_t u(t,\cdot)\|_{\dot{H}_D^{-1}(\tilde{\Omega})} + &\|\beta(\cdot)u(t,\cdot)\|_{L^2(\tilde{\Omega})} \\
&= \|\beta(\cdot)\partial_t^2 v\|_{\dot{H}_D^{-1}(\tilde{\Omega})} + \|\beta(\cdot)\partial_t v(t,\cdot)\|_{L^2(\tilde{\Omega})} \\
&= \|\beta(\cdot)\Delta v(t,\cdot)\|_{\dot{H}^{-1}(\tilde{\Omega})} + \|\beta(\cdot)\partial_t v(t,\cdot)\|_{L^2(\tilde{\Omega})}.
\end{aligned}
$$

Since, also,

$$
\begin{aligned}
\|\beta(\cdot)\Delta v(t,\cdot)\|_{\dot{H}_D^{-1}(\tilde{\Omega})} &\le \|\beta(\cdot)v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|(\Delta\beta(\cdot))v(t,\cdot)\|_{\dot{H}_D^{-1}(\tilde{\Omega})} \\
&\quad + \sum_j \|\beta_j(\cdot)\partial_j v(t,\cdot)\|_{\dot{H}_D^{-1}(\tilde{\Omega})} \\
&\le \|\beta(\cdot)v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|(\Delta\beta(\cdot))v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} \\
&\quad + \sum_j \left(\|\partial_j(\beta_j(\cdot)v(t,\cdot))\|_{\dot{H}_D^{-1}(\tilde{\Omega})} + \|\beta_{jj}(\cdot)v(t,\cdot)\|_{\dot{H}_D^{-1}(\tilde{\Omega})}\right) \\
&\le \|\beta(\cdot)v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|(\Delta\beta(\cdot))v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} \\
&\quad + \sum_j \left(\|\beta_j(\cdot)v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|\beta_{jj}(\cdot)v(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})}\right),
\end{aligned}
$$

(2.5) yields

$$
\|\beta(\cdot)u(t,\cdot)\|_{L^2(\tilde{\Omega})} + \|\beta(\cdot)\partial_t u(t,\cdot)\|_{\dot{H}_D^{-1}(\tilde{\Omega})} \le Ct^{-n/2}\left(\|f\|_{L^2(\tilde{\Omega})} + \|g\|_{\dot{H}_D^{-1}(\tilde{\Omega})}\right). \tag{2.6}
$$

Since $[\Box, \partial_t] = 0$ and ∂_t preserves the support of the data and the boundary condition, we have that $u_t(t,x)$ is a solution of

$$
\begin{cases}
\Box u_t(t,x) = 0\,, \quad (t,x) \in \mathbb{R}\times\Omega\,,\\
u_t(0,x) = g(x)\,,\\
\partial_t u_t(0,x) = \Delta f(x)\,,\\
u(t,x) = 0\,, \quad x \in \partial\Omega\,.
\end{cases}
$$

Thus, by (2.4) and the fact that $\Box u = 0$, we have

$$
\begin{aligned}
\|\beta(\cdot)u_t(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} + &\|\beta(\cdot)\Delta u(t,\cdot)\|_{L^2(\tilde{\Omega})} \\
&= \|\beta(\cdot)u_t(t,\cdot)\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|\beta(\cdot)u_{tt}(t,\cdot)\|_{L^2(\tilde{\Omega})} \\
&\le Ct^{-n/2}\left(\|g\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|\Delta f\|_{L^2(\tilde{\Omega})}\right) \\
&= Ct^{-n/2}\left(\|g\|_{\dot{H}_D^{1}(\tilde{\Omega})} + \|f\|_{\dot{H}_D^{2}(\tilde{\Omega})}\right).
\end{aligned} \tag{2.7}
$$

Since

$$\begin{aligned}\|\beta(\cdot)u(t,\cdot)\|_{\dot{H}^2_D(\tilde{\Omega})} &\leq \|(\Delta\beta)(\cdot)u(t,\cdot)\|_{L^2(\tilde{\Omega})} + \|\beta(\cdot)\Delta u(t,\cdot)\|_{L^2(\tilde{\Omega})} \\ &\quad + \sum_j \|\beta_j(\cdot)\partial_j u(t,\cdot)\|_{L^2(\tilde{\Omega})} \\ &\leq \|(\Delta\beta)(\cdot)u(t,\cdot)\|_{L^2(\tilde{\Omega})} + \|\beta(\cdot)\Delta u(t,\cdot)\|_{L^2(\tilde{\Omega})} \\ &\quad + \sum_j \left(\|\beta_j(\cdot)u(t,\cdot)\|_{\dot{H}^1_D(\tilde{\Omega})} + \|\beta_{jj}(\cdot)u(t,\cdot)\|_{L^2(\tilde{\Omega})}\right),\end{aligned}$$

(2.4),(2.7), and the monotonicity in γ of the norms $\|\cdot\|_{\dot{H}^\gamma_D(\tilde{\Omega})}$ yield

$$\|\beta(\cdot)u(t,\cdot)\|_{\dot{H}^2_D(\tilde{\Omega})} + \|\beta(\cdot)\partial_t u(t,\cdot)\|_{\dot{H}^1_D(\tilde{\Omega})} \leq Ct^{-n/2}\left(\|f\|_{\dot{H}^2_D(\tilde{\Omega})} + \|g\|_{\dot{H}^1_D(\tilde{\Omega})}\right).$$

Similarly, by looking at u_{tt}, u_{ttt}, etc., we see that

$$\|\beta(\cdot)u(t,\cdot)\|_{\dot{H}^s_D(\tilde{\Omega})} + \|\beta(\cdot)\partial_t u(t,\cdot)\|_{\dot{H}^{s-1}_D(\tilde{\Omega})} \leq Ct^{-n/2}\left(\|f\|_{\dot{H}^s_D(\tilde{\Omega})} + \|g\|_{\dot{H}^{s-1}_D(\tilde{\Omega})}\right) \quad (2.8)$$

for any nonnegative integer $s > 1$. Thus, by complex interpolation between (2.4),(2.6), and (2.8), we have

$$\|\beta(\cdot)u(t,\cdot)\|_{\dot{H}^\gamma_D(\tilde{\Omega})} + \|\beta(\cdot)\partial_t u(t,\cdot)\|_{\dot{H}^{\gamma-1}_D(\tilde{\Omega})} \leq Ct^{-n/2}\left(\|f\|_{\dot{H}^\gamma_D(\tilde{\Omega})} + \|g\|_{\dot{H}^{\gamma-1}_D(\tilde{\Omega})}\right) \quad (2.9)$$

for any $\gamma \geq 0$. Finally, by the characterization of the homogeneous Sobolev spaces given above, this is equivalent to (2.1) for any $0 \leq \gamma < n/2$. □

3 Weighted Minkowski Estimates

In this section we show that weighted versions of the Minkowski Strichartz estimates for solutions to the homogeneous wave equation can be obtained when the initial data are compactly supported. Specifically, we are looking at the homogeneous free wave equation

$$\begin{cases} \Box w(t,x) = \partial_t^2 w(t,x) - \Delta w(t,x) = 0\,, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n\,, \\ w(0,x) = f(x) \in \dot{H}^\gamma(\mathbb{R}^n)\,, \\ \partial_t w(0,x) = g(x) \in \dot{H}^{\gamma-1}(\mathbb{R}^n)\,. \end{cases} \tag{3.1}$$

where the Cauchy data f, g are supported in $\{x \in \mathbb{R}^n : |\,x\,| < R\}$.

3.1 Weighted Energy Estimates

We begin by showing that one can obtain weighted versions of the energy inequality. For the case $n \geq 3$, we need only slightly modify the arguments of Hörmander [9] (Lemma 6.3.5, p. 101) and Lax-Philips [14] (Appendix 3).

Lemma 3.1. *Suppose that $n \geq 3$. Let $w(t,x)$ be a solution to the homogeneous Minkowski wave equation* (3.1) *with smooth initial data f, g supported in $\{|x| \leq R\}$. Then, the following estimate holds*

$$\int (t-|x|)^2 \left(\left|\nabla_x w(t,x)\right|^2 + (\partial_t w(t,x))^2 \right) \, dx \leq C_R \left(\int \left|\nabla f\right|^2 + |g|^2 \, dx \right)$$

Proof. It is not difficult to check that

$$\mathrm{div}_x p + \partial_t q = N(w) \Box w$$

where

$$\begin{aligned} N(w) &= 4t(x \cdot \nabla w) + 2(r^2 + t^2) w_t + 2(n-1) t w \\ p &= -2t w_t^2 x - 4t(x \cdot \nabla w) \nabla w + 2t \left|\nabla w\right|^2 x \\ &\qquad\qquad - 2(r^2+t^2) w_t \nabla w - 2(n-1) t w \nabla w \\ q &= 4t(x \cdot \nabla w) w_t + (r^2+t^2)\left(\left|\nabla w\right|^2 + w_t^2\right) + 2(n-1) t w w_t - (n-1) w^2. \end{aligned}$$

If we integrate over a cylinder $[0,T] \times \{x \in \mathbb{R}^n : |x| \leq \bar{R}\}$ for $\bar{R}$ sufficiently large, Huygens' principle and the divergence theorem gives us that:

$$\int_{t=T} q\, dx - \int_{t=0} q\, dx = 0. \tag{3.2}$$

Here, since the initial data are compactly supported, we have

$$\begin{aligned}\int_{t=0} q\, dx &= \int_{t=0} r^2\left(\left|\nabla_x w(0,x)\right|^2 + w_t(0,x)^2\right) - (n-1)w(0,x)^2\, dx \\ &\leq C_R\left(\int \left|\nabla f\right|^2 + |g|^2\, dx\right).\end{aligned} \tag{3.3}$$

Now, let us introduce the standard invariant vector fields

$$Z_0 = t\partial_t + \sum_{j=1}^n x_j\partial_j, \quad Z_{0k} = t\partial_k + x_k\partial_t, \quad Z_{jk} = x_k\partial_j - x_j\partial_k$$

for $j, k = 1, 2, ..., n$. Notice that

$$\int_{t=T} q\, dx = \int_{t=T}\left(\left|Z_0 w\right|^2 + \sum_{0\leq j<k\leq n}\left|Z_{jk}w\right|^2 + 2(n-1)tww_t - (n-1)w^2\right) dx \tag{3.4}$$

Applying Lemma 6.3.5 of Hörmander [9] (p. 101), we see that (3.2)-(3.4) yield

$$\|Z_0 w(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \sum_{j<k}\|Z_{jk}w(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \leq C_R\left(\int\left|\nabla f\right|^2 + |g|^2\, dx\right). \tag{3.5}$$

Thus, we see that in order to complete the proof, it suffices to show that

$$\int (t-r)^2\left(w_t^2 + \left|\nabla w\right|^2\right) dx \leq \|Z_0 w\|^2_{L^2(\mathbb{R}^n)} + \sum_{0\leq j<k\leq n}\|Z_{jk}w\|^2_{L^2(\mathbb{R}^n)}.$$

Since the Cauchy-Schwarz inequality gives us that $\left|\nabla w\right| \geq w_r$ and since $4trw_tw_r \geq -2tr(w_t^2 + w_r^2)$, we have

$$\begin{aligned}\|Z_0 w\|^2_{L^2(\mathbb{R}^n)} + \sum_{0\leq j<k\leq n}\|Z_{jk}w\|^2_{L^2(\mathbb{R}^n)} &= \int (t^2+r^2)\left(w_t^2 + \left|\nabla w\right|^2\right) + 4trw_tw_r\, dx \\ &= \int (t-r)^2\left(w_t^2 + \left|\nabla w\right|^2\right) + 2trw_t^2 + 2tr\left|\nabla w\right|^2 + 4trw_tw_r \\ &\geq \int (t-r)^2\left(w_t^2 + \left|\nabla w\right|^2\right)\end{aligned}$$

as desired. □

We can use a different argument to extend the above result to the case $n = 2$.

Lemma 3.2. *Suppose that $n = 2$. Let $w(t,x)$ be a solution to the homogeneous Minkowski wave equation* (3.1) *with smooth initial data f, g supported in $\{|x| \le R\}$. Then, the following estimate holds*

$$\|(t-|x|)^\theta \partial_t w(t,x)\|_{L^2_x(\mathbb{R}^2)} \le C_R \left(\|f\|_{\dot{H}^1(\mathbb{R}^2)} + \|g\|_{L^2(\mathbb{R}^2)} \right)$$

for any $\theta < 1$.

Proof. For $t - |x| \le 2R$, the inequality follows trivially from standard conservation of energy. We will, thus, focus on $S_t = \{x \in \mathbb{R}^2 : t - |x| > 2R\}$.

We know that w is given by

$$w(t,x) = \partial_t \left(\frac{t}{2\pi} \int_{|y|<1} f(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right) + \left(\frac{t}{2\pi} \int_{|y|<1} g(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right).$$

Since

$$v(t,x) = \frac{t}{2\pi} \int_{|y|<1} f(x+ty) \frac{dy}{\sqrt{1-|y|^2}}$$

is a solution to the initial value problem $\Box v = 0$; $v(0,x) = 0$, $\partial_t v(0,x) = f(x)$, we have

$$\begin{aligned} \partial_t w(t,x) = \Delta_x &\left(\frac{t}{2\pi} \int_{|y|<1} f(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right) \\ &+ \partial_t \left(\frac{t}{2\pi} \int_{|y|<1} g(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right) \\ = \frac{t}{2\pi} &\int_{|y|<1} (\Delta f)(x+ty) \frac{dy}{\sqrt{1-|y|^2}} + \frac{1}{2\pi} \int_{|y|<1} g(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \\ &+ \frac{t}{2\pi} \int_{|y|<1} (\nabla g)(x+ty) \cdot y \frac{dy}{\sqrt{1-|y|^2}}, \end{aligned}$$

and thus,

$$\begin{aligned} \|(t-|x|)^\theta \partial_t w(t,x)\|_{L^2_x(S_t)} \le &\left\| (t-|x|)^\theta \frac{t}{2\pi} \int_{|y|<1} (\Delta f)(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right\|_{L^2_x(S_t)} \\ &+ \left\| (t-|x|)^\theta \frac{1}{2\pi} \int_{|y|<1} g(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \right\|_{L^2_x(S_t)} \\ &+ \left\| (t-|x|)^\theta \frac{t}{2\pi} \int_{|y|<1} (\nabla g)(x+ty) \cdot y \frac{dy}{\sqrt{1-|y|^2}} \right\|_{L^2_x(S_t)}. \end{aligned} \tag{3.6}$$

We will examine the three quantities on the right separately.

For the first, a change of variables and integration by parts (using the fact that f is compactly supported) gives:

$$\begin{aligned}\Big|(t-|x|)^\theta t\int_{|y|<1}(\Delta f)(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\Big| \\ &= C\left|(t-|x|)^\theta \frac{1}{t}\int_{|y|<t}(\Delta f)(x-y)\frac{dy}{\sqrt{1-|\frac{y}{t}|^2}}\right| \\ &= C\left|(t-|x|)^\theta \int_{|y|<t}(\Delta f)(x-y)\frac{dy}{\sqrt{t^2-|y|^2}}\right| \\ &\le C\big|t-|x|\big|^\theta \int_{|y|<t}|\nabla f|(x-y)\frac{|y|\,dy}{(t^2-|y|^2)^{3/2}}.\end{aligned}$$

Since we have that $|y|\le|x|+R$ on the support of f,

$$\frac{1}{(t-|y|)^\theta}\le\frac{1}{(t-|x|-R)^\theta}.$$

Thus, on S_t,

$$\begin{aligned}\left|(t-|x|)^\theta t\int_{|y|<1}(\Delta f)(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\right| \\ &\le C\int_{|y|\le t-R}|\nabla f|(x-y)\frac{|y|\,dy}{(t-|y|)^{3/2-\theta}(t+|y|)^{3/2}}.\end{aligned}$$

Applying Young's inequality to the convolution on the right yields

$$\begin{aligned}\left\|(t-|x|)^\theta \frac{t}{2\pi}\int_{|y|<1}(\Delta f)(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\right\|_{L^2_x(S_t)} \\ &\le C\|\nabla f\|_{L^1(\mathbb{R}^2)}\left\|\frac{y}{(t-|y|)^{3/2-\theta}(t+|y|)^{3/2}}\right\|_{L^2(\{|y|<t-R\})} \\ &\le C\|\nabla f\|_{L^2(\mathbb{R}^2)}\left\|\frac{y}{(t-|y|)^{3/2-\theta}(t+|y|)^{3/2}}\right\|_{L^2(\{|y|<t-R\})}.\end{aligned}$$

The last inequality is a result of Hölder's inequality (since f is compactly supported). We, thus, need to show that the second norm on the right is bounded independent of

t. To do so, we write this integral in polar coordinates and evaluate. Since $\theta < 1$,

$$\begin{aligned}\int_{|y|<t-R} \frac{|y|^2}{(t-|y|)^{3-2\theta}(t+|y|)^3}\, dy &= \int_0^{t-R}\int_0^{2\pi} \frac{\rho^3}{(t-\rho)^{3-2\theta}(t+\rho)^3}\, d\theta\, d\rho \\ &\le C\int_0^{t-R} \frac{\rho^3}{(t^2-\rho^2)^{3-2\theta}(t+\rho)^{2\theta}}\, d\rho \\ &\le C\frac{t^2}{t^{2\theta}}\int_0^{t-R} \frac{\rho}{(t^2-\rho^2)^{3-2\theta}}\, d\rho \\ &\le Ct^{2-2\theta}\big[(2tR-R^2)^{-2+2\theta} - (t^2)^{-2+2\theta}\big] \\ &\le C\frac{t^2}{t^{2\theta}(2Rt-R^2)^{2-2\theta}} \le C\end{aligned}$$

as desired. The last inequality follows from the fact that we are assuming $t \ge t-|x| > 2R$.

For the second piece on the right side of (3.6), a change of variables and considerations of the support of g, as above, yield

$$\begin{aligned}\left|(t-|x|)^\theta \int_{|y|<1} g(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\right| &\le C\left|\frac{(t-|x|)^\theta}{t}\int_{|y|<t} g(x-y)\frac{dy}{\sqrt{t^2-|y|^2}}\right| \\ &\le C\frac{1}{t^{1-\theta}}\int_{|y|<t-R} |g(x-y)|\frac{dy}{\sqrt{t^2-|y|^2}}.\end{aligned}$$

Applying Young's inequality and Hölder's inequality to the norm of the compactly supported function g yields

$$\begin{aligned}&\left\|(t-|x|)^\theta \int_{|y|<1} g(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\right\|_{L^2(S_t)} \\ &\qquad\le \frac{C}{t^{1-\theta}}\|g\|_{L^1(\mathbb{R}^2)}\left\|\frac{1}{\sqrt{t^2-|y|^2}}\right\|_{L^2(\{|y|\le t-R\})} \\ &\qquad\le \frac{C}{t^{1-\theta}}\|g\|_{L^2(\mathbb{R}^2)}\left\|\frac{1}{\sqrt{t^2-|y|^2}}\right\|_{L^2(\{|y|\le t-R\})}.\end{aligned}$$

We, thus, need to prove that the second norm is bounded by $Ct^{1-\theta}$. Expanding the square of this norm and writing in polar coordinates gives

$$\begin{aligned}\int_{|y|<t-R}\frac{dy}{t^2-|y|^2} &= \int_0^{t-R}\int_0^{2\pi}\frac{\rho}{t^2-\rho^2}\, d\theta\, d\rho \\ &\le C\int_0^{t-R}\frac{\rho}{t^2-\rho^2}\, d\rho \\ &= C\left(\ln t^2 - \ln(2Rt-R^2)\right) \le Ct^{2-2\theta}\end{aligned}$$

for any $\theta < 1$, as desired.

For the last piece on the right side of (3.6), we again do a change of variables and integrate by parts

$$\begin{aligned}
&\left|(t-|x|)^\theta t \int_{|y|<1} (\nabla g)(x+ty)\cdot y \frac{dy}{\sqrt{1-|y|^2}}\right| \\
&\quad = C\left|\frac{(t-|x|)^\theta}{t}\int_{|y|<t-R} g(x-y)\left[\frac{1}{\sqrt{t^2-|y|^2}} + \frac{|y|^2}{(t^2-|y|^2)^{3/2}}\right] dy\right| \\
&\quad \le C\frac{1}{t^{1-\theta}}\int_{|y|<t-R} |g(x-y)|\frac{dy}{\sqrt{t^2-|y|^2}} \\
&\qquad\quad + C\frac{(t-|x|)^\theta}{t}\int_{|y|<t-R} |g(x-y)|\frac{|y|^2}{(t^2-|y|^2)^{3/2}}\, dy \\
&\quad \le C\frac{1}{t^{1-\theta}}\int_{|y|<t-R} |g(x-y)|\frac{dy}{\sqrt{t^2-|y|^2}} \\
&\qquad\quad + C(t-|x|)^\theta\int_{|y|<t-R} |g(x-y)|\frac{|y|}{(t^2-|y|^2)^{3/2}}\, dy.
\end{aligned}$$

By the previous step, the first piece is bounded by $C\|g\|_{L^2(\mathbb{R}^2)}$. Replacing ∇f by g in the argument used for the first step of this proof, the second integral is bounded by $C\|g\|_{L^2(\mathbb{R}^2)}$.

Using the bounds that we have just demonstrated for each piece on the right side of (3.6), we have that

$$\|(t-|x|)^\theta \partial_t w(t,x)\|_{L^2(\mathbb{R}^2)} \le C\left(\|f\|_{\dot{H}^1(\mathbb{R}^2)} + \|g\|_{L^2(\mathbb{R}^2)}\right)$$

as desired. □

3.2 Weighted Dispersive Inequality

Next, we look at the weighted analog of the dispersive inequality when the initial data have compact supports. Here we refer to well-known stationary phase estimates.

Lemma 3.3. *Suppose $n \ge 2$. Let w be a solution to the homogeneous Minkowski wave equation* (3.1) *with initial data f, g supported in $\{|x| \le R\}$. Then, we have*

$$\|(|t|-|x|)\partial_t w(t,x)\|_{L^\infty_x(\{|t|-|x|\ge 2R\})} \le \frac{C_R}{|t|^{(n-1)/2}}\left(\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\right).$$

Proof. By scaling, we may assume that $R = 1$. For simplicity, we will demonstrate the result for $t > 0$.

Begin by writing $w = w_1 + w_2$, where w_1 is a solution of the homogeneous Minkowski wave equation (3.1) with Cauchy data $(w, w_t)|_{t=0} = (f, 0)$ and w_2 is a solution of the Minkowski wave equation (3.1) with Cauchy data $(w, w_t)|_{t=0} = (0, g)$. It will, thus, suffice to show that the estimate holds for w_1 and w_2 separately. Since the arguments are the same for each piece, we will restrict our attention to showing that the estimate holds for w_2, the more technical piece.

Since $\partial_t w_2$ is a linear combination of $e^{\pm it\sqrt{-\Delta}} g$, it will be enough to prove that:

$$\left\|(t - |x|)e^{it\sqrt{-\Delta}} g\right\|_{L^\infty(\{t-|x|>2\})} \le \frac{C}{t^{(n-1)/2}} \|g\|_{L^2(\mathbb{R}^n)}$$

when g is supported in $\{|x| < 1\}$.

We begin by fixing a smooth, radial cutoff function χ such that $\chi(\xi) = 1$ for $\{|\xi| \le 1\}$ and $\chi(\xi) = 0$ for $\{|\xi| \ge 2\}$. Then, set

$$\beta(\xi) = \chi(\xi) - \chi(2\xi).$$

Thus, β is a compactly supported, smooth, radial function supported away from 0. In fact, it is not hard to show that $\operatorname{supp} \beta \subset \{1/2 \le |\xi| \le 2\}$, and that we have a partition of unity

$$\chi(\xi) + \sum_{j=1}^{\infty} \beta\left(\frac{\xi}{2^j}\right) = 1$$

for all $\xi \neq 0$.

We decompose $e^{it\sqrt{-\Delta}} g$ as follows:

$$\begin{aligned} &\|(t - |x|)e^{it\sqrt{-\Delta}} g\|_{L^\infty(\{t-|x|>2\})} \\ &\qquad \le \|(t - |x|)e^{it\sqrt{-\Delta}} \chi(\sqrt{-\Delta}) g\|_{L^\infty(\{t-|x|>2\})} \\ &\qquad\quad + \sum_{j=1}^{\infty} \|(t - |x|)e^{it\sqrt{-\Delta}} \beta\left(\frac{\sqrt{-\Delta}}{2^j}\right) g\|_{L^\infty(\{t-|x|>2\})}. \end{aligned} \tag{3.7}$$

We, then, want to examine the pieces on the right side of the decomposition.

Since g is supported in $\{|x| \leq 1\}$, we see that

$$\begin{aligned}
&\left|(t-|x|)e^{it\sqrt{-\Delta}}\beta\left(\frac{\sqrt{-\Delta}}{2^j}\right)g\right| \\
&\qquad \leq \big|t-|x|\big|\int\Big|\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\beta\left(\frac{\xi}{2^j}\right)\,d\xi\Big||g(y)|\,dy \\
&\qquad \leq \big|t-|x|\big|\sup_{|y|\leq 1}\left(\left|\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\beta\left(\frac{\xi}{2^j}\right)\,d\xi\right|\right)\|g\|_{L^1(\mathbb{R}^n)} \\
&\qquad \leq \big|t-|x|\big|\sup_{|y|\leq 1}\left(\left|\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\beta\left(\frac{\xi}{2^j}\right)\,d\xi\right|\right)\|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left|(t-|x|)e^{it\sqrt{-\Delta}}\chi(\sqrt{-\Delta})g\right| \\
&\qquad\qquad \leq \big|t-|x|\big|\sup_{|y|\leq 1}\left(\left|\int e^{i(x-y)\cdot\xi}e^{it|\xi|}\chi(\xi)\,d\xi\right|\right)\|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Set

$$K_j(t,x;y) = \int e^{i(x-y)\cdot\xi}e^{it|\xi|}\beta\left(\frac{\xi}{2^j}\right)\,d\xi$$

for $j = 1, 2, 3, ...$ and set

$$K_0(t,x;y) = \int e^{i(x-y)\cdot\xi}e^{it|\xi|}\chi(\xi)\,d\xi.$$

For $j > 0$, using polar coordinates, we see that we can rewrite this as

$$\begin{aligned}
K_j(t,x;y) &= \int_0^\infty\int_{S^{n-1}} e^{i\rho[(x-y)\cdot\omega+t]}\beta\left(\frac{\rho}{2^j}\right)\rho^{n-1}\,d\sigma(\omega)\,d\rho \\
&= 2^{jn}\int_0^\infty\int_{S^{n-1}} e^{i\rho[2^j((x-y)\cdot\omega+t)]}a(\rho)\,d\sigma(\omega)\,d\rho
\end{aligned}$$

where a is the smooth function that is compactly supported away from 0 given by $\beta(\rho)\rho^{n-1}$. For $j = 0$, we similarly have

$$K_0(t,x;y) = \int_0^\infty\int_{S^{n-1}} e^{i\rho[((x-y)\cdot\omega+t)]}a_0(\rho)\,d\sigma(\omega)\,d\rho$$

where a_0 is given by $\rho^{n-1}\chi(\rho)$. While $a_0(0) = 0$, it is not identically zero in a neighborhood of the origin.

There are four different cases that we must examine separately. For the first two cases, we assume that $t \geq 2|x-y|$. Then, for $j = 1, 2, 3, ...$, set

$$I_j = 2^{jn}\int_0^\infty e^{i\rho[2^j(x-y)\cdot\omega+2^jt]}a(\rho)\,d\rho.$$

Integrating by parts N times yields (using the fact that a is compactly supported away from 0):

$$|I_j| \leq C \frac{2^{jn}}{2^{jN} \big| t - |x-y| \big|^N} \leq C \frac{2^{jn}\, 2^{(n-1)/2}}{2^{jN} |t|^{(n-1)/2}} \frac{1}{\big| t - |x-y| \big|^{N-(n-1)/2}}$$

for any $N = 0, 1, 2, \ldots$. Thus, if we choose $N > n$, we see that:

$$|K_j(t,x;y)| \leq C 2^{-jm} \frac{1}{t^{(n-1)/2}} \frac{1}{\big| t - |x-y| \big|} \tag{3.8}$$

where $m > 0$.

For $j = 0$, set

$$I_0 = \int_0^\infty e^{i\rho[(x-y)\cdot\omega + t]} a_0(\rho)\, d\rho.$$

Here, since a_0 is not supported away from zero, we need to be a bit more careful with the boundary terms. Since $\frac{d^N}{d\rho^N} a_0 = \frac{d^N}{d\rho^N}\big(\rho^{n-1}\chi(\rho)\big) = 0$ for $N < n-1$, we can integrate by parts n times (on the n^{th} time we get a boundary term). Using the fact that a_0 is compactly supported, this yields:

$$\begin{aligned}
|I_0| &\leq \left| \frac{\chi(0)}{(t + \omega\cdot(x-y))^n} \right| + \left| \int_0^\infty \frac{e^{i\rho[(x-y)\cdot\omega+t]}}{(t+\omega\cdot(x-y))^n} a_0^{(n)}(\rho)\, d\rho \right| \\
&\leq C \frac{1}{\big| t - |x-y| \big|^n} \\
&\leq C \frac{1}{t^{(n-1)/2}} \frac{1}{\big| t - |x-y| \big|^{(n+1)/2}}.
\end{aligned}$$

Thus, we have

$$|K_0(t,x;y)| \leq C \frac{1}{t^{(n-1)/2}} \frac{1}{\big| t - |x-y| \big|}. \tag{3.9}$$

For the last two cases, we have that $t < 2|x-y|$. Here, for $j = 1, 2, 3, \ldots$, we write

$$K_j(t,x;y) = 2^{jn} \int_0^\infty \widehat{d\sigma}(2^j \rho(x-y)) e^{i2^j\rho t} a(\rho)\, d\rho$$

and for $j = 0$,

$$K_0(t,x;y) = \int_0^\infty \widehat{d\sigma}(\rho(x-y)) e^{i\rho t} a_0(\rho)\, d\rho.$$

By Theorem 1.2.1 of Sogge [25], we have that

$$\widehat{d\sigma}(\eta) = e^{-i|\eta|} a_1(\eta) + e^{i|\eta|} a_2(\eta)$$

where a_1, a_2 satisfy the bounds:

$$\left|\left(\frac{\partial}{\partial \eta}\right)^{\mu} a_i(\eta)\right| \leq C_\mu (1+|\eta|)^{-(n-1)/2-|\mu|}$$

for $i = 1, 2$. Thus, we have, for $j = 1, 2, 3...$:

$$\begin{aligned}|K_j(t,x;y)| = 2^{jn} &\left|\int_0^\infty e^{i\rho[2^j(t-|x-y|)]} a_1(2^j\rho(x-y))a(\rho)\, d\rho\right| \\ &+ 2^{jn}\left|\int_0^\infty e^{i\rho[2^j(t+|x-y|)]} a_2(2^j\rho(x-y))a(\rho)\, d\rho\right|\end{aligned}$$

and similarly,

$$\begin{aligned}|K_0(t,x;y)| = &\left|\int_0^\infty e^{i\rho[(t-|x-y|)]} a_1(\rho(x-y))a_0(\rho)\, d\rho\right| \\ &+ \left|\int_0^\infty e^{i\rho[(t+|x-y|)]} a_2(\rho(x-y))a_0(\rho)\, d\rho\right|.\end{aligned}$$

Since supp $a(\rho) \subset \{1/2 \leq \rho \leq 2\}$, integrating by parts N times and the estimates for a_1, a_2 yield

$$\begin{aligned}&|K_j(t,x;y)| = \\ &\quad \leq C\, 2^{jn} \frac{1}{2^{jN}(2^j)^{(n-1)/2}} \frac{1}{|x-y|^{(n-1)/2}} \left[\frac{1}{\big|t-|x-y|\big|^N} + \frac{1}{\big|t+|x-y|\big|^N}\right] \\ &\quad \leq C(2^j)^{(n+1)/2-N} \frac{1}{t^{(n-1)/2}} \frac{1}{\big|t-|x-y|\big|^N}.\end{aligned}$$

If we choose $N > \frac{(n+1)}{2}$ and use the fact that we are focusing on $t - |x| \geq 2$ and $|y| \leq 1$, we see that:

$$|K_j(t,x;y)| \leq C 2^{-jm} \frac{1}{t^{(n-1)/2}} \frac{1}{\big|t-|x-y|\big|} \tag{3.10}$$

where $m > 0$.

Again, for $j = 0$, we need to be careful with the boundary terms when integrating by parts. Since $a_0^{(M)}(0) = 0$ for $M < n-1$ and a_0 is compactly supported, integrating by parts N times (for $N < n$) yields

$$\begin{aligned}|K_0(t,x;y)| = &\left|\frac{1}{(t-|x-y|)^N}\int_0^\infty e^{i\rho[(t-|x-y|)]}\left(\frac{\partial}{\partial\rho}\right)^N (a_1(\rho(x-y))a_0(\rho))\, d\rho\right| \\ &+ \left|\frac{1}{(t+|x-y|)^N}\int_0^\infty e^{i\rho[(t+|x-y|)]}\left(\frac{\partial}{\partial\rho}\right)^N (a_2(\rho(x-y))a_0(\rho))\, d\rho\right|.\end{aligned}$$

Each of these integrals are composed of pieces of the form:

$$\begin{aligned}
i_{k,j} &= C_k \left| \int_0^\infty e^{i\rho(t\pm|x-y|)} \left(\frac{\partial}{\partial\rho}\right)^k [a_m(\rho(x-y))] \left(\frac{\partial}{\partial\rho}\right)^{N-k} (\rho^{n-1}\chi(\rho))\, d\rho \right| \\
&= C_{k,j} \left| \int_0^\infty e^{i\rho(t\pm|x-y|)} \rho^{(n-1)-j} \left(\frac{\partial}{\partial\rho}\right)^k [a_m(\rho(x-y))] \left(\frac{\partial}{\partial\rho}\right)^{N-k-j} (\chi(\rho))\, d\rho \right|
\end{aligned}$$

for $m = 1, 2$.

When $j \neq N-k$, $\left(\frac{\partial}{\partial\rho}\right)^{N-k-j} (\chi(\rho))$ is supported away from zero and the above argument gives the desired bound

$$\begin{aligned}
i_{k,j} &\leq C \left| \int_1^2 e^{i\rho(t\pm|x-y|)} \frac{|x-y|^k}{(1+\rho|x-y|)^{(n-1)/2+k}} \rho^{n-1-j}\, d\rho \right| \\
&\leq C|x-y|^{-(n-1)/2}
\end{aligned}$$

When $j = N-k$, we have, by a change of variables,

$$\begin{aligned}
i_{k,N-k} &= C \left| \int_0^\infty e^{i\rho(t\pm|x-y|)} \rho^{n-1-N+k} \chi(\rho) \left(\frac{\partial}{\partial\rho}\right)^k [a_m(\rho(x-y))]\, d\rho \right| \\
&\leq C \int_0^2 \frac{|x-y|^k}{(1+\rho|x-y|)^{\frac{n-1}{2}+k}} \rho^{n-1-N+k}\, d\rho \\
&\leq C|x-y|^{N-n} \int_0^{2|x-y|} \frac{\rho^{n-1-N+k}}{(1+\rho)^{\frac{n-1}{2}+k}}\, d\rho \\
&\leq C|x-y|^{-(n-1)/2}
\end{aligned}$$

for $N < \frac{n+1}{2}$. Since we are assuming that $n \geq 2$, we, in particular, get the bound for $N = 1$.

Plugging these estimates for $i_{k,j}$ into the equation for K_0 when $N = 1$ and using the fact that we are in the case $t < 2|x-y|$, we see that

$$|K_0(t, x; y)| \leq C \frac{1}{t^{(n-1)/2}} \frac{1}{\big|t - |x-y|\big|} \tag{3.11}$$

on $\{t - |x| \geq 2\} \cap \{|y| \leq 1\}$.

Using the estimates (3.8),(3.10) for K_j (j=0,1,2,...), we now see that:

$$\begin{aligned}
&\left|(t-|x|)e^{it\sqrt{-\Delta}}\beta\left(\frac{\sqrt{-\Delta}}{2^j}\right)g\right| \\
&\qquad\qquad \leq C2^{-jm}\frac{1}{t^{(n-1)/2}}\sup_{|y|\leq 1}\left(\frac{\big|t-|x|\big|}{\big|t-|x-y|\big|}\right)\|g\|_{L^2(\mathbb{R}^n)} \\
&\qquad\qquad \leq C2^{-jm}\frac{1}{t^{(n-1)/2}}\frac{\big|t-|x|\big|}{\big|t-|x|-1\big|}\|g\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

where $m > 0$. Hence, we have:

$$\|(t-|x|)e^{it\sqrt{-\Delta}}\beta\left(\frac{\sqrt{-\Delta}}{2^j}\right)g\|_{L^\infty(\{t-|x|>2\})} \leq C2^{-jm}\frac{1}{t^{(n-1)/2}}\|g\|_{L^2(\mathbb{R}^n)}.$$

Similarly, using (3.9), (3.11), we have

$$\|(t-|x|)e^{it\sqrt{-\Delta}}\chi(\sqrt{-\Delta})g\|_{L^\infty(\{t-|x|>2\})} \leq C\frac{1}{t^{(n-1)/2}}\|g\|_{L^2(\mathbb{R}^n)}.$$

Plugging these into (3.7), we see that we get the desired bound for w_2 since 2^{-jm} is summable in j for $m > 0$. □

3.3 Weighted Strichartz Estimates

From the previous three lemmas, we are able to derive a weighted Strichartz estimate for solutions to the Minkowski wave equation with compactly supported initial data.

Theorem 3.4. *Suppose $n \geq 2$ and p, q, γ are admissible. Let w be a solution to the homogeneous Minkowski wave equation* (3.1) *with Cauchy data f, g supported in $\{|x| \leq R\}$. Then, for any $\theta < 1$, we have the following estimate:*

$$\|(|t|-|x|)^\theta w(t,x)\|_{L^p_t L^q_x(\mathbb{R}^{1+n})} \leq C_R\big(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\big).$$

Proof. By the Global Minkowski Strichartz estimate (1.3) and Huygens' principle, it will suffice to show the estimate in the case $|t| - |x| \geq 2R$. We will, also, stick to the case $t \geq 0$. Let $S_t = \{x : t - |x| \geq 2R\}$.

By Lemma 3.1 ($n \geq 3$), Lemma 3.2 ($n = 2$), and Lemma 3.3, we have:

$$\begin{aligned}
\|(t-|x|)^\theta \partial_t w(t,x)\|_{L^2_x(S_t)} &\leq C\left(\|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\right) \\
\|(t-|x|)^\theta \partial_t w(t,x)\|_{L^\infty_x(S_t)} &\leq \frac{C}{t^{(n-1)/2}}\left(\|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\right).
\end{aligned}$$

In the second inequality, we have used the fact that f is compactly supported and the monotonicity in γ of $\dot{H}^\gamma$ for such f. By Riesz-Thorin interpolation, we have

$$\|(t-|x|)^\theta \partial_t w(t,x)\|_{L^q_x(S_t)} \le C\,\left(\frac{1}{t^{(n-1)/2}}\right)^{\left(1-\frac{2}{q}\right)} \left(\|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\right).$$

Since by (1.4)

$$p\cdot\frac{n-1}{2}\left(1-\frac{2}{q}\right) > \frac{p}{2}\left(\frac{n-1}{2}\left(1-\frac{2}{q}\right)\right) \ge 1,$$

we see that taking the L^p_t norm of both sides yields

$$\|(t-|x|)^\theta \partial_t w(t,x)\|_{L^p_t L^q_x(\{t\ge 2R\}\times S_t)} \le C\left(\|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}\right). \tag{3.12}$$

If we now argue as we did in obtaining (2.6) from (2.5), we see that (3.12) yields

$$\|(t-|x|)^\theta w(t,x)\|_{L^p_t L^q_x(\{t\ge 2R\}\times S_t)} \le C\left(\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1}(\mathbb{R}^n)}\right). \tag{3.13}$$

The result, then, follows from the monotonicity of the Sobolev norms since the data f, g are compactly supported and $\gamma \ge 0$.

□

4 Mixed Estimates in Minkowski Space

In this section, we prove a couple of results that follow from the fact that

$$\sup_{\xi} |\xi|^{2\gamma} \left[\int \left|\widehat{\beta}(\xi-\eta)\right| \, \delta(\tau - |\eta|\,) \, d\eta \right] \le C_{n,\gamma,\beta} \, \tau^{2\gamma} \tag{4.1}$$

if β is a smooth function supported in $\{|x| \le 1\}$ and $0 \le \gamma \le \frac{n}{2}$.

Proof of Equation (4.1). Expanding in polar coordinates, we see that

$$|\xi|^{2\gamma} \int |\hat{\beta}(\xi-\eta)|\delta(\tau - |\eta|) \, d\eta = |\xi|^{2\gamma} \int_{S^{n-1}} |\hat{\beta}(\xi - \tau\omega)| \tau^{n-1} \, d\sigma(\omega). \tag{4.2}$$

On the set $|\xi| \ge 2\tau$, since $\hat{\beta}$ is Schwarz class, we have

$$|\hat{\beta}(\xi - \tau\omega)| \le \frac{C}{(1 + |\xi - \tau\omega|)^{n+1}} \le \frac{C}{(1+|\xi|)^{n+1}}.$$

Thus,

$$\sup_{\xi \ge 2\tau} |\xi|^{2\gamma} \left[\int \left|\widehat{\beta}(\xi-\eta)\right| \, \delta(\tau - |\eta|\,) \, d\eta \right] \le C \sup_{|\xi| \ge 2\tau} \frac{|\xi|^{2\gamma}}{(1+|\xi|)^{n+1}} \tau^{n-1}.$$

Since $\frac{|\xi|^{2\gamma}}{(1+|\xi|)^{n+1}}$ is a decreasing function in $|\xi|$, we have that the right side is bounded by

$$C \frac{\tau^{n-1}}{(1+2\tau)^{n+1}} \tau^{2\gamma} \le C\tau^{2\gamma}$$

as desired.

For the case $|\xi| \le 2\tau$, using the fact that $\hat{\beta}$ is Schwarz class, we have

$$\begin{aligned}
\int_{S^{n-1}} |\hat{\beta}(\xi - \tau\omega)| \tau^{n-1} \, d\sigma(\omega) &\le C \int_{S^{n-1}} \frac{\tau^{n-1}}{(1 + |\xi - \tau\omega|)^{n+1}} \, d\sigma(\omega) \\
&\le C \int_{\tau}^{\tau+1} \int_{S^{n-1}} \frac{\rho^{n-1}}{(1 + |\xi - \rho\omega|)^{n+1}} \, d\sigma(\omega) \, d\rho \\
&\le C \int_{\mathbb{R}^n} \frac{1}{(1+|z|)^{n+1}} \, dz < \infty.
\end{aligned}$$

Together with (4.2), this yields

$$\sup_{|\xi| \le 2\tau} |\xi|^{2\gamma} \left[\int |\hat{\beta}(\xi-\eta)| \delta(\tau - |\eta|) \, d\eta \right] < C\tau^{2\gamma}$$

which proves (4.1). □

The first of the results that we shall prove follows with very little change from [23]. The second result requires a different argument from [23] in order to avoid needing sharp Huygens' principle.

Lemma 4.1. *Let β be a smooth function supported in $\{|x| \le 1\}$. Suppose $0 \le \gamma \le \frac{n}{2}$. Then*

$$\int_{-\infty}^{\infty} \|\beta(\cdot)e^{it\sqrt{-\Delta}}f(\cdot)\|^2_{\dot H^\gamma(\mathbb{R}^n)}\,dt \le C_{n,\gamma,\beta}\,\|f\|^2_{\dot H^\gamma(\mathbb{R}^n)}$$

Proof. By Plancherel's theorem in t and x, we can write

$$\int_{-\infty}^{\infty} \|\beta(\cdot)e^{it\sqrt{-\Delta}}f(\cdot)\|^2_{\dot H^\gamma(\mathbb{R}^n)}\,dt = C\int_0^\infty \int |\xi|^{2\gamma}\left|\int \hat\beta(\xi-\eta)\delta(\tau-|\eta|)\hat f(\eta)\,d\eta\right|^2 d\xi\,d\tau.$$

By the Schwarz inequality (in η), this can be bounded by

$$C\int_0^\infty \int |\xi|^{2\gamma}\left(\int |\hat\beta(\xi-\eta)|\,\delta(\tau-|\eta|)\,d\eta\right)\left(\int |\hat\beta(\xi-\eta)|\,\delta(\tau-|\eta|)|\hat f(\eta)|^2\,d\eta\right)d\xi\,d\tau.$$

Now, applying (4.1), we see that

$$\begin{aligned}\int_{-\infty}^{\infty} &\|\beta(\cdot)e^{it\sqrt{-\Delta}}f(\cdot)\|^2_{\dot H^\gamma(\mathbb{R}^n)}\,dt\\ &\le C\int_0^\infty \int \tau^{2\gamma}\left(\int |\hat\beta(\xi-\eta)|\delta(\tau-|\eta|)|\hat f(\eta)|^2\,d\eta\right)d\xi\,d\tau\\ &\le C\int\int |\hat\beta(\xi-\eta)||\eta|^{2\gamma}|\hat f(\eta)|^2\,d\eta\,d\xi.\end{aligned}$$

However, by Young's inequality, this is bounded by

$$C\int |\eta|^{2\gamma}|\hat f(\eta)|^2\,d\eta = C\|f\|^2_{\dot H^\gamma(\mathbb{R}^n)}.$$

□

Lemma 4.2. *Let w be a solution to the Cauchy problem for the Minkowski wave equation*

$$\begin{cases}\Box w(t,x) = \partial_t^2 w(t,x) - \Delta w(t,x) = F(t,x)\,, & (t,x)\in\mathbb{R}\times\mathbb{R}^n\,,\\ w(0,x) = f(x)\,,\\ \partial_t w(0,x) = g(x)\,.\end{cases}$$

Suppose that the global Minkowski Strichartz estimate (1.3) *holds, that $\gamma \le \frac{n}{2}$, and that $r < 2$. Then, for β a smooth function supported in $\{\,|x|\le 1\,\}$, we have*

$$\sup_{|\alpha|\le 1}\int_{-\infty}^{\infty}\|\,\beta(\cdot)\,\partial_x^\alpha w(t,\cdot)\,\|^2_{\dot H^{\gamma-1}(\mathbb{R}^n)}\,dt \le C\left(\,\|f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L_t^rL_x^s(\mathbb{R}^{1+n})}\,\right)^2.$$

Proof. If $F = 0$, the result follows from Lemma 4.1. Thus, it will suffice to show that

$$\int_0^\infty \| \, \beta(\cdot) \, w(t,\cdot) \|^2_{\dot H^\gamma(\mathbb{R}^n)} \, dt \le C \, \|F\|^2_{L^r_t L^s_x(\mathbb{R}^{1+n})}$$

when the initial data f, g are assumed to vanish.

We begin by establishing that

$$TF(t,x) = \Lambda^\gamma \beta(\cdot) \int \frac{\sin(t-s)\Lambda}{\Lambda} F(s,\cdot) \, ds$$

is bounded from $L^r_t L^s_x(\mathbb{R}^{1+n}_+)$ to $L^2_t L^2_x(\mathbb{R}^{1+n})$. In other words, we want to show that

$$\int \|\beta(\cdot) \int \frac{\sin(t-s)\Lambda}{\Lambda} F(s,\cdot) \, dx\|^2_{\dot H^\gamma(\mathbb{R}^n)} \, dt \le C \|F\|^2_{L^r_t L^s_x(\mathbb{R}^{1+n})} \tag{4.3}$$

when F is assumed to vanish for $t < 0$.

By Strichartz estimate (1.3), we have

$$\begin{aligned} \int \frac{|\eta|^{2\gamma}}{|\eta|^2} |\tilde F(|\eta|,\eta)|^2 \, d\eta &= \int \frac{|\eta|^{2\gamma}}{|\eta|^2} \left| \int e^{-is|\eta|} \hat F(s,\eta) \, ds \right|^2 d\eta \\ &\le \sup_t \int |\eta|^{2\gamma} \left| \int_0^t \frac{e^{i(t-s)|\eta|}}{|\eta|} \hat F(s,\eta) \, ds \right|^2 d\eta \\ &\le \sup_t \left(\|w(t,\cdot)\|^2_{\dot H^\gamma(\mathbb{R}^n)} + \|\partial_t w(t,\cdot)\|^2_{\dot H^{\gamma-1}(\mathbb{R}^n)} \right) \\ &\le C \|F\|^2_{L^r_t L^s_x(\mathbb{R}^{1+n})} \end{aligned} \tag{4.4}$$

where $\tilde F$ denotes the space-time Fourier transform of F.

By Plancherel's theorem in t, x, we have

$$\int \|\beta(\cdot) \int \frac{e^{i(t-s)\Lambda}}{\Lambda} F(s,\cdot) \, ds\|^2_{\dot H^\gamma(\mathbb{R}^n)} \, dt = \\ \int_0^\infty \int |\xi|^{2\gamma} \left| \int \hat\beta(\xi-\eta) \delta(\tau - |\eta|) \frac{1}{|\eta|} \tilde F(|\eta|,\eta) \, d\eta \right|^2 d\xi \, d\tau.$$

By the Schwarz inequality in η, this can be bounded by

$$\int_0^\infty \int |\xi|^{2\gamma} \left(\int |\hat\beta(\xi-\eta)| \delta(\tau - |\eta|) \, d\eta \right) \\ \times \left(\int |\hat\beta(\xi-\eta)| \delta(\tau-|\eta|) \frac{1}{|\eta|^2} |\tilde F(|\eta|,\eta)|^2 \, d\eta \right) d\xi \, d\tau.$$

Applying (4.1), (4.4), and Young's inequality, we see that

$$\begin{aligned}
\int \Big\| \beta(\cdot) \int \frac{e^{i(t-s)\Lambda}}{\Lambda} F(s,\cdot)\, ds \Big\|^2_{\dot H^\gamma(\mathbb{R}^n)}\, dt
&\le \int_0^\infty \int \tau^{2\gamma} \int |\hat\beta(\xi-\eta)| \delta(\tau - |\eta|) \frac{1}{|\eta|^2} |\tilde F(|\eta|,\eta)|^2 \, d\eta\, d\xi\, d\tau \\
&= \int\int |\hat\beta(\xi-\eta)| \frac{|\eta|^{2\gamma}}{|\eta|^2} |\tilde F(|\eta|,\eta)|^2 \, d\eta\, d\xi \\
&\le C \int \frac{|\eta|^{2\gamma}}{|\eta|^2} |\tilde F(|\eta|,\eta)|^2\, d\eta \\
&\le C \|F\|^2_{L^r_t L^s_x(\mathbb{R}^{1+n})}.
\end{aligned}$$

By a similar argument, we can show that the same bound holds for

$$\int \Big\| \beta(\cdot) \int \frac{e^{-i(t-s)\Lambda}}{\Lambda} F(s,\cdot)\, ds \Big\|^2_{\dot H^\gamma(\mathbb{R}^n)}\, dt$$

which establishes (4.3). By duality, this is equivalent to having

$$T^*F : L^2_t L^2_x(\mathbb{R}^{1+n}) \to L^{r'}_t L^{s'}_x(\mathbb{R}^{1+n})$$

bounded, where

$$T^*F = \int \frac{\sin(s-t)\Lambda}{\Lambda} \beta(\cdot) \Lambda^\gamma F(s,\cdot)\, ds.$$

We wanted to show, instead, that

$$WF : L^r_t L^s_x(\mathbb{R}^{1+n}) \to L^2_t L^2_x(\mathbb{R}^{1+n})$$

is bounded, where

$$WF(t,x) = \Lambda^\gamma \beta(\cdot) \int_0^t \frac{\sin(t-s)\Lambda}{\Lambda} F(s,\cdot)\, ds.$$

By duality, this is equivalent to showing that

$$W^*F : L^2_t L^2_x(\mathbb{R}^{1+n}) \to L^{r'}_t L^{s'}_x(\mathbb{R}^{1+n})$$

where

$$W^*F(t,x) = \int_t^\infty \frac{\sin(s-t)\Lambda}{\Lambda} \beta(\cdot)\Lambda^\gamma F(s,\cdot)\, ds.$$

This, however, follows from (4.3) after an application of the following lemma of Christ and Kiselev [6] (see also [23]). □

Lemma 4.3. *Let X and Y be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in $B(X,Y)$, the space of bounded linear mappings from X to Y. Suppose that $-\infty \le a < b \le \infty$ and $1 \le p < q \le \infty$. Set*

$$Tf(t) = \int_a^b K(t,s)f(s)\,ds$$

and

$$Wf(t) = \int_a^t K(t,s)f(s)\,ds.$$

Suppose that

$$\|Tf\|_{L^q([a,b],Y)} \le C\|f\|_{L^p([a,b],X)}.$$

Then,

$$\|Wf\|_{L^q([a,b],Y)} \le C\|f\|_{L^p([a,b],X)}.$$

5 Strichartz Estimates in the Exterior Domain

By scaling, we may take $R = \frac{1}{2}$ in the sequel. We begin by proving a weighted version of Theorem 1.1 when the data and forcing terms are compactly supported.

Lemma 5.1. *Suppose u is a solution to the Cauchy problem* (1.1) *with the forcing term F replaced by $F + G$, where F, G are supported in $\{|t| \leq 1\} \times \{|x| \leq 1\}$ and the initial data f, g are supported in $\{|x| \leq 1\}$. Then, for admissible p, q, r, s, γ, there exist a positive, finite constant C and a $\theta > 1/2$, so that the following estimate holds:*

$$\begin{aligned}\left\|(|t| - |x| + 2)^\theta \, u(t,x)\right\|_{L^p_t L^q_x(\mathbb{R}\times\Omega)} \\ \leq C \left(\|f\|_{\dot{H}^\gamma_D(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)} + \|F\|_{L^r_t L^s_x(\mathbb{R}\times\Omega)} + \int \|G(t,\cdot\,)\|_{\dot{H}^{\gamma-1}_D(\Omega)}\, dt \right).\end{aligned}$$

Proof: We will establish the result for $t \geq 0$. We begin this proof in the same manner as in Smith-Sogge [23]. Start by observing that by (1.2) and Duhamel's principle, the result holds for $t \in [0,1]$ and by (1.2)

$$\begin{aligned}\|u(1,\cdot)\|_{\dot{H}^\gamma_D(\Omega)} + \|\partial_t u(1,\cdot)\|_{\dot{H}^{\gamma-1}_D(\Omega)} \\ \leq C \left(\|f\|_{\dot{H}^\gamma_D(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)} + \|F\|_{L^r_t L^s_x(\mathbb{R}\times\Omega)} + \int \|G(t,\cdot)\|_{\dot{H}^{\gamma-1}_D(\Omega)}\, dt \right).\end{aligned} \quad (5.1)$$

By considering $t \geq 1$, Duhamel's principle and support considerations allow us to take $F = G = 0$ with f, g supported in $\{|x| \leq 2\}$.

We now fix a smooth β with $\beta(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\beta(x) = 0$ for $|x| \geq 1$. We, then, write u as $u = \beta u + (1-\beta)u$ and will examine these pieces separately.

We begin by looking at βu. Notice that

$$\Box(\beta u) = \sum_{j=1}^n b_j(x)\partial_{x_j} u + c(x)u \equiv \tilde{G}(t,x)$$

where b_j, c are supported in $\frac{1}{2} \leq |x| \leq 1$. Since $|t| - |x| \leq |t|$, it will suffice to show

$$\|(t+2)^\theta \beta(x) u(t,x)\|_{L^p_t L^q_x([1,\infty)\times\Omega)} \leq C \left(\|f\|_{\dot{H}^\gamma_D(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)} \right).$$

By the local Strichartz estimate (1.2), Duhamel's principle, and local energy decay

(2.1), we have

$$\begin{aligned}
&\|(t+2)^{\theta}\beta(x)u(t,x)\|^p_{L^p_tL^q_x([1,\infty)\times\Omega)} \\
&\qquad \le \sum_{j=1}^{\infty}(j+3)^{p\theta}\|\beta(x)u(t,x)\|^p_{L^p_tL^q_x([j,j+1]\times\Omega)} \\
&\qquad \le C\sum_{j=1}^{\infty}(j+3)^{p\theta}\big(\|\beta(\cdot)u(j,\cdot)\|_{\dot H^{\gamma}_D(\Omega)}+\|\beta(\cdot)\partial_t u(j,\cdot)\|_{\dot H^{\gamma-1}_D(\Omega)} \\
&\qquad\qquad\qquad + \int_j^{j+1}\|\tilde G(s,\cdot)\|_{\dot H^{\gamma-1}_D(\Omega)}\,ds\big)^p \\
&\qquad \le C\sum_{j=1}^{\infty}\frac{(j+3)^{p\theta}}{(j+3)^{p(n/2)}}\left(\|f\|_{\dot H^{\gamma}_D(\Omega)}+\|g\|_{\dot H^{\gamma-1}_D(\Omega)}\right)^p \\
&\qquad = C\left(\|f\|_{\dot H^{\gamma}_D(\Omega)}+\|g\|_{\dot H^{\gamma-1}_D(\Omega)}\right)^p
\end{aligned}$$

so long as $\frac{n}{2}-\theta>\frac{1}{p}$. For $n\ge 4$, since $p\ge 2$, we have the above inequality provided $\theta<\frac{n-1}{2}$. When $n=2$, by (1.4), we must have $p\ge 4$. Thus, the above inequality holds for any $\theta<\frac{3}{4}$.

For the $v(t,x)=(1-\beta)(x)u(t,x)$ piece, we have that v satisfies the Minkowski wave equation

$$\begin{cases}\Box v(t,x)=-\tilde G(t,x)\\ v(0,x)=(1-\beta)(x)f(x)\\ \partial_t v(0,x)=(1-\beta)(x)g(x).\end{cases}$$

Write $v=v_0+v_1$ where v_0 solves the homogeneous wave equation with the same Cauchy data as v and v_1 solves the inhomogeneous wave equation with vanishing Cauchy data. Then, by Theorem 3.4, we have

$$\begin{aligned}
&\|(t-|x|+2)^{\theta}(1-\beta)(x)u(t,x)\|_{L^p_tL^q_x(\mathbb{R}\times\Omega)} \\
&\qquad \le C\left(\|(1-\beta)(x)f\|_{\dot H^{\gamma}(\mathbb{R}^n)}+\|(1-\beta)(x)g(x)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}\right) \\
&\qquad\qquad + \|(t-|x|+2)^{\theta}v_1(t,x)\|_{L^p_tL^q_x(\mathbb{R}\times\Omega)}.
\end{aligned}$$

When $n\ge 4$, we can handle the last piece easily using local energy decay. By Duhamel's principle, write

$$\|(t-|x|+2)^{\theta}v_1(t,x)\|_{L^p_tL^q_x(\mathbb{R}\times\Omega)}=\left\|(t-|x|+2)^{\theta}\int_0^t v_1(s;t-s,x)\,ds\right\|_{L^p_tL^q_x(\mathbb{R}\times\Omega)}$$

where $v_1(s;\cdot,\cdot)$ solves

$$\begin{cases} \Box v_1(s;t,x) = 0 \\ v_1(s;0,x) = 0 \\ \partial_t v_1(s;0,x) = -\tilde{G}(s,x). \end{cases}$$

Applying Minkowski's integral inequality, we have

$$\|(t-|x|+2)^\theta v_1(t,x)\|_{L^p_t L^q_x(\mathbb{R}\times\Omega)} \le C \int s^\theta \|(t-s-|x|+2)^\theta v_1(s;t-s,x)\|_{L^p_t L^q_x(\mathbb{R}\times\Omega)}\, ds.$$

Thus, by Theorem 3.4, the right side is bounded by

$$\int s^\theta \|\tilde{G}(s,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\, ds.$$

Finally, by Calderon's extension [5] discussed in Section 2 and local energy decay (2.1), we have that this is bounded by

$$C\int s^{\theta-\frac{n}{2}} \left(\|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)}\right) ds \le C\left(\|f\|_{\dot{H}^\gamma(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)}\right)$$

for $\theta < 1$.

For $n = 2$, the situation is a bit more delicate. In the remainder of this proof, we will assume that $q < p$. A simple modification of the proof will yield the case $q \ge p \ge 4$. We begin by setting

$$G_j(t,x) = -\chi_{[j,j+1]}(t)\tilde{G}(t,x)$$

where χ_A is the characteristic function of the set A. Let $v_{1,j}$ be the forward solution to

$$\Box v_{1,j}(t,x) = G_j(t,x)$$

with vanishing Cauchy data. By Hölder's inequality, we then have

$$\begin{aligned}(t-|x|+2)^\theta v_1(t,x) &= \sum_{j=3}^{\infty} (t-x+2)^\theta v_{1,j}(t,x) \\ &\le C \sum_{j<\frac{1}{2}(t-|x|+2)} (t-j-|x|+2)^\theta v_{1,j}(t,x) + C \sum_{j\ge\frac{1}{2}(t-|x|+2)} j^\theta v_{1,j}(t,x) \\ &\le C\left(\sum_j |(t-j-|x|+2)^{\theta+\frac{1}{4}} v_{1,j}(t,x)|^{5/4}\right)^{4/5} \\ &\qquad + C\left(\sum_j |j^\theta (t-|x|-j+2)^{1-\varepsilon} v_{1,j}(t,x)|^q\right)^{1/q}\end{aligned}$$

provided $\varepsilon = \varepsilon(q)$ is sufficiently small. Thus, by Minkowski's integral inequality

$$
\begin{aligned}
&\|(|t|-|x|+2)^{\theta} v_1(t,x)\|_{L_t^p L_x^q(\mathbb{R}\times\Omega)} \\
&\qquad \le C\left(\sum_j \|(t-j-|x|+2)^{\theta+\frac{1}{4}} v_{1,j}(t,x)\|^{5/4}_{L_t^p L_x^q(\mathbb{R}\times\Omega)}\right)^{4/5} \\
&\qquad\quad + C\left(\sum_j j^{q\theta}\|(t-j-|x|+2)^{1-\varepsilon} v_{1,j}(t,x)\|^{q}_{L_t^p L_x^q(\mathbb{R}\times\Omega)}\right)^{1/q} \qquad (5.2)
\end{aligned}
$$

By Duhamel's principle, write

$$v_{1,j}(t,x) = \int_0^t v_{1,j}(s; t-s, x)\, ds$$

where $v_{1,j}(s;\cdot,\cdot)$ solves

$$
\begin{cases}
\Box v_{1,j}(s;t,x) = 0 \\
v_{1,j}(s;0,x) = 0 \\
\partial_t v_{1,j}(s;0,x) = G_j(s,x)
\end{cases}
$$

For the first term on the right side of (5.2), if we apply Theorem 3.4, we have

$$
\begin{aligned}
&\sum_j \|(t-j-|x|+2)^{\theta+\frac{1}{4}} v_{1,j}(t,x)\|^{5/4}_{L_t^p L_x^q(\mathbb{R}\times\Omega)} \\
&\qquad = C\sum_j \left\|(t-j-|x|+2)^{\theta+\frac{1}{4}} \int_j^{j+1} v_{1,j}(s;t-s,x)\, ds\right\|^{5/4}_{L_t^p L_x^q(\mathbb{R}\times\Omega)} \\
&\qquad \le C\sum_j \left(\int_j^{j+1} \|(t-s-|x|+2)^{\theta+\frac{1}{4}} v_{1,j}(s;t-s,x)\|_{L_t^p L_x^q(\mathbb{R}\times\Omega)}\, ds\right)^{5/4} \\
&\qquad \le C\sum_j \left(\int_j^{j+1} \|G(s,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^2)}\, ds\right)^{5/4} \\
&\qquad \le C\int \|G(s,\cdot)\|^{5/4}_{\dot H^{\gamma-1}(\mathbb{R}^2)}\, ds.
\end{aligned}
$$

By Calderon's extension [5] (see Proposition 2.1) and local energy decay (2.1), this is bounded by

$$C\int s^{-5/4}\left(\|f\|_{\dot H^{\gamma}(\Omega)} + \|g\|_{\dot H_D^{\gamma-1}(\Omega)}\right)^{5/4} ds \le C\left(\|f\|_{\dot H^{\gamma}(\Omega)} + \|g\|_{\dot H_D^{\gamma-1}(\Omega)}\right)^{5/4}$$

as desired.

For the second term on the right side of (5.2), we proceed similarly. Applying Theorem 3.4, we observe

$$\begin{aligned}
\sum_j j^{q\theta}\|(t-j-|x|+2)^{1-\varepsilon}v_{1,j}(t,x)\|^q_{L^p_tL^q_x(\mathbb{R}\times\Omega)}
&= C\sum_j j^{q\theta}\Big\|(t-j-|x|+2)^{1-\varepsilon}\int_j^{j+1} v_{1,j}(s;t-s,x)\,ds\Big\|^q_{L^p_tL^q_x(\mathbb{R}\times\Omega)}\\
&\le C\sum_j\Big(\int_j^{j+1} s^\theta\|(t-s-|x|+2)^{1-\varepsilon}v_{1,j}(s;t-s,x)\|_{L^p_tL^q_x(\mathbb{R}\times\Omega)}\,ds\Big)^q\\
&\le C\sum_j\Big(\int_j^{j+1} s^\theta\|G(s,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^2)}\,ds\Big)^q\\
&\le C\int s^{q\theta}\|G(s,\cdot)\|^q_{\dot H^{\gamma-1}(\mathbb{R}^2)}.
\end{aligned}$$

Again, by Calderon's extension [5] (see Proposition 2.1) and local energy decay (2.1), this is bounded by

$$C\int s^{-q+q\theta}\left(\|f\|_{\dot H^\gamma(\Omega)}+\|g\|_{\dot H_D^{\gamma-1}(\Omega)}\right)^q\,ds\le C\left(\|f\|_{\dot H^\gamma(\Omega)}+\|g\|_{\dot H_D^{\gamma-1}(\Omega)}\right)^q$$

provided $\theta < 1-\frac{1}{q}$. Since $q>2$ for $n=2$, we see that we may choose $\theta > 1/2$ as desired. □

We are now ready to prove the main theorem.

Proof of Theorem 1.1. By the previous lemma, it will suffice to show the result when f and g vanish for $\{|x|\le 1\}$.

We begin by decomposing u into

$$u(t,x)=u_0(t,x)-v(t,x)$$

where u_0 solves the Minkowski wave equation

$$\begin{cases}\Box u_0(t,x)=F(t,x)\\ u_0(0,x)=f(x)\\ \partial_t u_0(0,x)=g(x).\end{cases}$$

Here F is assumed to be 0 on $\mathbb{R}^n\backslash\Omega$.

We now fix a smooth compactly supported β such that $\beta(x) = 1$ for $|x| \leq 1/2$ and $\beta(x) = 0$ for $x \geq 1$. Then, further decompose u into

$$u(t,x) = u_0(t,x) - v(t,x) = (1-\beta)(x)u_0(t,x) + (\beta(x)u_0(t,x) - v(t,x)).$$

By the Global Minkowski Strichartz estimate (1.3), $(1-\beta)(x)u_0(t,x)$ satisfies the desired estimate. Thus, we may focus on $\beta(x)u_0(t,x) - v(t,x)$.

We have that $\beta(x)u_0(t,x) - v(t,x)$ satisfies

$$\Box(\beta(x)u_0(t,x) - v(t,x)) = \beta(x)F(t,x) + G(t,x)$$

with zero Cauchy data (since we are assuming that f, g vanish for $|x| \leq 1$). Here

$$G(t,x) = \sum_{j=1}^{n} b_j(x)\partial_{x_j}u_0(t,x) + c(x)u_0(t,x).$$

where b_j, c vanish for $|x| \geq 1$. By Lemma 4.2,

$$\int_{-\infty}^{\infty} \|G(t,\cdot)\|^2_{\dot{H}_D^{\gamma-1}(\Omega)} \leq C\left(\|f\|_{\dot{H}_D^{\gamma}(\Omega)} + \|g\|_{\dot{H}_D^{\gamma-1}(\Omega)} + \|F\|_{L_t^r L_x^s(\mathbb{R}\times\Omega)}\right)^2 \tag{5.3}$$

Let

$$\begin{aligned} F_j(t,x) &= \chi_{[j,j+1]}(t)F(t,x) \\ G_j(t,x) &= \chi_{[j,j+1]}(t)G(t,x), \end{aligned}$$

and write (for $t > 0$)

$$\beta u_0 - v = \sum_{j=0}^{\infty} u_j(t,x)$$

where $u_j(t,x)$ is the forward solution to

$$\Box u_j(t,x) = \beta(x)F_j(t,x) + G_j(t,x)$$

with zero Cauchy data.

Thus, by Lemma 5.1, we have

$$\begin{aligned} &\|(t-j-|x|+2)^{\theta}u_j(t,x)\|_{L_t^p L_x^q(\mathbb{R}\times\Omega)} \\ &\qquad \leq C\left(\|F_j(t,x)\|_{L_t^r L_x^s(\mathbb{R}\times\Omega)} + \int_j^{j+1} \|G(t,\cdot)\|_{\dot{H}_D^{\gamma-1}(\Omega)}\,dt\right). \end{aligned} \tag{5.4}$$

Since u_j is supported in the region $t - j - |x| + 2 \geq 1$, an application of the Cauchy-Schwartz inequality yields

$$
\begin{aligned}
|\beta(x)u_0(t,x) - v(t,x)| &\leq \sum_{j=0}^{\infty} |u_j(t,x)| \\
&\leq \left(\sum_{j=0}^{\infty} (t - j - |x| + 2)^{-2\theta} \right)^{1/2} \left(\sum_{j=0}^{\infty} [(t - j - |x| + 2)^{\theta} u_j(t,x)]^2 \right)^{1/2} \\
&\leq C \left(\sum_{j=0}^{\infty} [(t - j - |x| + 2)^{\theta} u_j(t,x)]^2 \right)^{1/2}
\end{aligned}
$$

since we can choose $\theta > 1/2$.

Since $1 \leq r, s \leq 2 \leq p, q$, Minkowski's integral inequality, (5.3), and (5.4) yield

$$
\begin{aligned}
&\|\beta(x)u_0(t,x) - v(t,x)\|^2_{L^p_t L^q_x(\mathbb{R}\times\Omega)} \\
&\quad \leq C \sum_{j=0}^{\infty} \|(t - j - |x| + 2)^{\theta} u_j\|^2_{L^p_t L^q_x(\mathbb{R}\times\Omega)} \\
&\quad \leq C \sum_{j=0}^{\infty} \|F_j\|^2_{L^r_t L^s_x(\mathbb{R}\times\Omega)} + C \sum_{j=0}^{\infty} \left(\int_j^{j+1} \|G(t,\cdot)\|_{\dot{H}^{\gamma-1}_D(\Omega)} \, dt \right)^2 \\
&\quad \leq C \sum_{j=0}^{\infty} \|F_j\|^2_{L^r_t L^s_x(\mathbb{R}\times\Omega)} + C \sum_{j=0}^{\infty} \left(\int_j^{j+1} \|G(t,\cdot)\|^2_{\dot{H}^{\gamma-1}_D(\Omega)} \, dt \right) \\
&\quad \leq C \|F\|^2_{L^r_t L^s_x(\mathbb{R}\times\Omega)} + C \int_0^{\infty} \|G(t,\cdot)\|^2_{\dot{H}^{\gamma-1}_D(\Omega)} \, dt \\
&\quad \leq C \left(\|f\|_{\dot{H}^{\gamma}_D(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_D(\Omega)} + \|F\|_{L^r_t L^s_x(\mathbb{R}\times\Omega)} \right)^2
\end{aligned}
$$

as desired. □

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Vita

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